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PROFESSOR:

It's time to get started. Today, we'll complete our discussion of nuclear magnetic resonance. And we started that last time. And after that, we will talk about multi-particle states, the kind of quantum mechanical states that you need if your system has more than one particle. That will bring us naturally to the idea of denser products of vector spaces, which is the key mathematical idea behind operations such as addition of angular momentum.

By introducing a little before, we really talk about addition of angular momentum, the idea of denser products of vector spaces. I really hope that this subject will look a lot less mysterious in a couple of weeks when we will really be adding angular momentum. So you will have seen the main mathematical structures. And you will know what you're supposed to do when you have a formula. So that should be of some help.

So last time, we were talking about nuclear magnetic resonance. And I described the problem. Here is a summary of where more or less we got last time. We said in the set up for nuclear magnetic resonance, we have a large magnetic field, the longitudinal magnetic field it's sometimes called, along the z direction. In addition to that longitudinal magnetic field, there's a rotating magnetic field in the xy plane. And it's rotating in the direction that you see here. As seen on the xy plane, it's clockwise on the plane.

In terms of a rotation around the z direction, it's negative in the sense that the positive rotation around the z plane would go like this. And this magnetic field is rotating in the other direction. So I wrote it here. The magnitude of this radio frequency component is B_1 . And typically, it's much smaller than B_0 in the problems of interest.

Then, as you recall, the spin Hamiltonian is always given by this constant gamma that relates to the dipole moment of a particle to its spin angular momentum $\mathbf{B} \cdot \mathbf{S}$, the S operator. And the S operator has components S_x, S_y, S_z multiplied by B. Well, the z component picks up a z, the x component S_x , the y component S_y .

And this is your Hamiltonian. And as we emphasized last time, this Hamiltonian is time dependent. Moreover, at different times, the Hamiltonians don't commute. So the HS at time T_1 , HS at time T_2 , they don't commute.

Therefore, any of the formulas we have for solving problems like this don't apply. The only formula that applies is this time ordered exponential that we spoke about that in general is very difficult to use. The time ordered exponential just never quits. And we're writing terms, term after term. And then, you have to add it all up. And adding it all up is difficult.

So most of the time, it's not a great help. So what did we say we would do? We wondered if we could somehow do quantum mechanics, or think of quantum mechanics, in the rotating frame, in a frame that is rotating with a magnetic field in such a way that perhaps in that frame the physics would simplify. And physics simplified would be that the Hamiltonian becomes time independent.

So we're trying to figure out, how would the Hamiltonian look in another frame? Now, we've not discussed in general very explicitly transformations of Hamiltonians within different frames. But we can sort of roughly try to guess what it would have to do.

So what we said last time is that we would imagine that you have an original system and no magnetic field, nothing whatsoever. So forget about all that-- no magnetic field. You put the spin state. It stays there forever, doesn't precess. It doesn't precess because there's no magnetic field.

Then, you jump into a rotating frame. So this was a static frame, no magnetic field. And then, you start rotating. And then, you look at the spins. And the spins are not moving. But you say, oh, I'm rotating. So for me, they seem to be precessing, in fact

precessing with the angular frequency that I am rotating. And therefore, in my rotating frame, I would have a Hamiltonian that I would call the rotating Hamiltonian, H_R . And how would it look?

We claim that in fact spin states in my rotating frame would have to be rotating really this way. Because I'm rotating with the magnetic field. So for me, they're rotating in the positive z direction with angular frequency ω . And this would be the operator that makes them rotate exactly that way. Because if you remember, an operator here minus i angle S_z over \hbar , this is the angle as a function of time. ωt , that's how these operators rotate states.

So this U of t is the U of t that rotates states the way I want them in my rotating frame. Now, you remember for time independent systems, the unitary operator is e to the minus iHt over \hbar . So you identify the Hamiltonian as ωS_z .

In general, given a unitary operator, the Hamiltonian is obtained this way. You take the time derivative of it multiplied by U dagger. And that's the formula for the Hamiltonian given the time evolution operator, something we did about a couple of weeks ago.

So that's where we stood-- some idea about the rotating frame. So the way we now make this complete is to say, well, I'm going to attempt to do the following. I'm going to act with U , this U of t , on this complicated state that I don't know how it behaves. And this I'm going to call the rotated wave function. It's a definition. We're going to use this operator that seems to induce a rotation of the frames and act it on the state to find this. And we're going to hope that the Schrodinger equation for this one is simpler.

So actually, last time, I actually did a computation and found what is the Schrodinger Hamiltonian for this, ψ_R . I want to do it in a slightly different way. I think it's a little easier perhaps, or maybe less familiar but more interesting. I write this like that-- U of t . And I say, look, this state evolves due to its own Hamiltonian. U of t is an extra thing I've put here. So here is U of t .

But how do I distinguish it? I put an S for the spin system, for H_S , ψ_0 . So ψ of t is being evolved by the Schrodinger Hamiltonian, the unitary operator associated to the Hamiltonian that we want times time of 0. So this is the total unitary operator that evolves this state. Therefore, I'm going to say, I'm going to recover the Hamiltonian using this formula.

