PROFESSOR: OK, last time we were talking about uncertainty. We gave a picture for uncertainty-- it was a neat picture, I think of the uncertainty, refer to the uncertainty measuring an operator A that was a Hermitian operator. And that uncertainty depended on the state that you were measuring. If the state was an eigenstate of A, there would be no uncertainty. If the state is not an eigenstate of A, there was an uncertainty. And this uncertainty was defined as the norm of A minus the expectation value of A acting on psi.

So that was our definition of uncertainty. And it had nice properties. In fact, it was zero if and only if the state was an eigenstate of the operator. We proved a couple of things as well-- that, in particular, one that is kind of practical is that delta A of psi squared is the expectation value of A squared on the state psi minus the expectation value of A on the state psi squared.

So that was also proven, which, since this number is greater than or equal to 0, this is greater than or equal to 0. And in particular, the expectation value of A squared is bigger than the expectation of A squared. So let's do a trivial example for a computation.

Suppose somebody tells you in an example that the spin is in an eigenstate of Sz. So the state psi it's what we called the plus state, or the z plus state. And you want to know what is uncertainty delta of Sx.

So you know if you're in an eigenstate of z, you are not in an eigenstate of x-- in fact, you're in a superposition of two eigenstates of Sx. Therefore, there should be some uncertainty here. And the question is, what is the quickest way in which you compute this uncertainty, and how much is it?

So many times, the simplest way is to just use this formula. So let's do that. So what is the expectation value of Sx in that state?

So it's Sx expectation value would be given by Sx on this thing. Now, actually, it's relatively clear to see that this expectation value is going to be 0, because Sx really in the state plus is equal amplitude to be Sx equal plus h bar over 2, or minus h bar over 2.

But suppose you don't remember that. In order to compute this, it may come handy to recall the matrix presentation of Sx, which you don't need to know by heart. So this state plus is the first state, and the basis state is the state 1 0.

And then we have Sx on plus is equal to h bar over 2 0 1 1 0, acting on 1 0. Zero and that's equal to h bar over 2. The first thing gives you 0, and the second one gives you 1. So that's, in fact, equal to h bar over 2, the state of minus.

So here you go to h bar over 2 plus minus, and you know plus and minus are orthogonal, so 0 is expected. Well, are we going to get zero uncertainty? No, because Sx squared, however, does have some expectation value. So what is the expectation value of Sx squared?

Well, there's an advantage here. You may remember that this Sx squared is a funny matrix. It's a multiple of the identity, because if you square this matrix, you get the multiple of the identity. So Sx squared is h over 2 squared times the identity matrix-- the two by two identity matrix.

So the expectation value of Sx squared is h bar over 2 squared times expectation value of the identity. And on any state, the expectation value on any normalized state, the expectation value of the identity will be equal to 1. So this is just h squared over 2 squared.

So back to our uncertainty, delta Sx squared would be equal to the expectation value of Sx squared minus the expectation value of Sx squared. This was 0. This thing was equal to h bar over 2 squared, and therefore, delta Sx is equal to h bar over 2.

So just I wanted to make you familiar with that. You can compute these things-- these norms and all these equations are pretty practical, and easy to use.

So today what we have to do is the following-- we're going to establish the uncertainty principle. We're going to just prove it. And then, once we have the uncertainty principle, we'll try to find some applications for it. So before doing an application, we will discuss the case of the energy time uncertainty principle, which is slightly more subtle and has interestingly connotations that we will develop today. And finally, we'll use the uncertainty principle to learn how to find bounds for energies of ground states. So we might make a rigorous application of the uncertainty principle.

So the uncertainty principle talks about two operators that are both Hermitian, and states the following-- so given the theorem, or uncertainty principle, given two Hermitian operators A and B, and a state psi normalized, then the following inequality holds. And we're going to write it in one way, then in another way.

Delta A psi squared times delta B-- sometimes people in order to avoid cluttering don't put the psi. I don't know whether to put it or not. It does look a little more messy with the psi there, but it's something you have to keep in mind. Each time you have an uncertainty, you are talking about some specific state that should not be forgotten. So maybe I'll erase it to make it look a little nicer.

Delta B squared-- now it's an inequality. So not just equality, but inequality. That product of uncertainties must exceed a number-- a computable number-- which is given by the following thing.

OK, so here it is. This is a number, is the expectation value of this strange operator in the state psi squared. So even such a statement is somewhat quite confusing, because you wish to know what kind of number is this.

