QUANTUM DYNAMICS

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November 4, 2013

Contents

1 Harmonic oscillator

The harmonic oscillator is an ubiquitous and rich example of a quantum system. It is a solvable system and allows the exploration of quantum dynamics in detail as well as the study of quantum states with classical properties.

The harmonic oscillator is a system where the classical description suggests clearly the definition of the quantum system. Classically a harmonic oscillator is described by the position

$x(t)$ of a particle of mass m and its momentum $p(t)$. The energy E of a particle with position x and momentum p is given by

$$E \;=\; \frac{p^2}{2m} + \frac{1}{2}m\omega^2\, x^2\,. \tag{1.1}$$

Here the constant ω, with units of inverse time, is related to the period of oscillation T by $\omega = 2\pi/T$. In the simplest application, the classical harmonic oscillator arises when a mass m free to move along the x axis is attached to a spring with spring constant k. The restoring force $F = -kx$ acting on the mass then results in harmonic motion with angular frequency $\omega = \sqrt{k/m}$.

The quantum system is easily defined. Instead of position and momentum dynamical variables we have hermitian operators $\hat{x}$ and $\hat{p}$ with commutation relation

$$[\,\hat{x}\,,\hat{p}\,] \;=\; i\hbar\,\mathbf{1}\,. \tag{1.2}$$

To complete the definition of the system we need a Hamiltonian. Inspired by the classical energy function (1.1) above we *define*

$$\hat{H} \;\equiv\; \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,. \tag{1.3}$$

The state space $\mathcal{H}$ is the space of square-integrable complex valued functions of x. The system so defined is the *quantum* harmonic oscillator.

In order to solve the quantum system we attempt to 'factorize' the Hamiltonian. This means finding an operator V such that we can rewrite the Hamiltonian as $\hat{H} = V^\dagger V$. This is not exactly possible, but with a small modification it becomes possible. We can find a V for which

$$\boxed{\hat{H} \;=\; V^\dagger V \;+\; E_0\,\mathbf{1}\,,} \tag{1.4}$$

where E_0 is a constant with units of energy that multiplies the identity operator. This extra diagonal contribution does not complicate our task of finding the eigenstates of the Hamiltonian, nor their energies. This factorization allows us to show that any energy eigenstate must have energy greater than or equal to E_0. Indeed it follows from the above equation that

$$\langle\psi|\hat{H}|\psi\rangle \;=\; \langle\psi|V^\dagger V|\psi\rangle + E_0\langle\psi|\psi\rangle \;= \langle V\psi|V\psi\rangle + E_0\,, \tag{1.5}$$

Since any norm must be greater than or equal to zero, we have shown that

$$\boxed{\langle\psi|\hat{H}|\psi\rangle \;\geq\; E_0\,.} \tag{1.6}$$

For a normalized energy eigenstate $|E\rangle$ of energy E: $\hat{H}|E\rangle = E|E\rangle$, and the above inequality yields, as claimed

$$\langle E|\hat{H}|E\rangle = E \geq E_0\,. \tag{1.7}$$

To factorize the Hamiltonian we first rewrite it as

$$\hat{H} \;=\; \frac{1}{2}m\omega^2\Big(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\Big)\,. \tag{1.8}$$

Motivated by the identity $a^2 + b^2 = (a - ib)(a + ib)$, holding for numbers a and b, we examine the product

$$\begin{aligned}\Big(\hat{x} - \frac{i\hat{p}}{m\omega}\Big)\Big(\hat{x} + \frac{i\hat{p}}{m\omega}\Big) &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x})\,,\\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\mathbf{1}\,,\end{aligned} \tag{1.9}$$

where the extra terms arise because $\hat{x}$ and $\hat{p}$, as opposed to numbers, do not commute. Letting

$$\begin{aligned}V &\equiv \hat{x} + \frac{i\hat{p}}{m\omega}\,,\\ V^\dagger &\equiv \hat{x} - \frac{i\hat{p}}{m\omega}\,,\end{aligned} \tag{1.10}$$

we rewrite (1.9) as

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} \;=\; V^\dagger V + \frac{\hbar}{m\omega}\mathbf{1}\,, \tag{1.11}$$

and therefore back in the Hamiltonian (1.8) we find,

$$\hat{H} \;=\; \frac{1}{2}m\omega^2\Big(V^\dagger V + \frac{\hbar}{m\omega}\mathbf{1}\Big) \;=\; \frac{1}{2}m\omega^2\, V^\dagger V \;+\; \frac{1}{2}\hbar\omega\mathbf{1}\,. \tag{1.12}$$

The constant E_0 defined in (1.4) is thus $\frac{1}{2}\hbar\omega$ and (1.6) implies that

$$\boxed{\langle\psi|\hat{H}|\psi\rangle \;\geq\; \frac{1}{2}\hbar\omega\,.} \tag{1.13}$$

This shows that $E \geq \frac{1}{2}\hbar\omega$ for any eigenstate of the oscillator.

It is convenient to scale the operators V and $V^\dagger$ so that they commute to give a simple, unit-free, constant. First we compute

$$\big[V\,, V^\dagger\,\big] \;=\; \Big[\hat{x} + \frac{i\hat{p}}{m\omega}\,, \hat{x} - \frac{i\hat{p}}{m\omega}\Big] \;=\; -\frac{i}{m\omega}[\hat{x}\,, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}] \;=\; \frac{2\hbar}{m\omega}\mathbf{1}\,. \tag{1.14}$$

This suggests the definition of operators

$$\begin{aligned}\hat{a} &\equiv \sqrt{\frac{m\omega}{2\hbar}}\,V\,,\\ \hat{a}^\dagger &\equiv \sqrt{\frac{m\omega}{2\hbar}}\,V^\dagger\,.\end{aligned} \tag{1.15}$$

Due to the scaling we have

$$\boxed{\left[\, \hat{a} \,, \hat{a}^\dagger \,\right] = \mathbf{1}\,.} \tag{1.16}$$

From the above definitions we read the relations between $(\hat{a}, \hat{a}^\dagger)$ and $(\hat{x}\,, \hat{p})$:

$$\boxed{\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \;, \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \;. \end{aligned}} \tag{1.17}$$

The inverse relations are many times useful as well,

$$\boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \;, \\ \hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,. \end{aligned}} \tag{1.18}$$

While neither $\hat{a}$ nor $\hat{a}^\dagger$ is hermitian (they are hermitian conjugates of each other), the above equations are consistent with the hermiticity of $\hat{x}$ and $\hat{p}$. We can now write the Hamiltonian in terms of the $\hat{a}$ and $\hat{a}^\dagger$ operators. Using (1.15) we have

$$V^\dagger V \;=\; \frac{2\hbar}{m\omega}\,\hat{a}^\dagger\hat{a}\,, \tag{1.19}$$

and therefore back in (1.12) we get

$$\boxed{\hat{H} \;=\; \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) \;=\; \hbar\omega\Big(\hat{N} + \frac{1}{2}\Big)\,, \quad \hat{N} \;\equiv\; \hat{a}^\dagger\hat{a}\,.} \tag{1.20}$$