So I will simply say that the rotating Hamiltonian is going to be H_R is $i\hbar \frac{d}{dt}$ of this whole thing, U of t U^\dagger of t , times its dagger. Because this is the general formula to obtain the Hamiltonian when you know the time evolution operator. So the dagger of this would be U^\dagger of t U of t dagger.

Now we have to evaluate this derivative. It's is not all that bad. If the time derivative hits the U , this one, then U and U^\dagger annihilate each other. So you're left with U^\dagger of t here. So you get $i\hbar \frac{d}{dt} U^\dagger$ of t .

Now, when it acts on this second one, what do we get? Well, I'll write it this way. Plus U of t , because this will be first. Then, $i\hbar \frac{d}{dt} U$. And then, you have U^\dagger of t , and then U of t . Now, this thing is how much? Well, in this thing is in fact this Hamiltonian associated to what we called U , which is just ωS_z .

So actually, I don't know if I have-- yeah, I have this notation here. I think it's useful-- H of U , the Hamiltonian associated to whatever U it is. So you get a nice formula-- plus U of t . This is the Hamiltonian $H_S U^\dagger$ of t .

So this formula we derived also last time by looking at the differential equations satisfied by ψ_R . So it's a nice formula. If somebody gives you something like this and says, here's a unitary operator, what is the new Hamiltonian? The new Hamiltonian is equal to the transformed old Hamiltonian plus the Hamiltonian associated to this time evolution operator. It's a nice formula.

Now, more concretely now, we can see what this is supposed to be. So here is the important crucial step. Let's see, H_R is supposed to be what? We now know we're going to choose U to be this thing. Therefore, H_U is ωS_z hat.

But now we have U of t , U^\dagger of t . So I have plus e to the minus i $\omega t S_z$

over \hbar times the Hamiltonian that we have over there, this full Hamiltonian H_S . So what is it? Minus γ $B_0 \hat{S}_z$ plus $B_1 \cos \omega t \hat{S}_x$ minus $\sin \omega t \hat{S}_y$ -- like that-- e to the $i \omega t \hat{S}_z$ over \hbar .

OK, that's supposed to be our new Hamiltonian. And unless it simplifies, we have not gained all that much. So before simplifying, are there questions, anything we've gotten so far? Yes.

AUDIENCE: So U of t is the unitary operator that enacts the rotation, and it's not corresponding to any Hamiltonian of the spin?

PROFESSOR: Right, U of t is the thing that essentially moves us into the rotating frame. This is the hope that we have. Any unitary operator, if it's what evolves your state, it corresponds to Hamiltonian. But that's not anything a priori to do with our original Hamiltonian. It's just another Hamiltonian. So you choose your U of t behind some Hamiltonian. But what you learn is that the Hamiltonian for your new system, or for the new wave function, will involve that other Hamiltonian. Yes?

AUDIENCE: H_R equals [INAUDIBLE]?

PROFESSOR: This is-- yeah. It's H_U . I think you are probably right. I should call it just H_U for-- yeah, it's a bad idea to call it H_R . U , H_U , much better-- thank you. Here, H_R is the Hamiltonian associated to the Schrodinger equation here. So the claim behind this calculation is that if you calculated $\hbar \frac{d}{dt}$ of ψ_R of t , you would find $H_R \psi_R$ of t . So that's the calculation I think we did last time in which we just calculated these derivatives. And you find this answer.

OK, so let's continue here and see what happens. Well, I have an \hat{S}_z and a \hat{S}_z similarity transformation here. But here is \hat{S}_z . So that \hat{S}_z doesn't care. And it can just go out. So what do we have? We have ω minus $\gamma B_0 \hat{S}_z$ hat like that.

Then, the rest would be what? I can take out the numbers γ , the B_1 . And then, I have this exponential-- e to the minus $i \omega t \hat{S}_z$ hat over \hbar . I'm just having here $\cos \omega t \hat{S}_x$ hat minus $\sin \omega t \hat{S}_y$ hat, e to the $i \omega t \hat{S}_z$ hat over \hbar .

OK, now there's two ways to do this. You have an exponential here and another exponential. This is the kind of thing you've done quite a few times in different ways. You could expand exponentials and just multiply. Or you can sort of do this usual formula of e to the AB e to the minus A for this term and for this term, and simplify, and do many things.

So let's do something different. When one looks at this, one could say, well, here's my problem. H_R -- I know U of t . I fixed it. So knowing ψ is knowing ψ_R . But to know ψ_R , I need to know H_R . And H_R still looks very complicated unless somehow this whole idea has worked out very well and this simplifies. And the hope is that it simplifies. Because in the rotating frame, the field is not rotating anymore.

So I'll call this M of t . And I think one way of doing this, doing something, if you're in a rush and you don't have a formula sheet or any of those things, is to just take its derivative, its time derivative. Now, if you have a function of time, and you don't know what it is, take its time derivative. And maybe you'll see a differential equation or see something.

So let's take the d/dt of M of t . OK, so what do we get? Well, we get the following-- e to the minus $i\omega t$ S_z over \hbar . And I'll put the big parentheses here. OK, here I have an object. This is something I think you are familiar. You've been computing these things for coherence state, for squeeze state, and for all that. So this will certainly sound familiar.