Could this be a complex number? If it were a complex number, why am I squaring? That doesn't make any sense. Inequalities-- these are real numbers. Deltas are defined to be real numbers. They're the norms.

So this is real positive. This would make no sense if this would be a complex number. So this number better be real.

And the way it's written, it seems to be particularly confusing, because there seems to be an i here. So at first sight, you might say, well, can it be real? But the thing that you should really focus here is this whole thing. This is some operator. And against all first impressions, this operator formed by taking the commutator of A and B-- this is the commutator A B minus B A-- is Hermitian, because, in fact, if you have two operators, and you take the commutator, if the two of them are Hermitian, the answer is not Hermitian. And that you know already-- x with p is equal to i h bar. These are Hermitian operators, and suddenly the commutator is not a Hermitian operator. You have the unit here. A Hermitian operator with a number here would have to be a real things. So there's an extra i, that's your first hint that this i is important.

So the fact is that this operator as defind here is Hermitian, because if you take 1 over i A B-- and we're going to try to take its Hermitian conjugate-- we have 1 over i A B minus B A. And we're taking the Hermitian conjugate.

Now, the i is going to get complex conjugated, so you're going to get 1 over minus i. The Hermitian conjugate of a product is the Hermitian conjugate in opposite order. So it would be B dagger A dagger minus A dagger B dagger.

And of course, these operators are Hermitian, so 1 over minus i is minus 1 over i. And here I get B A minus A B. So with a minus sign, this is 1 over i A B again.

So the operator is equal to its dagger-- its adjoint. And therefore, this operator is Hermitian. And as we proved, the expectation value of any Hermitian operator is real.

And we're in good shape. We have a real number. This could be negative. And a number, when you square it, is going to be a positive number. So this makes sense. We're writing something that at least makes sense.

Another way, of course, to write this equation, if you prefer-- this inequality, I mean-- is to take the square root. So you could write it delta A times delta B. Since this is a real number, I can take the square root and write just this as absolute value of psi, 1 over 2i i A B psi. And these bars here are absolute value. They're not norm of a vector. They are not norm of a complex number. They are just absolute value, because the thing inside is a real thing.

So if you prefer, whatever you like better, you've got here the statement of the uncertainty principle. So the good thing about this uncertainty principle formulated this way is that it's completely precise, because you've defined uncertainties precisely. Many times, when you first study the uncertainty principle, you don't define uncertainties precisely, and the uncertainty principle is something that goes with [? sim ?] is approximately equal to this. And you make statements that are intuitively interesting, but are not thoroughly precise. Yes, question, yes.

AUDIENCE: Should that be greater or equal?

PROFESSOR: Greater than or equal to, yes-- no miracles here. Other question? Other question?

So we have to prove this. And why do you have to prove this? This is a case, actually, in which many interesting questions are based on the proof. Why would that be the case?

Well, a question that is always of great interest is reducing uncertainties. Now, if two operators commute, this right-hand side is 0 and it just says that the uncertainty could be made perhaps equal to 0. It doesn't mean that the uncertainty is 0. It may depend on the state, even if the operators commute. This is just telling you it's bigger than 0, and perhaps by being clever, you can make it equal to 0.

Similarly, when you have two operators that just don't commute, it is of great importance to try to figure out if there is some states for which the uncertainty relation is saturated. So this is the question that, in fact, you could not answer if you just know this theorem written like this, because there's no statement here of what are the conditions for which this inequality is saturated.

So as we'll do the proof, we'll find those conditions. And in fact, they go a little beyond what the Schwarz inequality would say. I mentioned last time that this is a classic example of something that looks like the Schwarz inequality, and indeed, that will be the central part of the demonstration. But there's one extra step there that we will have to do. And therefore, if you want to understand when this is saturated, when do you have minimum uncertainty states, then you need to know the proof.

So before we do, of course, even the proof, there's an example-- the classic illustration that should be mentioned-- A equal x and B equals p, xp equal i h bar. That's the identity. So delta x squared delta p squared is greater or equal than psi 1 over 2i-- the commutator-- i h bar 1 psi squared.

And what do we get here? We get the i's cancel, the h bar over 2 goes out, gets squared, and everything else is equal to 1, because h is normalized. So the precise version of the uncertainty principle is this one for x and p.

And we will, of course, try to figure out when we can saturate this. What kind of wave functions saturate them? You know the ones that are just sort of strange-- if x is totally localized, the uncertainty of momentum must be infinite, because if delta x is 0, well, to make this something that at least doesn't contradict the identity, delta p better be infinite.