In here we have dropped the identity operator, which is usually understood. We have also introduced the *number* operator $\hat{N}$. This is, by construction, a hermitian operator and it is, up to a scale and an additive constant, equal to the Hamiltonian. An eigenstate of $\hat{H}$ is also an eigenstate of $\hat{N}$ and it follows from the above relation that the respective eigenvalues E and N are related by

$$E \;=\; \hbar\omega\left(N + \frac{1}{2}\right). \tag{1.21}$$

From the inequality (1.13) we have already shown that for any state

$$E \;\geq\; \frac{1}{2}\,\hbar\omega\,, \quad N \geq 0\,. \tag{1.22}$$

There cannot exist states with negative number. This can be confirmed directly. If $|\psi\rangle$ is a state of negative number we have

$$\hat{a}^\dagger\hat{a}|\psi\rangle = -\alpha^2|\psi\rangle\,, \quad \alpha > 0\,. \tag{1.23}$$

Multiplying by the state bra $\langle\psi|$ and noticing that $\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = \langle\hat{a}\psi|\hat{a}\psi\rangle$ we get

$$\langle\hat{a}\psi|\hat{a}\psi\rangle = -\alpha^2\langle\psi|\psi\rangle\,. \tag{1.24}$$

This is a contradiction, for if $|\psi\rangle$ is not the zero vector, the right-hand side is negative, which cannot be since the left hand side is also a norm-squared and thus positive.

Exercise. Prove the following commutation relations

$$\begin{aligned} \big[\,\hat{H}\,,\,\hat{a}\,\big] &= \;-\,\hbar\omega\,\hat{a}\,, \\ \big[\,\hat{H}\,,\,\hat{a}^\dagger\big] &= \;+\,\hbar\omega\,\hat{a}^\dagger\,. \end{aligned} \tag{1.25}$$

To derive the spectrum of the oscillator we begin by assuming that one normalizable eigenstate $|E\rangle$ of energy E exists:

$$\hat{H}|E\rangle \;=\; E|E\rangle\,, \quad \langle E|E\rangle \;>\; 0\,. \tag{1.26}$$

Note that the state must have positive norm-squared, as indicated above. The state $|E\rangle$ also an eigenstate of the number operator, with eigenvalue N_E given by

$$\hat{N}|E\rangle \;=\; N_E|E\rangle\,, \quad \text{with} \;\; N_E \;=\; \frac{E}{\hbar\omega} - \frac{1}{2}\,. \tag{1.27}$$

We will now define two states

$$\begin{aligned} |E_+\rangle &= \hat{a}^\dagger|E\rangle\,, \\ |E_-\rangle &= \hat{a}\,|E\rangle\,. \end{aligned} \tag{1.28}$$

Let us assume, for the time being that both of these states exist – that is, they are not zero nor they are inconsistent by having negative norm-squared. We can then verify they are energy eigenstates

$$\begin{aligned} \hat{H}|E_+\rangle &= \hat{H}\hat{a}^\dagger|E\rangle = \big([\hat{H},\hat{a}^\dagger] + \hat{a}^\dagger\hat{H}\big)|E\rangle = (\hbar\omega + E)\,\hat{a}^\dagger|E\rangle = (E+\hbar\omega)|E_+\rangle\,, \\ \hat{H}|E_-\rangle &= \hat{H}\hat{a}\,|E\rangle = \big([\hat{H},\hat{a}\,] + \hat{a}\,\hat{H}\big)|E\rangle = (-\hbar\omega + E)\,\hat{a}\,|E\rangle = (E-\hbar\omega)|E_-\rangle\,, \end{aligned} \tag{1.29}$$

As we label the states with their energies, this shows that

$$\begin{aligned} E_+ &= E + \hbar\omega\,, \quad & N_{E_+} &= N_E + 1\,, \\ E_- &= E - \hbar\omega\,, \quad & N_{E_-} &= N_E - 1\,. \end{aligned} \tag{1.30}$$

We call $\hat{a}^\dagger$ the *creation* or *raising* operator because it adds energy $\hbar\omega$ to the eigenstate it acts on, or raises the number operator by one unit. We call $\hat{a}$ the *annihilation* or *lowering* operator because it subtracts energy $\hbar\omega$ to the eigenstate it acts on, or lowers the number operator by one unit. One more computation is needed: we must find the norm-squared of the $|E_\pm\rangle$ states:

$$\begin{aligned} \langle E_+|E_+\rangle &= \langle E|\hat{a}\hat{a}^\dagger|E\rangle = \langle E|(\hat{N}+1)|E\rangle = (N_E+1)\langle E|E\rangle\,, \\ \langle E_-|E_-\rangle &= \langle E|\hat{a}^\dagger\hat{a}|E\rangle = \langle E|\hat{N}|E\rangle = N_E\,\langle E|E\rangle\,. \end{aligned} \tag{1.31}$$

We can summarize this as

$$\begin{aligned} \langle \hat{a}^\dagger E|\hat{a}^\dagger E\rangle &= (N_E+1)\langle E|E\rangle\,, \\ \langle \hat{a}E\,|\,\hat{a}E\,\rangle &= N_E\,\langle E|E\rangle\,. \end{aligned} \tag{1.32}$$

These equations tell us an interesting story. Since the state $|E\rangle$ is assumed to exist we must have $N_E \geq 0$ (see (1.22)) . We claim that as long as we act with $\hat{a}^\dagger$ on this state we do not obtain inconsistent states. Indeed the first equation above shows that norm-squared of $|\hat{a}^\dagger E\rangle$ is positive, as it should be. If we act again with $\hat{a}^\dagger$, since the number of $|\hat{a}^\dagger E\rangle$ is N_E+1 we find

$$\langle \hat{a}^\dagger\hat{a}^\dagger E|\hat{a}^\dagger\hat{a}^\dagger E\rangle = (N_E+2)\langle \hat{a}^\dagger E|\hat{a}^\dagger E\rangle = (N_E+2)(N_E+1)\langle E|E\rangle\,, \tag{1.33}$$

which is also positive. We *cannot* find an inconsistent negative norm-squared however many times we act with the raising operator.

The lowering operator, however, requires more care. Assume we have a state $|E\rangle$ with *integer* positive number N_E. The number eigenvalue goes down in steps of one unit each time we apply an $\hat{a}$ operator to the state. As long as the number of a state is positive, the *next* state having an extra $\hat{a}$ has positive norm-squared because of the relation $\langle \hat{a}E\,|\,\hat{a}E\,\rangle = N_E\,\langle E|E\rangle$. So no complication arises until we hit a state $|E'\rangle$ with number $N_{E'}=0$, in which case it follows that

$$\langle \hat{a}E'\,|\,\hat{a}E'\,\rangle = N_E'\,\langle E'|E'\rangle = 0\,. \tag{1.34}$$

Having zero norm, the state $|\hat{a}E'\rangle$ must be the zero vector and we cannot continue to apply lowering operators. We thus avoid inconsistency.

If the original $|E\rangle$ state has a positive *non-integer* number N_E we can lower the number by acting with $\hat{a}$'s until we get a state $|E'\rangle$ with number between zero and one. The next state $|\hat{a}E'\rangle$ has negative number and this is an inconsistency – as we showed before these cannot exist. This contradiction can only mean that the original assumptions cannot be true. So one of the following must be true

1. There is no state with non-integer positive number.

2. There is a state with non-integer positive number but the repeated application of $\hat{a}$ gives a vanishing state before we encounter states with negative number.

Option 2 actually cannot happen. For a state $|\psi\rangle$ of non-zero number $\hat{a}^\dagger\hat{a}|\psi\rangle \sim |\psi\rangle$ and therefore $\hat{a}$ cannot kill the state. We conclude that there are no states in the spectrum with non-integer number.