If you take a time derivative of this exponential, it will bring an operator down. When you take the time derivative of this operator, it will bring the operator down but with a different sign. And therefore, it will form the commutator with the thing like this in the middle.

So the first thing in this thing is the commutator of-- you take the time derivative of this-- minus $i\omega S_z$ hat over \hbar with the rest of this thing, with cosine $\omega t S_x$ minus sine $\omega t S_y$. Because the first derivative brings down the minus $i\omega$ over that. So I'll actually take the minus $i\omega$ over \hbar out.

OK, that's the first term that you get from differentiating this and that. And then, you have to differentiate the term in the middle. So then, you get minus ω sine ωt \hat{S}_x minus ω cosine ωt \hat{S}_y .

And now, we can close the big parentheses and put the other exponential like that. OK, so let's see this. \hat{S}_z with \hat{S}_x is $i\hbar \hat{S}_y$. So you get here minus $i\omega$ over \hbar , $i\hbar \hat{S}_y$ cosine ωt . Because \hat{S}_z with \hat{S}_x -- you should remember that I think. Otherwise, you'll waste time. x , y , and z , when you have \hat{S}_z with \hat{S}_x , you get \hat{S}_y . \hat{S}_x with \hat{S}_y , you get a z . And that order helps.

So $i\hbar$ this, and then the other term would be minus sine ωt . But now, \hat{S}_z with \hat{S}_y is minus $i\hbar$, so plus $i\hbar \hat{S}_x$. OK, I won't copy the other term. And let's see if it worked out. Minus i and i is a plus. The \hbar cancels, so plus $\omega \hat{S}_y$ cosine ωt minus ω cosine ωt \hat{S}_y . So this term cancels with this. And this-- there's an $i\hbar$. The sign is the same, so it's a plus ω sine ωt \hat{S}_x minus ω sine ωt .

So these two cancel. And happily, this whole thing is 0. So that is the signal your strategy worked. Why? Because M of t looks like it has lots of time dependence. But in fact, it has none. Its derivative is 0. So if its derivative is 0, you can evaluate it at any time you want.

Well, the best time to evaluate it is time equals 0 now. And you put here 0-- disappears, disappears. This term disappears. The whole thing is just \hat{S}_x , nothing else. So you may want to do it the other way by doing this similarity on this and on that, or do it whichever way you want. But this is good enough. So we have that this whole Hamiltonian has become very simple.

So H_R has become-- and I'll copy it here, I want to make sure-- minus γB_0 plus $\omega \hat{S}_z$ minus $\gamma B_1 \hat{S}_x$. Very nice Hamiltonian-- just two pieces. The \hat{S}_z part of the Hamiltonian that used to be this got this extra piece from the rotation. That's our $\omega \hat{S}_z$. And this piece over here is all that is left of the rotating one. Because in the rotating frame, the magnetic field is always along the x -

axis as it was at time equals 0.

So it's a perfectly nice Hamiltonian. Let's factor a few things. Let's remember the notation that ω_0 is [INAUDIBLE] more frequency associated to B_0 . So I'll write here, this is equal to minus γB_0 minus ω_0 over γS_z plus $B_1 S_x$ hat, which is minus γB_0 minus ω_0 over $\omega_0 S_z$ plus $B_1 S_x$ hat. So let me look at what we've done.

First step, I just factor a γ here, so nothing else. And the same γ went out. Next step was to use this equation, ω_0 equals γB_0 . [INAUDIBLE] to eliminate γ . So you eliminated γ by that. So it's equal to ω_0 over B_0 . So the B_0 went out. And you have 1 minus this thing.

And then, you can think of this H_R as minus γB_1 times S_x . In the usual notation for spin Hamiltonians, minus $\gamma B_1 S_x$, this is the rotating B , the so-called rotating B . But of course, it's not rotating anymore, happily. This is just B_0 times 1 minus ω_0 over $\omega_0 S_z$ plus $B_1 S_x$.

And then, how about our answer? What is the answer to this problem? Well, the answer is the ψ and t . So $\psi(t)$ is U dagger times ψ of t . And U dagger was e to the $i \omega_0 t S_z$ over \hbar -- $\omega_0 t$, yeah.

And what is the other U ? Well, the other U associated to a time evolution is just e to the minus $i H_R t$ over \hbar . Because H_R is time independent. So this equation that tells us how ψ evolves is with H_R . H_R is time independent.

And therefore, you'll have here e to the minus $i H_R t$, which is this. So it becomes plus $i \gamma B_1 S_x$ over $\hbar t$ acting on the state ψ at time equals 0. You see, the state ψ at time equals 0 is the same as the state ψ at time equals 0 because the unitary operator is the same.

So let me box this as well. This is the complete solution for the problem of the rotating spins. So the quantum mechanical problem is you've gotten this time dependent magnetic field. You want to know the spin evolution.

Well, the spin evolution is a little complicated. It's this whole thing. And let me just make sure everything is good here. Perfect. Fine. All right. Questions. Yes?

AUDIENCE: So you have x hat, so x_c hat and x_s hat--

AUDIENCE: BR.