Similarly, if you have an eigenstate of p, which is a wave, is totally delocalized, and you have infinite here and 0 here. Well, they're interesting states that have both, and we're going to try to find the ones of minimum uncertainty.

So OK, we've stated the principle. We've given an example. We've calculated an uncertainty. Let us prove the theorem. So as we mentioned before, this idea that the uncertainty is a norm, is a good one. So let's define two auxilliary variables-- f, a state f, which is going to be A minus the expectation value of A on psi. And we can put the ket here. And g, which is going to be B minus the expectation value of B, psi.

Now what do we know about this? Well the uncertainties are the norms of these states, so the norm squared of these states are the uncertainty squared. So delta A squared is f f, the norm squared. And delta B squared is g g.

And Schwarz' inequality says that the norm of f times the normal of g is greater than or equal than the absolute value of the inner product of f with g. So squaring this thing, which is convenient perhaps at this moment, we have f f-- norm squared of f-- norm squared of g must be greater than or equal than f g squared, absolute value squared. So this is Schwarz.

And this is going to just make a note-- here we know when this is saturated. It will be saturated if f is parallel to g. If these two vectors are parallel to each other, the Schwarz inequality is saturated. So that's something to keep in mind. We'll use it soon enough.

But at this moment, we can simply rewrite this as delta A squared times delta B squared-- after all, those were definitions-- are greater than or equal-- and this is going to be a complex number in general, so f g in Schwarz' inequality is just a complex number. So this is real of f g squared, plus the imaginary part of f g squared.

So that's what we have-- real and imaginary part. So let's try to get what f g is. So what is f g? Let's compute it.

Well we must take the bra corresponding to this, so this is psi. Since the operator is Hermitian, you have A minus expectation value of A, and here you have B minus expectation value of B psi. Now we can expand this, and it will be useful to expand.

But at the same time, I will invent a little notation here. I'll call this A check, and this B check. And for reference, I'll put that this is psi A check B check psi. On the other hand, let's just compute what we get.

So what do we get? Well, let's expand this. Well, the first term is A times B on psi psi, and we're not going to be able to do much about that-- A B psi.

And then we start getting funny terms-- A cross with B, and that's-- if you think about it a second, this is just going to be equal to the expectation value of A times the expectation of B, because the expectation value of B is a number, and then A is sandwich between two psi. So from this cross product, you get expectation value of A, expectation value of B, with a minus sign. From this cross product, you get the expectation value of A and expectation value of B-- another one with a minus sign. And then one with a plus sign.

So the end result is a single one with a minus sign. So expectation value of A, expectation value of B. Now, if I change f and g, I would like to compute not only fg inner product, but gf inner product. And you may say why? Well, I want it because I need the real part and the imaginary parts, and gf is the complex conjugate of f g, so might as well compute it.

So what is gf? Now you don't have to do the calculation again, because basically you change g to f or f to g by exchanging A and B. So I can just say that this is psi B A psi minus A B. And if I write it this way, I say it's just psi B check A check psi.

OK so we've done some work, and the reason we've done this work is because we actually need to write the right-hand side of the inequality. And let's, therefore, explore what these ones are. So for example, the imaginary part of f g is 1 over 2i f g minus its complex conjugate-- gf.

Imaginary part of a complex number is z minus z star divided by 2i. now, fg minus gf is actually simple, because this product of expectation values cancel, and this gives me the commutator of A with B. So this is 1 over 2i, and you have psi expectation value of A B commutator.

So actually, that looks exactly like what we want. And we're not going to be able to simplify it more. We can put the 1 over 2i inside. That fine. It's sort of in the operator. It can go out, but we're not going to do better than that.

You already recognize, in some sense, the inequality we want to prove, because if this is that, you could ignore this and say, well, it's anyway greater than this thing.

And that's this term.

But let's write the other one, at least for a little while. Real of fg would be 1/2 of fg plus gf. And now it is your choice how you write this. There's nothing great that you can do.

The sum of these two things have AB plus BA and then twice of this expectation value, so it's not nothing particularly inspiring. So you put these two terms and just write it like this-- 1/2 of psi anti-commutator off A check with B check. Anti-commutator, remember, is this combination of operators in which you take the product in one way, and add the product in the other way.

So I've used this formula to write this, and you could write it as an anti-commutator of A and B minus 2 times the expectation values, or whichever way you want it. But at the end of the day, that's what it is. And you cannot simplify it much.

So your uncertainty principle has become delta A squared delta B squared greater than or equal to expectation value of psi 1 over 2i A B psi squared plus expectation value of psi 1 over 2 A check B check psi squared. And some people call this the generalized uncertainty principle.