What are the energy eigenstates annihilated by $\hat{a}$? Assume there is such state $|E\rangle$:

$$\hat{a}\,|E\rangle = 0\,. \tag{1.35}$$

Acting with $\hat{a}^\dagger$ we find $\hat{a}^\dagger a|E\rangle = \hat{N}|E\rangle = 0$, so such state must have zero number and thus lowest energy:

$$N_E = 0\,, \quad E = \frac{1}{2}\,\hbar\omega\,. \tag{1.36}$$

To show that the state annihilated by $\hat{a}$ exists and is unique we solve the differential equation implicit in (1.35). We act with a position bra to find

$$\langle x|\hat{a}\,|E\rangle \;=\; 0 \quad \rightarrow \quad \sqrt{\frac{m\omega}{2\hbar}}\,\langle x|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)|E\rangle \;=\; 0\,. \tag{1.37}$$

The prefactor is irrelevant and we have, with $\psi_E(x) \equiv \langle x|E\rangle$,

$$\Big(x + \frac{\hbar}{m\omega}\frac{d}{dx}\Big)\psi_E(x) \;=\; 0 \quad \rightarrow \quad \frac{d\psi_E}{dx} \;=\; -\frac{m\omega}{\hbar}\,x\psi_E\,. \tag{1.38}$$

The solution of the first-order differential equation is unique (up to normalization)

$$\psi_E(x) \;=\; N_0 \exp\Big(-\frac{m\omega}{2\hbar}\,x^2\Big)\,, \quad N_0 \;=\; \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4}\,. \tag{1.39}$$

We have found a single state annihilated by $\hat{a}$ and it has number zero. The $\psi_E(x)$ above is the normalized wavefunction for the ground state of the simple harmonic oscillator.

In the following we denote states as $|n\rangle$ where n is the eigenvalue of the number operator $\hat{N}$:

$$\hat{N}|n\rangle \;=\; n|n\rangle\,. \tag{1.40}$$

In this language the ground state is the non-degenerate state $|0\rangle$ (do not confuse this with the zero vector or a state of zero energy!). It is annihilated by $\hat{a}$:

$$\text{SHO ground state } |0\rangle: \quad \hat{a}\,|0\rangle \;=\; 0\,, \qquad \hat{N}|0\rangle \;=\; 0\,, \quad \hat{H}|0\rangle \;=\; \frac{1}{2}\hbar\omega|0\rangle\,. \tag{1.41}$$

The ground state wavefunction was determined above

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\,. \tag{1.42}$$

Excited states are obtained by the successive action of $\hat{a}^\dagger$ on the ground state. The first excited state is

$$|1\rangle \;\equiv\; \hat{a}^\dagger|0\rangle \tag{1.43}$$

This state has number equal to one. Indeed, since $\hat{N}$ kills the ground state,

$$\hat{N}\hat{a}^\dagger|0\rangle \;=\; [\hat{N}, \hat{a}^\dagger]|0\rangle \;=\; \hat{a}^\dagger|0\rangle\,. \tag{1.44}$$

Moreover the state is properly normalized

$$\langle 1|1\rangle \;=\; \langle 0|\hat{a}\hat{a}^\dagger|0\rangle \;=\; \langle 0|[\hat{a}\,,\hat{a}^\dagger]|0\rangle \;=\; \langle 0|0\rangle = 1\,. \tag{1.45}$$

The next excited state is

$$|2\rangle \;=\; \frac{1}{\sqrt{2}}\,\hat{a}^\dagger\hat{a}^\dagger|0\rangle\,. \tag{1.46}$$

This state has number equal to two, as desired. The normalization is checked as follows:

$$\langle 2|2\rangle \;=\; \frac{1}{2}\langle 0|\,\hat{a}\hat{a}\,\hat{a}^\dagger\hat{a}^\dagger|0\rangle \;=\; \frac{1}{2}\langle 0|\,\hat{a}\,[\hat{a}\,,\hat{a}^\dagger\hat{a}^\dagger]\,|0\rangle \;=\; \frac{1}{2}\langle 0|\,\hat{a}\,(2\hat{a}^\dagger)|0\rangle \;=\; \langle 0|\,\hat{a}\,\hat{a}^\dagger|0\rangle \;=\; 1\,. \tag{1.47}$$

In order to get the general state it is useful to consider (1.32) in the new notation

$$\begin{aligned}\langle\hat{a}^\dagger n|\hat{a}^\dagger n\rangle \;&=\; (n+1)\langle n|n\rangle \;=\; n+1\,,\\ \langle\hat{a}n\,|\,\hat{a}n\,\rangle \;&=\; \;n\,\langle n|n\rangle \;=\; n\,.\end{aligned} \tag{1.48}$$

The first means that $\hat{a}^\dagger|n\rangle$ is a state of norm-squared $n+1$ and $\hat{a}|n\rangle$ is a state of norm-squared n. Since we know that $\hat{a}^\dagger|n\rangle \sim |n+1\rangle$ and $\hat{a}|n\rangle \sim |n-1\rangle$ we conclude that

$$\boxed{\begin{aligned}\hat{a}^\dagger|n\rangle \;&=\; \sqrt{n+1}\,|n+1\rangle\;,\\ \hat{a}\,|n\rangle \;&=\; \sqrt{n}\;\;|n-1\rangle\;.\end{aligned}} \tag{1.49}$$

The signs chosen for the square roots are consistent as you can check by using the two equations above to verify that $\hat{a}^\dagger\hat{a}\,|n\rangle = n|n\rangle$. From the top equation we have

$$|n\rangle \;=\; \frac{1}{\sqrt{n}}\,\hat{a}^\dagger|n-1\rangle\,. \tag{1.50}$$

Using that equation again for the rightmost ket, and then repeatedly, we find

$$\begin{aligned}|n\rangle \;&=\; \frac{1}{\sqrt{n}}\,\hat{a}^\dagger\frac{1}{\sqrt{n-1}}\hat{a}^\dagger|n-2\rangle \;=\; \frac{1}{\sqrt{n(n-1)}}\,(\hat{a}^\dagger)^2|n-2\rangle\\ &=\; \frac{1}{\sqrt{n(n-1)(n-2)}}\,(\hat{a}^\dagger)^3|n-3\rangle \;=\; \ldots\\ &=\; \frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^n|0\rangle\,.\end{aligned} \tag{1.51}$$

It is a good exercise to verify explicitly that $\langle n|n\rangle = 1$. In summary, the energy eigenstates are an orthonormal basis

$$\boxed{|n\rangle \;=\; \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \qquad \langle m|n\rangle = \delta_{mn}\,.} \tag{1.52}$$

You can verify by explicit computation that $\langle m|n\rangle = 0$ for $m \neq n$, but you can be sure this is true because these are eigenstates of the hermitian operator $\hat{N}$ with different eigenvalues (recall that theorem?).