BARTON Oh. I'm sorry. It should be BR. I'm sorry. What is the mistake here? Sorry. This is
ZWIEBACH: terrible. z hat, x hat. Thank you. That's a magnetic field. Another question. Yes?

AUDIENCE: This is a question [INAUDIBLE]. But when we take d/dt of f of t , how can we take the computation relation with h_u in that computation? Why is that the Hamiltonian that we take in the computation?

BARTON I don't understand. When we take the tan derivative of m of t , here was m of t from
ZWIEBACH: here to here.

AUDIENCE: [INAUDIBLE].

BARTON Why is this thing here?

ZWIEBACH:

AUDIENCE: Yeah, ω sc.

BARTON It's because when you take the time derivative of this-- let me give you a formula
ZWIEBACH: that you should check. t_A minus t_A . d/dt of this is equal to e to the t_A , A , B , e to the minus t_A . That's the formula we used. It should be all right.

So we have our time dependent states, and let's analyze it and see what it's doing. Yes? One more question.

AUDIENCE: So [INAUDIBLE] that we have these magnetic fields, but somehow, in our Hamiltonian, it all boils down to just the spin operating in the x direction.

BARTON Well, they're still in the z direction.

ZWIEBACH:

AUDIENCE: There's still the z component, but the time variant components in the y and x direction--

BARTON Has become something like in the x direction. Right.

ZWIEBACH:

AUDIENCE: It's hard to say physically, but what does the system actually look like?

BARTON That's what we're going to do right now.

ZWIEBACH:

AUDIENCE: All right.

BARTON This formula is very pleasant to look at. You say, oh, I'm very accomplished. I

ZWIEBACH: solved it. But what does that do? What does that describe? That's what we have to do now. And that's the interesting part, of course.

So applications always have B_1 much smaller than B_0 . This is the longitudinal component, and that's the case. Let me describe for you what this is doing in case one, ω much smaller than ω_0 .

You see, given B_0 , there is an ω_0 , the Larmor one over there. And B_0 is very large, so ω_0 is very large as well. So ω very slow is reasonable. The magnetic field is rotating much smaller than whatever rotation this other magnetic field would create on the things.

Moreover, if ω is much smaller than ω_0 , B_R is sort of like $B_0 z$ plus $B_1 x$, roughly that. So what does this do? Well, this is a magnetic field mostly along the z-axis. So here is x, y, z. This is a magnetic field that's like this, B_R .

Now, this B_R , let's assume also for an assumption that at t equals 0, the spin of whatever spin state you have is up plus z. So what does this magnetic field do to the spin? Well, you have it here.

So this is going to rotate the spin states around the axis of B_R . The rotation operator here, this is minus the angular velocity, so there's a minus sign here that

you always must think about. But forget about it for a second.

This is rotating around BR, so this, supposed to have a spin state here in the z direction, is going to start to rotating in a little cone like that. It's very close to the BR, so it's just going to rotate around it like that. And that's what this part is going to do.

But you say, but that's not the whole thing. The whole thing is this as well. Well, intuitively, this rotation that this produces a rotation around the z-axis, but it's much slower than this rotation because this BR field, its magnitude is like B_0 roughly. Therefore, it produces a very fast rotation.

So what you must imagine is this little spin state generating this cone here and rotating a million times a second, but this whole thing is still being rotated by this exponential around the z-axis. So this whole cone, in some sense, is precessing around the z-axis.

Now, if you want to know where the spin is at any instant of time, you do the following thing. You say, time equals one second. So you come here and say, one second, a billion turns and a quarter. It ends up here. But in one second, this rotates it by five degrees, so then it rotates a little more. You could do it that way. That's a good way to think about it. But in the approximation, which is so fast, basically, this cone has now, by the second exponential, been rotated like that.

Now let's do the case that is really the important one, the case, as you could imagine in which we know some resonance and ω is set equal to ω_0 . So the physicists know what ω_0 is for the spins, and then they make the radio frequency coincide with that ω_0 . In that case, you lose completely the z component here of this BR. So BR is going to be $B_1 x$.

And then, here is what's going to happen. You have a $B_1 x$ here, a B_1 over here in the x direction. The spin state is sitting at time equals 0 here. And since normal exponentials that do time evolutions have a minus sign here, I claim that instead of rotating the spin around B_1 , it's really rotating it around minus B_1 . So the spin will just go down here. Rotate, rotate, rotate, rotate.

Now, it's rotating with B_1 , which is much smaller than B_0 . So it's rotating with an angular frequency that is much smaller than ω_0 , but it's rotating down here. And now, what's really happening is the following. Let me draw this if I can.

Here it is. The spin is up, and it begins to go into the y -axis. Here is z . B_1 is turning into the y -axis, so it rotates a little. So let me ask you, what is the second exponential going to do? It's going to rotate it further, but this time around the z -axis.

So this one goes down a little bit in some time, but then this z exponential, the other exponential, is going to rotate it around the z -axis. So this is going to go down a little but rotate. So actually, what's going to happen is that it's going to do a spiral. As it begins to go down, the other one rotates. It goes down a little, the other one rotates it.

In fact, this, since B_1 is much smaller than B_0 or ω_0 , here now, ω is equal to ω_0 , so this is rotating around the z direction with ω_0 . But B_1 is much smaller than B_0 , so this rotation is very slow. As it goes down a little, it's rotating very fast and fills out the spiral until it gets down here.