You may find some textbooks that tell you "Prove the generalized uncertainty principle," because that's really what you get if you follow the rules and Schwarz' inequality. So it is of some interest. It is conceivable that sometimes you may want to use this.

But the fact is that this is a real number. This is a Hermitian operator as well. This is a real number. This is a positive number.

So if you ignore it, you still have the inequality holding. And many times-- and that's the interesting thing-- you really are justified to ignore it. In fact, I don't know of a single example-- perhaps somebody can tell me-- in which that second term is useful.

So what you say at this moment is go ahead, drop that term, and get an inequality.

So it follows directly from that, from this inequality, that delta A squared delta B squared is greater than or equal-- you might say, well, how do you know it's equal? Maybe that thing cannot be 0. Well, it can be 0 in some examples. So it's still greater than or equal to psi 1 over 2i A B psi squared. And that's by ignoring the positive quantity.

So that is really the proof of the uncertainty principle. But now we can ask what are the things that have to happen for the uncertainty principle to be saturated? That you really have delta A delta B equal to this quantity, so when can we saturate?

OK, what do we need? First we need Schwarz inequality saturation. So f and g must be states that are proportional to each other.

So we need one, that Schwarz is saturated. Which means that g is some number times f, where beta is a complex number. This is complex vector space, so parallel means multiply by a complex number. That's still a parallel vector.

So this is the saturation of Schwarz. Now, what else do we need? Well, we need that this quantity be 0 as well, that the real part of this thing is equal to 0. Otherwise, you really cannot reach it.

The true inequality is this, so if you have Schwarz, you've saturated. This thing is equal to this thing. The left-hand side is equal to this whole right-hand side. Schwarz buys you that.

But now we want this to be just equal to that. So this thing must be 0, so the real part of f overlap g-- of fg must be 0. What does that mean? It means that fg plus gf has to be 0.

But now we know what g is, so we can plug it here. So g is beta times f. Beta goes out, and you get beta f f. Now when you form the bra g, beta becomes beta star. So you get beta star f f equals 0.

And since f need not have zero norm, because there is some uncertainty presumably, you have that beta plus beta star is equal to 0, or real of beta is equal to 0. So that said, it's not that bad. You need two things-- that the f and g vectors be parallel with a complex constant, but actually, that constant must be purely imaginary.

So beta is purely imaginary-- that this beta is equal to i lambda, with lambda real. And we then are in shape. So for saturation, we need just g to be that, and g to be beta f.

So let me write that equation over here. So g-- what was g? It's B, B minus absolute value of B on psi, which is g, must be equal to beta, which is i lambda A minus absolute value of A on psi.

Condition-- so this is the final condition for saturation. now, that's a strange-looking equation. It's not all that obvious how you're even supposed to begin solving it.

Why is that? Well, you're trying to look for a psi, and you have a constraint on the psi. The psi must satisfy this. I actually will tell both Arum and Will to discuss some of these things in recitation-- how to calculate minimum uncertainty wave packets based on this equation, and what it means.

But in principle, what do you have to do? You have some kind of differential equation, because you have, say, x and p, and you want to saturate. So this is x, and this is p. Since p, you want to use a coordinate representation, this will be a derivative, and this will be a multiplication, so you'll get a differential equation on the wave function.

So you write an answer for the wave function. You must calculate the expectation value of B. You must calculate the expectation value of A, and then plug into this equation, and try to see if your answer allows a solution-- and a solution with some number here, lambda.

At least one thing I can tell you before you try this too hard-- this lambda is essentially fixed, because we can take the norm of this equation. And that's an interesting fact-- take the norm. And what is the norm of this? This is delta B, the norm of this state.

And the norm of i lambda--, well norm of i is 1. Norm of lambda is absolute value of lambda, because lambda was real. And you have delta A here of psi, of course.

So lambda can be either plus or minus delta B of psi over delta A of psi. So that's not an arbitrary constant. It's fixed by the equation already, in terms of things that you know. And therefore, this will be a subject of problems in a little bit of your recitation, in which you, hopefully, discuss how to find minimum uncertainty packets.

All right, so that's it for the proof of the uncertainty principle. And as I told you, the proof is useful in particular to find those special states of saturated uncertainty. We'll have a lot to say about them for the harmonic oscillator later on, and in fact throughout the course.

So are there any questions? Yes.