Their energies are given by

$$\boxed{H|n\rangle = E_n|n\rangle = \hbar\omega\Big(n+\frac{1}{2}\Big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,.} \tag{1.53}$$

One can prove that there are no additional excited states. If there were, they would have to have integer number and thus be degenerate with some of the above states. It can be shown (homework) that any such degeneracy would imply a degeneracy of the ground state, something we have ruled out explicitly. Therefore we have shown that the state space has the direct sum decomposition into one-dimensional $\hat{N}$-invariant subspaces U_n:

$$\mathcal{H} = U_0 \oplus U_1 \oplus U_2 \oplus \cdots\,, \qquad U_n \equiv \{\alpha|n\rangle,\ \alpha\in\mathbb{C},\ \hat{N}|n\rangle = n|n\rangle\}\,. \tag{1.54}$$

The algebra of $\hat{a}$ and $\hat{a}^\dagger$ operators allows simple computation of expectation values. For example,

$$\begin{aligned} \langle n|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle n|(\hat{a}+\hat{a}^\dagger)|n\rangle \;=\; 0\,, \\ \langle n|\hat{p}|n\rangle &= i\sqrt{\frac{m\omega\hbar}{2}}\langle n|(\hat{a}^\dagger-\hat{a})|n\rangle \;=\; 0\,. \end{aligned} \tag{1.55}$$

In here we used that $\langle n|\hat{a}|n\rangle \sim \langle n|n-1\rangle = 0$ and $\langle n|\hat{a}^\dagger|n\rangle \sim \langle n|n+1\rangle = 0$. For the quadratic operators, both $\hat{a}\hat{a}$ and $\hat{a}^\dagger\hat{a}^\dagger$ have zero diagonal matrix elements and therefore

$$\begin{aligned} \langle n|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\,\langle n|(\hat{a}+\hat{a}^\dagger)^2|n\rangle \;=\; \frac{\hbar}{2m\omega}\langle n|(\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a})|n\rangle\,, \\ \langle n|\hat{p}^2|n\rangle &= -\frac{m\omega\hbar}{2}\langle n|(\hat{a}^\dagger-\hat{a})^2|n\rangle \;=\; \frac{m\omega\hbar}{2}\langle n|(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)|n\rangle\,. \end{aligned} \tag{1.56}$$

But $\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 1 + \hat{N} + \hat{N} = 1 + 2\hat{N}$ so therefore

$$\begin{aligned} \langle n|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\,\big(1+2n\big) \;=\; \frac{\hbar}{m\omega}\,\Big(n+\frac{1}{2}\Big)\,, \\ \langle n|\hat{p}^2|n\rangle &= \frac{m\omega\hbar}{2}\,\big(1+2n\big) \;=\; m\hbar\omega\Big(n+\frac{1}{2}\Big)\,. \end{aligned} \tag{1.57}$$

It follows that in the state $|n\rangle$ we have the uncertainties

$$\begin{aligned} (\Delta x)^2 &= \frac{\hbar}{m\omega}\,\Big(n+\frac{1}{2}\Big) \\ (\Delta p)^2 &= m\hbar\omega\Big(n+\frac{1}{2}\Big)\,. \end{aligned} \tag{1.58}$$

As a result

$$\text{On the state } |n\rangle: \quad \Delta x\,\Delta p \;=\; \hbar\Big(n+\frac{1}{2}\Big)\,. \tag{1.59}$$

Only for the ground state $n = 0$ product of uncertainties saturates the lower bound given by the Heisenberg uncertainty principle.

2 Schrödinger dynamics

The state space of quantum mechanics –the Hilbert space $\mathcal{H}$ of states – is best thought as a space with time-independent basis vectors. There is no role for time in the definition of the state space $\mathcal{H}$. In the Schrödinger "picture" of the dynamics, the state that represents a quantum system depends on time. Time is viewed as a parameter: at different times the state of the system is represented by different states in the Hilbert space. We write the state vector as

$$|\Psi, t\rangle\,, \tag{2.1}$$

and it is a vector whose components along the basis vectors of $\mathcal{H}$ are time dependent. If we call those basis vectors $|u_i\rangle$, we have

$$|\Psi, t\rangle \;=\; \sum_i |u_i\rangle c_i(t)\,, \tag{2.2}$$

where the $c_i(t)$ are some functions of time. Since a state must be normalized, we can imagine $|\Psi, t\rangle$ as a unit vector whose tip, as a function of time, sweeps a trajectory in $\mathcal{H}$. We will first discuss the postulate of unitary time evolution and then show that the Schrödinger equation follows from it.

2.1 Unitary time evolution

We declare that for any quantum system there is a *unitary* operator $\mathcal{U}(t, t_0)$ such that for *any* state $|\Psi, t_0\rangle$ of the system at time t_0 the state at time t is obtained as

$$\boxed{|\Psi, t\rangle \;=\; \mathcal{U}(t, t_0)|\Psi, t_0\rangle\,, \quad \forall\; t, t_0\,.} \tag{2.3}$$

It must be emphasized that the operator $\mathcal{U}$ generates time evolution for *any* possible state at time t_0 –it does *not* depend on the chosen state at time t_0. A physical system has a single operator $\mathcal{U}$ that generates the time evolution of all possible states. The above equation is valid for all times t, so t can be greater than, equal to, or less than t_0. As defined, the operator $\mathcal{U}$ is unique: if there is another operator $\mathcal{U}'$ that generates exactly the same evolution then $(\mathcal{U} - \mathcal{U}')|\Psi, t_0\rangle = 0$ and since the state $|\Psi, t_0\rangle$ is arbitrary we must have that the operator $\mathcal{U} - \mathcal{U}'$ vanishes, showing that $\mathcal{U} = \mathcal{U}'$.

The unitary property of $\mathcal{U}$ means that

$$(\mathcal{U}(t, t_0))^\dagger \mathcal{U}(t, t_0) \;=\; \mathbf{1}\,. \tag{2.4}$$

In order to avoid extra parenthesis, we will write

$$\mathcal{U}^\dagger(t, t_0) \;\equiv\; (\mathcal{U}(t, t_0))^\dagger\,, \tag{2.5}$$

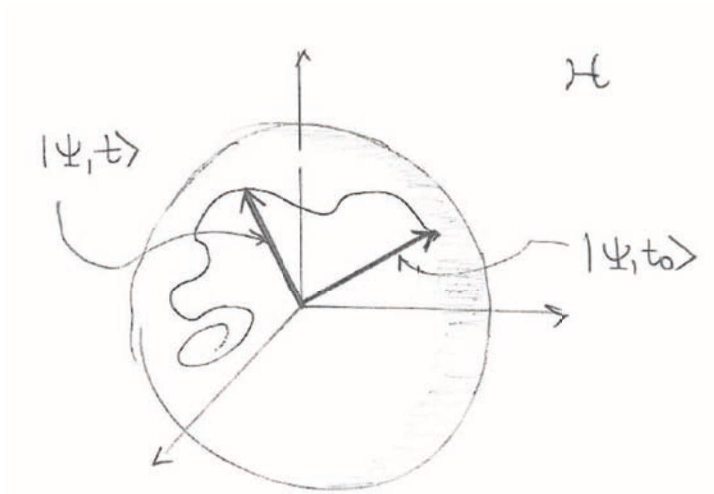

Figure 1: The initial state $|\Psi, t_0\rangle$ can be viewed as a vector in the complex vector space $\mathcal{H}$. As time goes by the vector moves, evolving by unitary transformations, so that its norm is preserved.

so that the unitarity property reads

$$\mathcal{U}^\dagger(t, t_0)\,\mathcal{U}(t, t_0) \;=\; \mathbf{1}\,. \tag{2.6}$$

Unitarity implies that the norm of the state is conserved[1]

$$\langle \Psi, t\,|\Psi, t\rangle \;=\; \langle \Psi, t_0\,|\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0)|\Psi, t_0\rangle \;=\; \langle \Psi, t_0|\,\Psi, t_0\rangle\,. \tag{2.7}$$

This is illustrated in Figure 1.