So that's what the poor spin is doing in this thing. It's actually of interest to time the radio frequency signals, to time the systems of B_1 so that you get it to go into the plane so that the spin is maximally perpendicular to the original direction. For that, you choose ω_1 times time, which is the Larmor frequency associated to B_1 times time, to be equal to $\pi/2$.

And ω_1 , just like any ω , is $\gamma B_1 \pi/2$. So t is $\pi/2 \gamma B_1$. It's called the half pulse, something like that. No, a 90 degree pulse. That's a normal name for it.

So if you keep the radio frequency on for this much time, then the spin, at the end of the day, it got to the equator of the sphere, and B_1 has disappeared. So B_1 is gone, B_R is gone, and then it keeps rotating because there is the other magnetic field still, the longitudinal magnetic field.

One exercise that you still have in the homework is to figure out the equation of the spiral that comes out of here. You will find it sounds complicated, but it's a couple of lines. It's very simple, actually, to do it, after you think about it for a second. That's one thing you'll have to do.

So this, in fact, is basically the solution of the problem, and let's just talk for a few minutes about what it's used for. So this is basically the technique that is exactly used for Magnetic Resonance Imaging, MRIs. If you have to say one of the interesting applications of quantum mechanics to technology, MRIs, Magnetic Resonance Imaging, is one of the great ones.

So how does it work? Well, it's a very useful device. In fact, it revolutionized medicine because it goes much beyond what you can do with x-rays. It's very popular. I imagine a good fraction of you have had an MRI. Let's see. How many people have had an MRI in their lives? We're pretty much, I think, 60% of people here.

So you remember getting into this cavity? Well, you've experienced there a large-- if you worry about cell phone radiation, well, there, you were with two Tesla, 20,000 Gauss, a big, big magnet. It's big enough that it has to be cooled with liquid helium at 3 degrees Kelvin so you can get the currents big enough and the magnetic field big enough.

So they put you into this cylinder, a solenoid, and there you go. And it's not dangerous unless you forget some metal or some device like that. In fact, it says in WebMD that if you have some sort of tattoos, they used iron ink, it can burn your skin or you could have some problems. Anyway, if you're claustrophobic, may have MRIs that are open air, but they're less powerful, less strong magnetic field.

So this is two Tesla, roughly. And so what happens is that basically, this thing just is trying to figure out the local concentration of water. It begins like that. The magnetic fields react with the hydrogen atoms, in fact, with the protons inside there. Each proton has a magnetic dipole moment. It gets roughly aligned to the magnetic field, to this B_0 . So the protons get aligned to the B_0 that is going in here, B_0 . So the

proton spin is up there.

In fact, because of temperature, not all of your protons in your body get aligned. Maybe one in a million does, but that's high enough. And then they send this 90 degree pulse.

So this thing starts spiraling, and then it finally goes into this direction and it rotates, rotates like crazy. Then, as it rotates like crazy, a rotating dipole moment is like a generator of electromagnetic waves. So it generates an electromagnetic wave, and there are detectors all over that pick up this signal.

The strength of that signal is proportional to the concentration of water. So it gives you an idea, not to distinguish solid matter versus soft matter, but all kinds of liquids. So it's very, very useful. You pick a signal over here and you get it from a receiver, and it tells you about the local concentration of water. And therefore, it allows you to distinguish different tissues. Some tissues have lots of water, some tissues have less water, so it begins to distinguish different tissues.

Now, there's two more things that people do. This pin is rotating very fast, but there's a relaxation time. After it rotates for a while, there's a time, T_2 , is relaxation of this rotation. Relaxation time. And then there's another time, T_1 , which is the time after this is turned off that it takes the spin to go back again to the original position.

So two times, the relaxation time in which you lose your rotation here because it's interacting with other spins. It's called spin, spin, relaxation. And then an interaction with a whole set of neighboring atoms that brings it eventually back up and aligns it to the magnetic field. So you measure two things, T_1 and T_2 , and those are very good clues because you can put any liquid in the machine and measure its T_1 and T_2 .

And then you place a table of T_1 's and T_2 's, and then if you want to figure out what kind of thing you have in your body, they look it up, and immediately, they may know what are the possible candidates. In fact, T_2 is actually enough to distinguish white matter, gray matter, and fluids in your brain. Totally different T_2 's. T_1 is helpful to

discuss all kinds of things as well. So basically, people have figured out how to do that.

Finally, one last thing. When you go into the machine, it sometimes makes noises. Big noises. And those are gradient magnets that are being moved. You see, basically, if this thing would be like that, you would pick up a signal and you wouldn't know where it comes from.

So what these people do now-- this has gotten extremely sophisticated. They put a gradient magnet which changes the value of B_0 locally by increasing it. For example, the B_0 is not really constant in z , but it changes. So the ω_0 of rotation of the spins changes as well. So with a sufficiently high gradient of magnetic field, they can have spatial resolution. If they pick a signal a little bit higher frequency, they know it's a little bit higher up in your body.