AUDIENCE: So if we have one of the states and an eigenstate, we know that [INAUDIBLE] is 0 and we then mandate that the uncertainty of the other variable must be infinite. But is it even possible to talk about the uncertainty? And if so, are we still guaranteed-- we know that it's infinite, but it's possible for 0 and an infinite number to multiply [INAUDIBLE]

PROFESSOR: Right, so you're in a somewhat uncomfortable position if you have zero uncertainty. Then you need the other one to be infinite. So the way, presumably, you should think of that, is that you should take limits of sequences of wave functions in which the uncertainty in x is going to 0, and you will find that as you take the limit, and delta x is going to 0, and delta p is going to infinity, you can still have that.

Other questions?

Well, having done this, let's try the more subtle case of the uncertainty principle for energy and time. So that is a pretty interesting subject, actually. And should I erase here? Yes, I think so.

Actually, [? Griffith ?] says that it's usually badly misunderstood, this energy-time uncertainty principle, but seldom your misunderstanding leads to a serious mistake. So you're saved. It's used in a hand-wavy way, and it's roughly correct, although people say all kinds of funny things that are not exactly right.

So energy time uncertainty-- so let me give a small motivation-- a hand-wavy motivation, so it doesn't get us very far, but at least it gives you a picture of what's going on. And these uncertainty relations, in some sense, have a basis on some simple statements that are totally classical, and maybe a little imprecise, but incontrovertible, about looking at waveforms, and trying to figure out what's going on.

So for example, suppose in time you detect a fluctuation that as time progresses, just suddenly turns on. Some wave that just dies off after a little while. And you have a good understanding of when it started, and when it ended. And there's a time T.

So whenever you have a situation like that, you can try to count the number of waves-- full waves that you see here. So the number of waves would be equal to-- or periods, number of full waves-- would be the total time divided by the period of this wave. So sometimes T is called the period. But here, T is the total time here, and the period is 2 pi over omega.

So we say this is omega t over 2 pi. Now, the problem with these waves that begin and end, is that you can't quite see or make sure that you've got the full wave here. So in the hand-wavy way, we say that even as we looked at the perfectly well-defined, and you know the shape exactly-- it's been measured-- you can't quite tell whether you've got the full wave here or a quarter of a wave more, so there's an uncertainty in delta n which is of order 1. You miss half on one side, and half on the other side.

So if you have an uncertainty here of order 1, and you have no uncertainty in T, you would claim that you have, actually, in some sense, an uncertainty in what omega is. Omega might be well measured here, but somehow towards the end you can't quite see. T we said was precise, so over 2 pi is equal to 1. I just took a delta of here, and I said P is precise, so it's delta omega.

So this is a classical statement. An electrical engineer would not need to know any quantum mechanics to say that's about right, and you can make it more or less precise. But that's a classical statement.

In quantum mechanics, all that happens is that something has become quantum, and the idea that you have something like this, we can associate it with a particle, a photon, and in which case, the uncertainty in omega is uncertainty in energy. So for a photon, the uncertainty is equal to h bar omega, so delta omega times h bar is equal to the uncertainty in energy.

So if you plug it in here, you multiply it by h bar here, and you would get delta E times T is equal to 2 pi h bar. And then you have to add words. What is T? Well, this T is the time it takes the photon to go through your detector. You've been seeing it. You saw a wave. You recorded it, and took a time T-- began, ended. And it so it's the time it took you to have the pulse go through.

And that time is related to an uncertainty in the energy of the photon. And that's sort of the beginning of a time energy uncertainty relationship. This is quantum, because the idea that photons carry energies and they're quantized-- this is a single photon-- and this connection with energy is quantum mechanics.

So this is good and reasonable intuition, perhaps. And it can be the basis of all kinds of things. But it points out the fact that the more delicate part here is T. How could I speak of a time uncertainty?

And the fact is that you can't speak of a time uncertainty really precisely. And the reason is, because there's no Hermitian operator for which we could say, OK the eigenstates of this Hermitian operator are times, and then you have a norm, and it's an uncertainty. So you can't do it. So you have to do something different this time. And happily, there's something you can do that is precise and makes sense. So we'll do it.

So what we have to do is just try to use the uncertainty principle that we have, and at least one operator. We can use something that is good for us. We want uncertainty in energy, and we have the Hamiltonian. It's an operator. So for that one, we can use it, and that's the clue.

So you'll take A to be the Hamiltonian, and B to be some operator Q that may depend on some things-- for example, x and p, or whatever you want. But the one thing I want to ask from this operator is that Q has no explicit time dependence-- no explicit time dependence whatsoever.