We now make a series of comments on this postulate.

1. For time $t = t_0$, equation (2.3) gives no time evolution

$$|\Psi, t_0\rangle \;=\; \mathcal{U}(t_0, t_0)|\Psi, t_0\rangle\,. \tag{2.8}$$

Since this equality holds for *any* possible state at $t = t_0$ the unitary evolution operator must be the unit operator

$$\mathcal{U}(t_0, t_0) \;=\; \mathbf{1}\,, \quad \forall t_0\,. \tag{2.9}$$

2. Composition. Consider the evolution from t_0 to t_2 as a two-step procedure, from t_0 to t_1 and from t_1 to t_2:

$$|\Psi, t_2\rangle \;=\; \mathcal{U}(t_2, t_1)|\Psi, t_1\rangle \;=\; \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)|\Psi, t_0\rangle\,. \tag{2.10}$$

It follows from this equation and $|\Psi, t_2\rangle \;=\; \mathcal{U}(t_2, t_0)|\Psi, t_0\rangle$ that

$$\mathcal{U}(t_2, t_0) \;=\; \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)\,. \tag{2.11}$$

[1]We also recall that any operator that preserves the norm of arbitrary states is unitary.

3. Inverses. Consider (2.11) and set $t_2 = t_0$ and $t_1 = t$. Then using (2.9) we get

$$\mathbf{1} \;=\; \mathcal{U}(t_0, t)\,\mathcal{U}(t, t_0)\,. \tag{2.12}$$

Thus we have

$$\mathcal{U}(t_0, t) \;=\; (\mathcal{U}(t, t_0))^{-1} \;=\; (\mathcal{U}(t, t_0))^{\dagger}\,, \tag{2.13}$$

where the first relation follows from (2.12) and the second by unitarity. Again, declining to use parenthesis that are not really needed, we write

$$\boxed{\mathcal{U}(t_0, t) \;=\; \mathcal{U}^{-1}(t, t_0) \;=\; \mathcal{U}^{\dagger}(t, t_0)\,.} \tag{2.14}$$

Simply said, inverses or hermitian conjugation of $\mathcal{U}$ reverse the order of the time arguments.

2.2 Deriving the Schrödinger equation

The time evolution of states has been specified in terms of a unitary operator $\mathcal{U}$ assumed known. We now ask the 'reverse engineering' question. What kind of differential equation do the states satisfy for which the solution is unitary time evolution? The answer is simple and satisfying: a Schrödinger equation.

To obtain this result, we take the time derivative of (2.3) to find

$$\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; \frac{\partial \mathcal{U}(t, t_0)}{\partial t}|\Psi, t_0\rangle\,. \tag{2.15}$$

We want the right hand side to involve the ket $|\Psi, t\rangle$ so we write

$$\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; \frac{\partial \mathcal{U}(t, t_0)}{\partial t}\,\mathcal{U}(t_0, t)|\Psi, t\rangle\,. \tag{2.16}$$

Finally, it is convenient to have the same kind of $\mathcal{U}$ operator appearing, so we trade the order of times in the second $\mathcal{U}$ for a dagger:

$$\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; \frac{\partial \mathcal{U}(t, t_0)}{\partial t}\,\mathcal{U}^{\dagger}(t, t_0)|\Psi, t\rangle\,. \tag{2.17}$$

This now looks like a differential equation for the state $|\Psi, t\rangle$. Let us introduce a name for the operator acting on the state in the right-hand side:

$$\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; \Lambda(t, t_0)|\Psi, t\rangle\,, \tag{2.18}$$

where

$$\Lambda(t, t_0) \;\equiv\; \frac{\partial \mathcal{U}(t, t_0)}{\partial t}\,\mathcal{U}^{\dagger}(t, t_0)\,. \tag{2.19}$$

The operator Λ has units of inverse time. Note also that

$$\Lambda^\dagger(t,t_0) \;=\; \mathcal{U}(t,t_0)\,\frac{\partial \mathcal{U}^\dagger(t,t_0)}{\partial t}\,, \tag{2.20}$$

since the adjoint operation changes the order of operators and does not interfere with the time derivative.

We now want to prove two important facts about Λ:

1. $\Lambda(t,t_0)$ is antihermitian. To prove this begin with the equation

$$\mathcal{U}(t,t_0)\mathcal{U}^\dagger(t,t_0) \;=\; \mathbf{1}\,, \tag{2.21}$$

   and take a derivative with respect to time to find,

$$\frac{\partial \mathcal{U}(t,t_0)}{\partial t}\mathcal{U}^\dagger(t,t_0) + \mathcal{U}(t,t_0)\,\frac{\partial \mathcal{U}^\dagger(t,t_0)}{\partial t} \;=\; 0\,. \tag{2.22}$$

   Glancing at (2.19) and (2.20) we see that we got

$$\Lambda(t,t_0) + \Lambda^\dagger(t,t_0) \;=\; 0\,, \tag{2.23}$$

   proving that $\Lambda(t,t_0)$ is indeed anti-hermitian.

2. $\Lambda(t,t_0)$ is actually independent of t_0. This is important because in the differential equation (2.17) t_0 appears nowhere except in Λ. To prove this independence we will show that $\Lambda(t,t_0)$ is actually equal to $\Lambda(t,t_1)$ for any other time t_1 different from t_0. So its value cannot depend on t_0. Or said differently, imagine $t_1 = t_0 + \epsilon$, then $\Lambda(t,t_0) = \Lambda(t,t_0+\epsilon)$ and as a result $\frac{\partial \Lambda(t,t_0)}{\partial t_0} = 0$. To prove the claim we begin with (2.19) and insert the unit operator in between the two factors

$$\begin{aligned}
\Lambda(t,t_0) \;&=\; \frac{\partial \mathcal{U}(t,t_0)}{\partial t}\mathcal{U}^\dagger(t,t_0)\\
&=\; \frac{\partial \mathcal{U}(t,t_0)}{\partial t}\Big(\mathcal{U}(t_0,t_1)\mathcal{U}^\dagger(t_0,t_1)\Big)\,\mathcal{U}^\dagger(t,t_0)\\
&=\; \frac{\partial}{\partial t}\Big(\mathcal{U}(t,t_0)\mathcal{U}(t_0,t_1)\Big)\mathcal{U}^\dagger(t_0,t_1)\,\mathcal{U}^\dagger(t,t_0)\\
&=\; \frac{\partial \mathcal{U}(t,t_1)}{\partial t}\mathcal{U}(t_1,t_0)\,\mathcal{U}(t_0,t) \;=\; \frac{\partial \mathcal{U}(t,t_1)}{\partial t}\mathcal{U}(t_1,t)\\
&=\; \frac{\partial \mathcal{U}(t,t_1)}{\partial t}\mathcal{U}^\dagger(t,t_1) \;=\; \Lambda(t,t_1)\,,
\end{aligned} \tag{2.24}$$

   as we wanted to prove.