So they get resolutions with these magnets of about how much? Very high resolutions. Half a millimeter. So at the end of the day, it's a great machine, and lots of mathematics in reconstructing images, lots of computation, lots of experimental analysis of constants, much of it phenomenological. It would be hard to predict those constants, but you can measure them. So it's very, very practical and very nice.

So it's a nice application, and we'll leave it at that. Are there any questions? Not that I know much more about that. But it's a junior lab experiment as well. So I believe the calculation is fairly straightforward and the technology is just amazing. It's great. Yes?

AUDIENCE: So about getting your spin back up along the z direction, is that mechanism the same as when you relax [INAUDIBLE]?

BARTON When you relax what?

ZWIEBACH:

AUDIENCE: So is the mechanism for getting your spin back along the z direction to find T_1 , is that the same mechanism as when you do all of this [INAUDIBLE]?

BARTON

ZWIEBACH:

No. It's sort of a different mechanism. We don't analyze it. It's not being driven by Schrodinger time evolution. All these relaxations are more complicated phenomena. In fact, if you maintain the magnetic field, this is supposed to go on forever and never stop rotating. But due to interactions with other things, it's sort of stops rotating at some stage, and then eventually, many of them also go back up. So these are not easily calculable things within our analysis. These are phenomenological constants that need to be measured or done by experiment.

So last part of today's lecture is multi-particle states and tensor products. Let's get on with that. Let's see we have there. So multi-particle states and tensor products.

The idea is the following. You have a system with more than one particle. Let's talk two particles. It won't matter at this moment whether they're distinguishable, not distinguishable. Those are things that come later, and we will probably not discuss much of that in this semester. This will be more in 806.

But let's consider if we have particle one, and we'll keep the possibility that this is completely distinguishable, in fact. Particle one. Its quantum mechanics is described by a complex vector space, v . And you have some operators, T_1 , T_2 . Particle two, complex vector space, w , and the operators, S_1 , S_2 , all that.

You see, the list of operators is something that you are already familiar with. Quantum mechanics operators can include momentum, position, if it's three dimensional, three positions, three momenta, angular momentum, spin, Hamiltonians, all those things, and those exist for both particles. So the question is, how do we describe the composite system, the system of the two particles that exist at the same time, that possibly could interact even with each other? So we need a description of these two things.

Well, a description of particle one is described by some vector, v , in the vector space. Description of particle two is some state, some vector ω , in the vector space w . So we imagine that it's a reasonable thing to give you those two vectors and say, look, that's what particle one is doing, that's what particle two is doing, and

that is correct. It's a bit more subtle than that, as we'll see now, but we could list v and list w , and this is the information about each particle.

And this is, in some ways, too naive, as we will see. It has the right idea, but the amazing possibilities that can take place when you have these two particles are not quite reflected on that thing yet. So to make this idea clearer, we'll use a notation.

So I will say that I will encode these things and I will write them as v tensor product w . It's reflecting that we're going to do something more than just view this as the possibility. You have a system. You know what the particle one is doing, you know what the second particle is doing. List those two, that's all that can happen. Not quite true.

So let's put it like that, and this will be the information.

BARTON

I'm not multiplying this in any obvious way. It's not that I'm supposed to multiply and do a calculation here. I'm just putting the two pieces of data but putting this curly ball here as to say that it's the two together, and we'll have some ways of dealing with this object.

ZWIEBACH:

And this will be said to be an element of a new, complex vector space, v tensor w . So what we are saying here, with v belonging to v and w belonging to w , this thing belongs to v plus w . Tensor product of the vector spaces.

At this moment, this is a little funny, but let me ask you some things about it. We have v w . So this is an element of that space. I put a vector of the first vector space here and I put a vector of the second vector space, and this will be a vector here. I'm not saying it's the most general vector there, or how we operate with them, but it's a vector there.

Now, you say, look, when I had states, you had states plus minus, you put constants in front of them. So I'll put a constant in front of this v , α . Well, is this something related? We had this vector that was a vector in the tensor product. How is this vector related to this vector?

Well, unless you declare that these things are somewhat related, this object is going to be very, very large because if this vector has nothing to do with this, this is going to be yet another vector linearly independent with this. So it's, again, your choice what you're going to declare. If you will be constructing what is called a direct product, not a tensor product, and that's apologies to [? Shanker. ?] He uses the word "direct products" wrong mathematically. Mathematicians don't call this a direct product. Direct product is something in which this has nothing to do with this.

But in physics, you're thinking of the amplitude to have a first particle doing that, and suddenly, that amplitude becomes twice as big and this other one doesn't change. Well, the amplitude defining this one here and that one there has become twice as big just because this one is twice more likely to be here and that's the same. So we're going to say that this is the same as that.

The alpha can go out and it's the same thing, and it's also the same thing as v tensor alpha w . So numbers go out, and they don't go out with a complex conjugate. They go out just like that. So that's one thing we declare that will happen with these things, and that's a property that we impose. It's what is usually understood to take place in tensor products.

Now, if this is a vector in this space, then if, for example, v_1 is a vector and v_2 cross w_2 is a vector in v cross w , then this is supposed to be a linear vector space. So it should be true that $\alpha v_1 + \beta v_2$ also belongs to this vector space.