So let's see what this gives us as an uncertainty relationship. Well, it would give us that delta H squared-- that's delta Q squared-- would be greater than or equal to the square of psi 1 over 2i H with Q psi.

OK, that's it. Well, but in order to get some intuition from here, we better be able to interpret this. This doesn't seem to have anything to do with energy and time. So is there something to do with time here?

That is, in fact, a very well-known result in quantum mechanics-- that somehow commutators with the Hamiltonian test the time derivative of operators. So whenever you see an H with Q commutator, you think ah, that's roughly dQ dt. And we'll see what happens with that. And say, oh, dQ dt, but it doesn't depend on T-- you said 0. No it's not 0. There's no explicit dependence, but we'll see what happens.

So at this moment, you really have to stop for one second and derive a familiar result-- that may or may not be that familiar to you from 804. I don't think it was all that emphasized. Consider expectation value of Q.

And then the expectation of Q-- let me write it as psi Q psi, like this. Now let's try to take the time derivative of this thing. So what is the time derivative of the expectation value of q?

And the idea being that look, the operator depends on some things, and it can have time-dependent expectation value, because the state is changing in time. So operators can have time-dependent expectation values even though the operators don't depend on time. So for example, this depends on x and p, and the x and p in a harmonic oscillator are time dependent. They're moving around, and this could have time dependence.

So what do we get from here? Well, if I have to take the time derivative of this, I have d psi dt here, Q psi, plus psi Q d psi dt. And in doing this, and not differentiating Q itself, I've used the fact that this is an operator and there's no time anywhere there. I didn't have to differentiate Q.

So how do we evaluate this? Well, you remember the Schrodinger equation. Here the Schrodinger equation comes in, because you have time derivatives of your state.

So i d psi dt, i H bar d psi dt is equal to H psi. That's a full time-dependent Schrodinger equation. So here, maybe, I should write this like that-- this is all time-dependent stuff.

At this moment, I don't ignore the time dependence. The states are not stationary states. If they would be stationary states, there would be no energy uncertainty.

So I have this, and therefore, I plug this in here, and what do we get? i H bar h psi Q psi plus psi Q i H bar H psi. Now, I got the i H in the wrong place-- sorry-- 1 over i H bar, and 1 over i H bar. Now the first term-- this thing comes out as its complex conjugate-- 1 minus i H bar, because it's on the first input.

H is Hermitian, so I can send it to the other side, so psi, HQ psi. Second term-- the 1 over i H just goes out, and I don't have to move anybody. QH is there, psi. So actually, this is i over H bar, because minus i down goes up with i.

And I have here psi HQ, and this is minus i over H bar, so I get HQ minus QH psi. So this is your final result-- the expectation value d dt of the expectation value of Q is equal to i over H bar, expectation value of the commutator of H with Q.

So this is neat, and it should always stick in your mind. This is true. We will see the

Heisenberg way of writing this equation in a little while-- not today, but in a couple of weeks. But maybe even write it even more briefly as i over H bar expectation value of HQ.

So what do we get from here? Well, we can go back to our uncertainty principle, and rewrite it, having learned that we have time derivative. So time finally showed up, and that's good news. So we're maybe not too far from a clear interpretation of the uncertainty principle.

So we're going back to that top equation, so that what we have now is delta H squared delta Q squared is that thing over there, the expectation value of 1 over 2i. There's some signs there, so what do we have-- equals 1 over 2i H bar over i d dt of Q.

So what I did here was to say that this expectation value was H bar over i d dt of Q, and I plugged it in there. So you square this thing, so there's not too much really to be done. The i don't matter at the end of the day. It's a minus 1 that gets squared.

So the H bar over 2-- I'm sorry-- the H bar over 2 does remain here, squared. And you have dQ dt squared. Q is a Hermitian operator. B was supposed to be Hermitian. The expectation value is real. The time derivative is real. It could be going up or down.

So at the end of the day, you have delta H delta Q is greater than or equal to H bar over 2, the absolute value of dQ over dt. There we go. This is, in a sense, the best you can do.

Let's try to interpret what we've got. Well, we've got something that still doesn't quite look like a time uncertainty relationship, but there's time in there. But it's a matter of a definition now.

You see, if you have delta Q, and you divide it by dQ dt, first it is some sort of time. It has the units of time. And we can define it, if you wish, to be sub delta t. And what physically, does this delta t represent?

Well, it's roughly-- you see, things change in time. The rate of change of the expectation value of Q may not be uniform. It make change fast, or it may change slowly. But suppose it's changing.