It follows that we can write $\Lambda(t) \equiv \Lambda(t, t_0)$, and thus equation (2.18) becomes

$$\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; \Lambda(t)|\Psi, t\rangle\,. \tag{2.25}$$

We can define an operator $H(t)$ by multiplication of Λ by $i\hbar$:

$$\boxed{H(t) \;\equiv\; i\hbar\Lambda(t) \;=\; i\hbar\,\frac{\partial \mathcal{U}(t, t_0)}{\partial t}\,\mathcal{U}^\dagger(t, t_0)\,.} \tag{2.26}$$

Since Λ is antihermitian and has units of inverse time, $H(t)$ is a *hermitian* operator with units of energy. Multiplying (2.25) by $i\hbar$ we find the Schrödinger equation:

$$\boxed{\text{Schrödinger equation:}\quad i\hbar\frac{\partial}{\partial t}|\Psi, t\rangle \;=\; H(t)\,|\Psi, t\rangle\,.} \tag{2.27}$$

This is our main result. Unitary time evolution implies this equation. In this derivation the Hamiltonian is obtained from the knowledge of $\mathcal{U}$, as shown in (2.26). In most familiar situations, we know the Hamiltonian and wish to calculate the time evolution operator $\mathcal{U}$.

There are basically two reasons why the quantity $H(t)$ appearing in (2.27) is identified with the Hamiltonian. First, in quantum mechanics the momentum operator is given by $\hbar/i$ times the derivative with respect to a spatial coordinate. In special relativity energy corresponds to the time component of the momentum four-vector and thus it is reasonable to view it as an operator proportional to a time derivative. Second, we have used (2.27) to derive an equation for the time evolution of expectation values of observables. For an observable Q this took the form

$$\frac{d\langle Q\rangle}{dt} \;=\; \frac{1}{i\hbar}\langle[Q, H]\rangle \tag{2.28}$$

This equation is a natural generalization of the classical mechanics Hamiltonian equations and $H(t)$ plays a role analogous to that of the classical Hamiltonian. Indeed, in classical mechanics one has Poisson brackets $\{\cdot\,,\cdot\}_{pb}$ defined for functions of x and p by

$$\{A, B\}_{pb} \;=\; \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} \tag{2.29}$$

It then turns out that for any observable function $Q(x, p)$, its time derivative is given by taking the Poisson bracket of Q with the Hamiltonian:

$$\frac{dQ}{dt} \;=\; \{Q, H\}_{pb} \tag{2.30}$$

The similarity to (2.28) is quite striking. In fact, one can view commutators as essentially $\hbar$ times Poisson brackets

$$[A, B] \quad\Longleftrightarrow\quad i\hbar\,\{A, B\}_{pb} \tag{2.31}$$

Indeed $[x,p] = i\hbar$ while $\{x,p\}_{pb} = 1$. While these reasons justify our calling of H in the Schrödinger equation the Hamiltonian, ultimately we can say that any Hermitian operator with units of energy has the right to be called a Hamiltonian, regardless of any connection to a classical theory.

The Schrödinger wavefunction $\Psi(x,t)$ is defined by

$$\Psi(x,t) \equiv \langle x|\Psi,t\rangle\,. \tag{2.32}$$

If we hit (2.27) with the position state $\langle x|$ from the left we get

$$i\hbar\,\frac{\partial\Psi(x,t)}{\partial t} = \langle x|H(t)|\Psi,t\rangle\,. \tag{2.33}$$

If, moreover,

$$H(t) = \frac{\hat{p}^2}{2m} + V(\hat{x})\,, \tag{2.34}$$

then the equation becomes

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\Big)\Psi(x,t)\,. \tag{2.35}$$

This is the familiar form of the Schrödinger equation for one-dimensional potentials.

2.3 Calculation of the unitary time evolution operator

The typical situation is one where the Hamiltonian $H(t)$ is known and we wish to calculate the unitary operator $\mathcal{U}$ that implements time evolution. From equation (2.26), multiplying by $\mathcal{U}(t,t_0)$ from the right gives

$$i\hbar\,\frac{\partial\mathcal{U}(t,t_0)}{\partial t} = H(t)\,\mathcal{U}(t,t_0)\,. \tag{2.36}$$

This is viewed as a differential equation for the *operator* $\mathcal{U}$. Note also that letting both sides of this equation act on $|\Psi,t_0\rangle$ gives us back the Schrödinger equation.

Since there is no possible confusion with the time derivatives, we do not need to write them as partial derivatives. Then the above equation takes the form

$$\frac{d\mathcal{U}}{dt} = -\frac{i}{\hbar}H(t)\,\mathcal{U}(t)\,. \tag{2.37}$$

If we view operators as matrices, this is a differential equation for the matrix $\mathcal{U}$. Solving this equation is in general quite difficult. We will consider three cases of increasing complexity.

Case 1. H is time independent. In this case, equation (2.37) is structurally of the form

$$\frac{d\mathcal{U}}{dt} = K\,\mathcal{U}(t)\,, \quad K = -\frac{i}{\hbar}\,H\,, \tag{2.38}$$

where $\mathcal{U}$ is a time dependent matrix, and K is a time-independent matrix. If the matrices were one-by-one, this reduces to the plain differential equation

$$\frac{du}{dt} = ku(t) \quad \to \quad u(t) \;=\; e^{kt}u(0)\,. \tag{2.39}$$

For the matrix case (2.38) we claim that

$$\mathcal{U}(t) \;=\; e^{tK}\,\mathcal{U}(0)\,. \tag{2.40}$$

Here we have the exponential of a matrix multiplied from the right by the matrix $\mathcal{U}$ at time equal zero. At $t = 0$ the ansatz gives the proper result, by construction. The exponential of a matrix is defined by the Taylor series

$$e^{tK} \;=\; 1 + tK + \frac{1}{2!}(tK)^2 + \frac{1}{3!}(tK)^3 + \cdots \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\,t^n K^n \tag{2.41}$$

Therefore it follows that the derivative takes the familiar simple form

$$\frac{d}{dt}e^{tK} \;=\; Ke^{tK} \;=\; e^{tK}K\,. \tag{2.42}$$

With this result we readily verify that (2.40) does solve (2.38):

$$\frac{d\mathcal{U}}{dt} \;=\; \frac{d}{dt}\,(e^{tK}\mathcal{U}(0)) \;=\; Ke^{tK}\mathcal{U}(0) \;=\; K\mathcal{U}(t)\,. \tag{2.43}$$

Using the explicit form of the matrix K the solution is therefore

$$\mathcal{U}(t,t_0) \;=\; e^{-\frac{i}{\hbar}Ht}\,\mathcal{U}_0\,, \tag{2.44}$$

where $\mathcal{U}_0$ is a constant matrix. Recalling that $\mathcal{U}(t_0,t_0) = \mathbf{1}$, we have $\mathcal{U}_0 = e^{iHt_0/\hbar}$ and therefore the full solution is

$$\boxed{\mathcal{U}(t,t_0) \;=\; \exp\Big[-\frac{i}{\hbar}H(t-t_0)\Big]\,, \quad \text{Time-independent } H\,.} \tag{2.45}$$

Exercise. Verify that the ansatz $\mathcal{U}(t) = \mathcal{U}(0)e^{tK}$, consistent for $t = 0$, would have not provided a solution of (2.38).

Case 2. $[H(t_1)\,, H(t_2)\,] = 0$ for all t_1, t_2. Here the Hamiltonian is time dependent but, despite this, the Hamiltonian at different times commute. One example is provided by the Hamiltonian for a spin in a magnetic field of time-dependent magnitude but constant direction.