And suddenly, you see why this was a little too naive. If you tell me, this is what particle one is doing and this is what particle is doing, end of story, this list, well, quantum mechanics seems to say that there's more interesting possibilities in which the state is a superposition now in which first particle is doing this, which may be a superposition itself, and this one is doing this, plus first particle is doing that and second particle is doing that.

So it's not enough to say to the state of two particles, say, you just need to know the state of one, state of the other, list the two. That is a possible state of the system,

but it's not the most general. The most general is a superposition in which particle one does something, particle two does something, plus another possibility, particle one is doing something else, particle two is doing another thing.

This is the origin of this famous idea of entanglement, as we will see you next time. But let's just continue here. These particles, roughly speaking, seem to be entangled, in which you can't say what particle one is doing without knowing what particle two is doing and things like that.

So something new has happened by taking this idea slowly and figuring out what's going on, but we need yet one more thing to cut down this tensor product. This tensor product is still a little bigger than what you may want it to be in the following sense.

Suppose you have v_1 plus v_2 , which is another vector, tensor w , another vector. Unless you tell me what this is, I don't know that this is actually v_1 plus v_2 unless you declare it. This is just another vector here and that. Why should that be equal to this?

Well, it's the way you feel about quantum mechanics. That's how you should think of the composite system. If the first state can be either of two possibilities while the other one is not, well it's a superposition in which, yes, the first state is doing this and the first state is doing something different with the second doing the same.

So this is part of also an axiom of the tensor product. Whenever you have tensor products, you will simplify in that way. It looks almost trivial, but it's certainly something that has to be stated. If you had a direct product, you form a vector space by just putting first vector, second vector, and you don't explain anything. This would not be true.

So similarly, we'll also have $v_1 w_1$ plus w_2 equals $v_1 w_1$ plus $v_1 w_2$. So these are our last operations. So with these two operations, we've defined completely how to calculate with this two particle Hilbert space, and how to define the states.

Let me add some intuition to this by stating the following. So what is this space, v tensor w , is the space spanned by things like this. So with these rules, here comes the thing that puts it all, in a sense, together.

If e_1 up to e_n is a basis for v , and f_1 up to f_m is a basis for w , this set of things, e_i tensor f_j , and how many of them are there? There are n of this and m of those. This for i from 1 up to n , j from 1 up to m , form a basis for v tensor w .

So basically, if you have a four dimensional vector space v and a 10 dimensional vector space w , v cross w is 40 dimensional. You multiply the dimensions. You don't sum them. In the tensor product, the dimensions multiply.

You see, the elements of this tensor product was one vector here, one vector there. So it behooves you that you would expect that, well, you can get all vectors that can sit here with all the e_i 's, all the vectors here, but only if you have these rules of superposition and linearity. So this is the whole basis of the vector space. And dimension of v cross w would be equal to the dimension of v times the dimension of w . Questions? Yes?

AUDIENCE: So for the first box thing up there, so v and w are two different vector spaces. Let's say that they're vector spaces over different fields. How do you know that--

BARTON Oh no. Different fields, no. Both are over complex numbers.

ZWIEBACH:

AUDIENCE: Oh, OK. They're both complex.

BARTON Both spaces are complex vector spaces. Did I say it? Complex vector space,

ZWIEBACH: complex vector space. Then v cross w is an element of a new complex vector space. So yes, everybody's complex here.

Let me continue a little more because we need to get a little further. I would like to get a little further here today. We still have about 10 minutes. These are really, in some ways simple but in some ways very subtle ideas. I hope you appreciate it. At some moments, you think, this is obvious. It's ridiculous. It's taking too long to

explain it. At some point, you step back and say, I'm now confused. What does that mean? It is subtle.

So let's see now what happens. We need to do operators on v cross w . So suppose T is an operator in v , and S is an operator in w . So we call them like that, T operators and the S operator on w . So here it is. We define an S tensor T that will be an operator on the tensor product, v cross w . Let's see how it is.

So if this is an operator on the tensor product, it should know how to act on v any element. In any basic element or element of this form, since all the elements can be written as superposition of these things, if this is a linear operator and I define it acting on this, we've got it.

Now, what I'm going to write in the right hand side is a matter of common sense. It's a definition, but it's hardly possible to have any other definition than what I'm going to do. Lights. Main screen, window shades. Oh, lights here. There we go.

Well, you know, I got the order wrong here, T cross S . And the only thing you could do is if T , you know how it acts on v . Well, let it act on v . And it will be a vector in v , and you know how S acts on w , so let it act on w , and that's the definition. Not that complicated, really, in a sense. Each operator acts on its thing, so you just let it act. Whoever has to act, acts wherever it can do it.

And it's linear operator, and if you act on more things, it will be linear. It will all be fine here. Now, this is a fine operator. Now, the operators that are a little more surprising, perhaps, at first sight are the following.

Suppose you have an operator, T_1 , that is an operator on v , and you want it to act on the tensor product. You say, well, I need another operator. This acts on v . I need another operator attached on the tensor product, but you don't give me one, so what am I to do? So this is what is called upgrading an operator that acts on one vector space to an operator that acts on the whole thing because you need everything to act now on the tensor product.