Roughly, this ratio, of this would be constant, is the time it takes the expectation value of Q to change by delta Q. It is like a distance divided by a velocity. So this is roughly the time needed for the expectation value of Q to change by delta Q, by the uncertainty.

So it's a measure of the time needed for a significant change, if the expectation value, if the uncertainty of Q is significant, and is comparable to Q. Well, this is the time needed for significant change. Now this is pretty much all you can do, except that of course, once you write it like that, you pull this down, and you go up now, delta H delta t is greater or equal than H bar over 2.

And this is the best you can do with this kind of approach. Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, I simply define this, which is a time that has some meaning if you know what the uncertainty of the operator is and how fast it's changing-- is the time needed for a change. Once I defined this, I simply brought this factor down here, so that delta Q over this derivative is delta t, and the equation just became this equation.

So we'll try to figure out a little more of what this means right away, but you can make a few criticisms about this thing. You can say, look, this delta time uncertainty is not universal. It depends which operator Q you took.

True enough. I cannot prove that it's independent of the operator Q, and many times I cannot even tell you which operator Q is the best operator to think about. But you can try.

And it does give you-- first, it's a mathematical statement about how fast things can change. And that contains physics, and it contains a very precise fact as well.

Actually, there's a version of the uncertainty principle that you will explore in the homework that is, maybe, an alternative picture of this, and asks the following thing-- if you have a state and a stationary state, nothing changes in the state. But if it's a stationary state, the energy uncertainty is 0, because the energy is an eigenstate of the energy. So nothing changes. So you have to wait infinite time for there to be a change, and this makes sense.

Now you can ask the following question-- suppose I have a state that is not an eigenstate of energy. So therefore, for example, the simplest thing would be a superposition of two eigenstates of different energies. You can ask, well, there will be time evolution and this state will change in time. So how can I get a constraint on changes? How can I approach changes?

And people discovered the following interesting fact-- that if you have a state, it has unit norm, and if it evolves, it may happen that at some stage, it becomes orthogonal to itself-- to the original one. And that is a big change. You become orthogonal to what you used to be. That's as big a change as can happen.

And then you can ask, is there a minimum time for which this can happen? What is the minimum time in which a state can change so much that it becomes orthogonal to itself? And there is such an uncertainty principle. It's derived a little differently from that.

And it says that if you take delta t to be the time it takes psi of x and t to become orthogonal to psi of x0, then this delta t times delta E-- the uncertainty of the energies is the uncertainty in h-- is greater than or equal to h bar over 4. Now a state may never become orthogonal to itself, but that's OK. Then it's a big number on the left-hand side.

But the quickest it can do it is that. And that's an interesting thing. And it's a version of the uncertainty principle.

I want to make a couple more remarks, because this thing is mysterious enough that it requires thinking. So let's make some precise claims about energy uncertainties and then give an example of what's happening in the physical situation. Was there a question? Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: You're going to explore that in the homework. Actually, I don't think you're going to show it, but--

AUDIENCE: [INAUDIBLE] H bar [INAUDIBLE] it's even less than the uncertainty [INAUDIBLE]

PROFESSOR: It's a different statement. It's a very precise way of measuring, creating a time. It's a precise definition of time, and therefore, there's no reason why it would have been the same.

So here is a statement that is interesting-- is that the uncertainty delta E in an isolated system is constant-- doesn't change. And by an isolated system, a system in which there's no influences on it, a system in which you have actually time independent Hamiltonians. So H is a time independent Hamiltonian.

Now that, of course, doesn't mean the physics is boring. Time- independent Hamiltonians are quite interesting, but you have a whole system. Let's take it to be isolated. There's no time dependent things acting on it, and H should be a time independent Hamiltonian.

So I want to use this statement to say the following-- if I take Q equals H in that theorem over there, I get that d dt of the expectation value of H would be what? It would be i over H bar. Since H is time independent-- the condition here was that Q had no time dependence.

But then I get H commutator with H. So I get here H commutator with H. And that commutator is 0. However complicated an operator is, it commutes with itself.

So the expectation value of the energy doesn't change. We call that energy conservation. But still, if you take Q now equal to H squared, the time derivative of the expectation value of H squared, you get i over H bar. You're supposed to be H commutator with Q, which is H squared, now. And that's also 0.

So no power of the expectation value of H vanishes. And therefore, we have that the time derivative of the uncertainty of H squared-- which is the time derivative of the expectation value of H squared minus the expectation value of H squared-- well, we've shown each one of the things on the right-hand side are 0, so this is 0. So delta H is constant.