We claim that the time evolution operator is now given by

$$\boxed{\mathcal{U}(t,t_0) \;=\; \exp\Big[-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\Big]\,.} \tag{2.46}$$

If the Hamiltonian is time independent, the above solution reduces correctly to (2.45). To prove that (2.46) solves the differential equation (2.37) we streamline notation by writing

$$R(t) \equiv -\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t') \quad \rightarrow \quad R' = -\frac{i}{\hbar} H(t)\,, \tag{2.47}$$

where primes denote time derivatives. We claim that $R'(t)$ and $R(t)$ commute. Indeed

$$\big[R'(t)\,, R(t)\big] = \Big[-\frac{i}{\hbar} H(t)\,, -\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\Big] = \Big(-\frac{i}{\hbar}\Big)^2 \int_{t_0}^{t} dt' \,\big[H(t)\,, H(t')\big] = 0\,. \tag{2.48}$$

The claimed solution is

$$\mathcal{U} = \exp R(t) = 1 + R(t) + \frac{1}{2}R(t)R(t) + \frac{1}{3!}R(t)R(t)R(t) + \ldots \tag{2.49}$$

We have to take the time derivative of $\mathcal{U}$ and this time we do it slowly(!):

$$\begin{aligned} \frac{d}{dt}\mathcal{U} = \frac{d}{dt}\exp R &= R' + \frac{1}{2}(R'R + RR') + \frac{1}{3!}(R'RR + RR'R + RRR') + \ldots\,, \\ &= R' + R'R + \frac{1}{2!}R'RR + \ldots = R' \exp(R) \end{aligned} \tag{2.50}$$

The lesson here is that the derivative of $\exp R$ is simple if R' commutes with R. We have thus obtained

$$\frac{d}{dt}\mathcal{U} = -\frac{i}{\hbar}H(t)\,\mathcal{U}\,, \tag{2.51}$$

which is exactly what we wanted to show.

Case 3. $[H(t_1)\,, H(t_2)\,] \neq 0$. This is the most general situation and there is only a series solution. We write it here even though it will not be needed in our work. The solution for $\mathcal{U}$ is given by the so-called 'time-ordered' exponential, denoted by the symbol T in front of an exponential

$$\begin{aligned} \mathcal{U}(t, t_0) = \mathrm{T}\exp\Big[-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\Big] &\equiv 1 + \Big(-\frac{i}{\hbar}\Big)\int_{t_0}^{t} dt_1 H(t_1) \\ &+ \Big(-\frac{i}{\hbar}\Big)^2 \int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2) \\ &+ \Big(-\frac{i}{\hbar}\Big)^3 \int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2) \int_{t_0}^{t_2} dt_3 H(t_3) \\ &+ \ldots\,. \end{aligned} \tag{2.52}$$

The term time-ordered refers to the fact that in the n-th term of the series we have a product $H(t_1)H(t_2)H(t_3)\ldots H(t_n)$ of *non-commuting* operators with integration ranges that force ordered times $t_1 \geq t_2 \geq t_3 \cdots \geq t_n$.

3 Heisenberg dynamics

The idea here is to confine the dynamical evolution to the operators. We will 'fold' the time dependence of the states into the operators. Since the objects we usually calculate are time-dependent expectation values of operators, this approach turns to be quite effective.

We will define time-dependent Heisenberg operators starting from Schrödinger operators. In fact, to any Schrödinger operator we can associate its corresponding Heisenberg operator. Schrödinger operators come in two types, time independent ones (like $\hat{x}, \hat{p}$) and time dependent ones (like Hamiltonians with time-dependent potentials). For each of those types of operators we will associate Heisenberg operators.

3.1 Heisenberg operators

Let us consider a Schrödinger operator $\hat{A}_S$, with the subscript S for Schrödinger. This operator may or may not have time dependence. We now examine a matrix element of $\hat{A}_S$ in between time dependent states $|\alpha, t\rangle$ and $|\beta, t\rangle$ and use the time-evolution operator to convert the states to time zero:

$$\langle\alpha, t|\hat{A}_S|\beta, t\rangle \;=\; \langle\alpha, 0|\,\mathcal{U}^\dagger(t,0)\,\hat{A}_S\,\mathcal{U}(t,0)\,|\beta, 0\rangle\,. \tag{3.1}$$

We simply define the Heisenberg operator $\hat{A}_H(t)$ associated with $\hat{A}_S$ as the object in between the time equal zero states:

$$\boxed{\hat{A}_H(t) \;\equiv\; \mathcal{U}^\dagger(t,0)\,\hat{A}_S\,\mathcal{U}(t,0)\,.} \tag{3.2}$$

Let us consider a number of important consequences of this definition.

1. At $t = 0$ the Heisenberg operator becomes equal to the Schrödinger operator:

$$\hat{A}_H(0) \;=\; \hat{A}_S\,. \tag{3.3}$$

   The Heisenberg operator associated with the unit operator is the unit operator:

$$\mathbf{1}_H \;=\; \mathcal{U}^\dagger(t,0)\,\mathbf{1}\,\mathcal{U}(t,0) \;=\; \mathbf{1}\,. \tag{3.4}$$

2. The Heisenberg operator associated with the product of Schrödinger operators is equal to the product of the corresponding Heisenberg operators:

$$\hat{C}_S \;=\; \hat{A}_S\hat{B}_S \quad\rightarrow\quad \hat{C}_H(t) \;=\; \hat{A}_H(t)\hat{B}_H(t)\,. \tag{3.5}$$

   Indeed,

$$\begin{aligned}\hat{C}_H(t) &= \mathcal{U}^\dagger(t,0)\,\hat{C}_S\,\mathcal{U}(t,0) \;=\; \mathcal{U}^\dagger(t,0)\,\hat{A}_S\hat{B}_S\,\mathcal{U}(t,0)\\ &= \hat{\mathcal{U}}^\dagger(t,0)\,\hat{A}_S\,\mathcal{U}(t,0)\mathcal{U}^\dagger(t,0)\,\hat{B}_S\,\mathcal{U}(t,0) \;=\; \hat{A}_H(t)\hat{B}_H(t)\,.\end{aligned} \tag{3.6}$$

3. It also follows from (3.5) that if we have a commutator of Schrödinger operators the corresponding Heisenberg operators satisfy the same commutation relations

$$[\hat{A}_S,\, \hat{B}_S\,] \;=\; C_S \quad\rightarrow\quad [\hat{A}_H(t),\, \hat{B}_H(t)\,] \;=\; \hat{C}_H(t)\,. \tag{3.7}$$

Since $\mathbf{1}_H = \mathbf{1}$, eqn. (3.7) implies that, for example,

$$[\hat{x}\,,\hat{p}\,] \;=\; i\hbar\,\mathbf{1} \quad\rightarrow\quad [\hat{x}_H(t)\,,\hat{p}_H(t)\,] \;=\; i\hbar\,\mathbf{1}\,. \tag{3.8}$$