So what you do is the following. You let the operator become T_1 that acts on v , and

you put tensor product with the identity operator. This is more or less what you would imagine. You have now an operator that belongs to the linear operators on the tensor product, but it's the operator you got times that. If you had an operator, S_1 , belonging to L of w , it would go into 1 times S_1 .

Now I ask you the question, do these operators commute? Strange question, but it's a fundamental question. It's really the basis of most of your intuition about two particle systems. Well, let's see if they commute. You have T_1 tensor 1 multiplied by 1 tensor S_1 . It's going to act on v tensor w . Or you have it in the other order, 1 S_1 T_1 .

Well, when it acts, the first part, it gives me T_1 tensor 1 , but now acting on this, this is v tensor Sw . And then when I act on this one, I get now T_1 on v tensor S_1 of w . And when I act here, the first step, it gives me a T_1 on v , and the second gives me the S_1 on w , so it gives me the same thing. T_1 on v , S_1 on w .

So these two operators, because they originated, they've operators of the first particle, operator of the second particle, they can act on the whole system, but they still commute. They don't know about each other, and the communication now is the calculation which you just have seen. What this is a good example of this thing is that when you try to write the Hamiltonian of the whole system, H total, you would say, oh, it's the Hamiltonian of the first system, tensor 1 , plus 1 tensor the Hamiltonian of the second system.

Time for an example. So the example is a famous one, and this is a great example because it's at the basis of combining angular momentum. So it's an example you're going to see now and see many times.

Two spin one $1/2$ particles. The first particle has a state plus and a state minus for the first article. The second particle has a state plus and a state minus.

So how do we form the tensor product? Well, we say, these are our basis states for the first Hilbert space, the basis states for the second Hilbert space. We're supposed to take the product state. So our tensor product is going to be spanned,

so two spins. The tensor vector space is spanned by a vector in the first times a vector in the second, the basis vectors. You could have plus in 1, minus in 2, minus in 1, plus in 2, and minus in 1, minus in 2.

So two spin states form a four-dimensional complex vector space. That's how you describe them, with those little products. And the most general state is the following. Most general state is a ψ , which is $\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3 + \alpha_4 \psi_4$ -- you could put 1 and 2 -- plus here's α_1 , α_2 , plus, minus, plus α_3 minus, plus, plus α_4 minus, minus.

Let's do just one simple computation, be done for today. I want you to try to figure out what is the result of acting with a total z component of angular momentum on this state, whatever that means. Total z component of angular momentum. So naively speaking, the total z component of angular momentum would be the z component of angular momentum of the first particle plus the z component of the angular momentum of the second particle, but you know that's not the way we should write it.

So how is it really? It should be S_z of the first particle tensor product with 1 plus 1 tensor product with S_z of the second particle. You can say 2 or 1 here. This is really what it is. You see, how do we upgrade an operator in the first Hilbert space to an operator on the tensor product? You just let it act on the first state but do nothing on the second.

So summing the two angular momenta really means constructing this new operator in the new, larger, more extended space in which it acts in this way. So let's just calculate what that is and we'll stop there.

Let's see. For example, we'll have the first term, S_z 1 tensor 1 acting on ψ on the whole state. Well, it acts on the first term, α_1 . Now, this operator is supposed to act on this thing, so this acts on that and that acts on that. So you get-- I'll write this whole thing-- S_z hat on plus tensor plus for the first term. This term just acted on that.

We can act with the second one now. Well, I was putting the first, so let me leave the first. Let's do the second term, plus alpha 2. It still acts on the first only, Sz hat plus tensor minus plus alpha 2 Sz hat on minus tensor plus, plus alpha 4 Sz hat on minus tensor minus.

Now, what is this? Well, this thing is h bar over 2 times plus, and the number, of course, goes out of the tensor product. You don't worry about that. There's h bar over 2 everywhere. Alpha 1 plus, plus. Here the same, another plus, so plus alpha 2 plus, minus. And here is minus, so minus alpha 3 minus, plus, minus alpha 4, also minus, minus, minus. So that's what it is.

If I do the other one, you could do it with me now quickly. 1 tensor Sz 2 on psi. Once you get accustomed, these are pretty direct. So I have to act with Sz hat on these ones, so I just act on the second one, on the second state. So I get h bar over 2.

For the first one, you get a plus, because it's acting on this thing, so you get alpha 1 plus, plus. For the second one, however, you get a minus because it's the second operator, so minus alpha 2 plus, minus. For the third one, it's a plus, so you get a plus alpha 3 minus, plus. And for the last one, it's a minus, so you get minus alpha 4 minus, minus.

So when you add them together, these two pieces, it's the total, Sz total, acting on the state, and what did you get? You get h bar, this one's at, alpha plus, plus alpha 1. These two cancel, these two cancel. Minus alpha 4 minus, minus. That's the whole action of Sz on this thing.

And if you wanted to have a state with total Sz equals 0, then you would have to put alpha 1 and alpha 4 to 0. You will, if you want, at some stage, calculate, maybe in recitation, how much is Sy at this state and how much is Sx at this state, and try to figure out if there is a state whose total spin angular momentum is 0. It's just a calculation like this. So next week, we'll continue with this and with teleportation and Bell inequalities.