So the uncertainty-- delta E or delta H of the system is constant. So what do we do with that? Well it helps us think a little about time dependent processes.

And the example we must have in mind is perhaps the one of a decay that leads to a radiation of a photon, so a transition that leads to a photon radiation. So let's consider that example. So we have an atom in some excited state, decays to the ground state and shoots out the photon.

Then it's an unstable state, because if it would be stable, it wouldn't change in time. And the excited state of an atom is an unstable state, decays into-- goes into the ground state. And it makes a photon.

Now this idea of the conservation of energy uncertainty at least helps you in this situation that you would typically do it with a lot of hand-waving, organize your thoughts. So what happens in such decay? There's a lifetime, which is a typical time you have to wait for that excited state to decay. And these lifetime is called tau.

And certainly as the lifetime goes through, and the decay happens, some observable changes a lot. Some observable Q must change a lot. Maybe a position of the electron in an orbit, or the angular momentum of it, or some squared of the momentum-- some observable that we could do an atomic calculation in more detail must change a lot.

So there will be associated with some observable that changes a lot during the lifetime, because it takes that long for this thing to change. There will be an energy uncertainty associated to a lifetime. So how does the energy uncertainty reflect itself?

Well, you have a ground state. And you have this excited state. But generally, when you have an excited state due to some interactions that produce instability, you actually have a lot of states here that are part of the excited state. So you have an excited state, but you do have, typically, a lot of uncertainty-- but not a lot-- some uncertainty of the energy here.

The state is not a particular one. If it would be a particular one, it would be a stationary state-- would stay there forever. Nevertheless, it's a combination of some things, so it's not quite a stationary state. It couldn't be a stationary state, because it would be eternal. So somehow, the dynamics of this atom must be such that there's interactions between, say, the electron and the nucleus, or possibly a radiation field that makes the state of this electron unstable, and associated to it an uncertainty in the energy.

So there's an uncertainty here, and this particle-- this electron goes eventually to the ground state, and it meets a photon. So there is, associated to this lifetime, an uncertainty delta E times tau, and I will put similar to H bar over 2. And this would be the delta E here, because your state must be a superposition of some states over there.

And then what happens later? Well, this particle goes to the ground state-- no uncertainty any more about what its energy is. So the only possibility at this moment consistent with the conservation of uncertainty in the system is that the photon carries the uncertainty. So that photon must have an uncertainty as well.

So delta energy of the photon will be equal to h bar delta omega, or h delta nu. So the end result is that in a physical decay process, there are uncertainties. And the uncertainty gets carried out, and it's always there-- the delta E here and the photon having some uncertainty.

Now one of the most famous applications of this thing is related to the hyperfine transition of hydrogen. And we're very lucky in physics. Physicists are very lucky. This is a great break for astronomy and cosmology, and it's all based on this uncertainty principle.

You have the hyperfine transition of hydrogen. So we will study later in this course that because of the proton and electron spins in the hydrogen atom, there's a splitting of energies having to do with the hyperfine interaction. It's a magnetic dipole interaction between the proton and the electron.

And there's going to be a splitting. And there's a transition associated with this splitting. So there's a hyperfine splitting-- the ground state of the hyperfine splitting of some states. And it's the top state and the bottom state.

And as the system decays, it emits a photon. This photon is approximately a 21 centimeter wavelength-- is the famous 21 centimeter line of hydrogen. And it corresponds to about 1420 megahertz.

So how about so far so good. There's an energy splitting here, 21 centimeters wavelength, 5.9 times 10 to the minus 6 eV in here. But that's not the energy difference that matters for the uncertainty, just like this is not the energy difference that matters for the uncertainty.

What matters for the uncertainty is how broad this state is, due to interactions that will produce the decay. It's a very funny, magnetic transition. And how long is the lifetime of this state? Anybody know? A second, a millisecond, a day? Nobody? Ten million years-- a long time-- 10 million years-- lifetime tau.

A year is about pi times 10 to the 7 seconds is pretty accurate. Anyway, 10 million years is a lot of time. It's such a large time that it corresponds to an energy uncertainty that is so extraordinarily small, that the wavelength uncertainty, or the frequency uncertainty, is so small that corresponding to this 1420, it's I think, the uncertainty in lambda-- and lambda is of the order of 10 to the minus 8.

The line is extremely sharp, so it's not a fussy line that it's hard to measure. It's the sharpest possible line. And it's so sharp because of this 10 million years lifetime, and the energy time uncertainty relationship.

That's it for today.