4. Schrödinger and Heisenberg Hamiltonians. Assume we have a Schrödinger Hamiltonian that depends on some Schrödinger momenta and position operators $\hat{p}$ and $\hat{x}$, as in

$$H_S(\hat{p}, \hat{x}\,;\, t)\,. \tag{3.9}$$

Since the $\hat{x}$ and $\hat{p}$ in H_S appear in products, property 2 implies that the associated Heisenberg Hamiltonian H_H takes the same form, with $\hat{x}$ and $\hat{p}$ replaced by their Heisenberg counterparts

$$H_H(t) \;=\; H_S(\hat{p}_H(t)\,, \hat{x}_H(t)\,;\, t)\,. \tag{3.10}$$

5. Equality of Hamiltonians. Under some circumstances the Heisenberg Hamiltonian is in fact equal to the Schrödinger Hamiltonian. Recall the definition

$$H_H(t) \;=\; \mathcal{U}^\dagger(t,0)\, H_S(t)\, \mathcal{U}(t,0)\,. \tag{3.11}$$

Assume now that $[H_S(t), H_S(t')] = 0$. Then (2.46) gives the time evolution operator

$$\mathcal{U}(t,0) \;=\; \exp\Big[-\frac{i}{\hbar}\int_0^t dt' H_S(t')\Big]\,. \tag{3.12}$$

Since the H_S at different times commute, $H_S(t)$ commutes both with $\mathcal{U}(t,0)$ and $\mathcal{U}^\dagger(t,0)$. Therefore the $H_S(t)$ can be moved, say to the right, in (3.11) giving us

$$H_H(t) \;=\; H_S(t)\,, \quad \text{when} \quad [H_S(t), H_S(t')] = 0\,. \tag{3.13}$$

The meaning of this relation becomes clearer when we use (3.10) and (3.9) to write

$$H_S(\hat{p}_H(t)\,, \hat{x}_H(t)\,;\, t) \;=\; H_S(\hat{p}\,, \hat{x}\,;\, t)\,. \tag{3.14}$$

Operationally, this means that if we take $\hat{x}_H(t)$ and $\hat{p}_H(t)$ and plug them into the Hamiltonian (left-hand side), the result is as if we had simply plugged $\hat{x}$ and $\hat{p}$. We will confirm this for the case of the simple harmonic oscillator.

6. Equality of operators. If a Schrödinger operator A_S commutes with the Hamiltonian $H_S(t)$ for all times then A_S commutes with $\mathcal{U}(t,0)$ since this operator (even in the most complicated of cases) is built using $H_S(t)$. It follows that $A_H(t) = A_S$; the Heisenberg operator is equal to the Schrödinger operator.

7. Expectation values. Consider (3.1) and let $|\alpha, t\rangle = |\beta, t\rangle = |\Psi, t\rangle$. The matrix element now becomes an expectation value and we have:

$$\boxed{\langle\Psi, t|\hat{A}_S|\Psi, t\rangle \;=\; \langle\Psi, 0|\,\hat{A}_H(t)\,|\Psi, 0\rangle\,.} \tag{3.15}$$

With a little abuse of notation, we simply write this equation as

$$\boxed{\big\langle\hat{A}_S\big\rangle \;=\; \big\langle\hat{A}_H(t)\big\rangle\,.} \tag{3.16}$$

You should realize when writing such an equation that on the left hand side you compute the expectation value using the time-dependent state, while on the right-hand side you compute the expectation value using the state at time equal zero. If you prefer you can write out the equation as in (3.15) in case you think there is a possible confusion.

3.2 Heisenberg equation of motion

We can calculate the Heisenberg operator associated with a Schrödinger one using the definition (3.2). Alternatively, Heisenberg operators satisfy a differential equation: the Heisenberg equation of motion. This equation looks very much like the equations of motion of classical dynamical variables. So much so, that people trying to invent quantum theories sometimes begin with the equations of motion of some classical system and they postulate the existence of Heisenberg operators that satisfy similar equations. In that case they must also find a Heisenberg Hamiltonian and show that the equations of motion indeed arise in the quantum theory.

To determine the equation of motion of Heisenberg operators we will simply take time derivatives of the definition (3.2). For this purpose we recall (2.36) which we copy here using the subscript S for the Hamiltonian:

$$i\hbar\,\frac{\partial\mathcal{U}(t,t_0)}{\partial t} \;=\; H_S(t)\,\mathcal{U}(t,t_0)\,. \tag{3.17}$$

Taking the adjoint of this equation we find

$$i\hbar\,\frac{\partial\mathcal{U}^\dagger(t,t_0)}{\partial t} \;=\; -\,\mathcal{U}^\dagger(t,t_0)\,H_S(t)\,. \tag{3.18}$$

We can now calculate. Using (3.2) we find

$$\begin{aligned} i\hbar\frac{d}{dt}\hat{A}_H(t) \;=\;& \Big(i\hbar\frac{\partial\mathcal{U}^\dagger}{\partial t}(t,0)\Big)\,\hat{A}_S(t)\,\mathcal{U}(t,0) \\ &+\mathcal{U}^\dagger(t,0)\,\hat{A}_S(t)\,\Big(i\hbar\frac{\partial\mathcal{U}}{\partial t}(t,0)\Big) \\ &+\mathcal{U}^\dagger(t,0)\,i\hbar\frac{\partial\hat{A}_S(t)}{\partial t}\;\mathcal{U}(t,0) \end{aligned} \tag{3.19}$$

Using (3.17) and (3.18) we find

$$
\begin{aligned}
i\hbar\frac{d}{dt}\hat{A}_H(t) \;=\;\; & -\mathcal{U}^\dagger(t,0)\, H_S(t)\, \hat{A}_S(t)\mathcal{U}(t,0) \\
& +\mathcal{U}^\dagger(t,0)\, \hat{A}_S(t)\, H_S(t)\mathcal{U}(t,0) \\
& +\mathcal{U}^\dagger(t,0)\, i\hbar\frac{\partial \hat{A}_S(t)}{\partial t}\; \mathcal{U}(t,0)
\end{aligned}
\qquad (3.20)
$$

We now use (3.5) and recognize that in the last line we have the Heisenberg operator associated with the time derivative of $\hat{A}_S$:

$$
i\hbar\frac{d}{dt}\hat{A}_H(t) \;=\; -H_H(t)\,\hat{A}_H(t) \;+\hat{A}_H(t)\,H_H(t) \;+i\hbar\Big(\frac{\partial \hat{A}_S(t)}{\partial t}\Big)_H \qquad (3.21)
$$

We now recognize a commutator on the right-hand side, so that our final result is

$$
\boxed{\, i\hbar\,\frac{d\hat{A}_H(t)}{dt} \;=\; \big[\,\hat{A}_H(t)\,,\,H_H(t)\big] \;+\, i\hbar\,\Big(\frac{\partial \hat{A}_S(t)}{\partial t}\Big)_H \,.} \qquad (3.22)
$$

A few comments are in order.

1. Schrödinger operators without time dependence. If the operator $\hat{A}_S$ has no explicit time dependence then the last term in (3.22) vanishes and we have the simpler

$$
i\hbar\,\frac{d\hat{A}_H(t)}{dt} \;=\; \big[\,\hat{A}_H(t)\,,\,H_H(t)\big]\,. \qquad (3.23)
$$

2. Time dependence of expectation values. Let A_S be a Schrödinger operator without time dependence. Let us now take the time derivative of the expectation value relation in (3.15):

$$
\begin{aligned}
i\hbar\frac{d}{dt}\langle\Psi,t|\hat{A}_S|\Psi,t\rangle \;=\; i\hbar\frac{d}{dt}\langle\Psi,0|\,\hat{A}_H(t)\,|\Psi,0\rangle \;&=\langle\Psi,0|\; i\hbar\,\frac{d\hat{A}_H(t)}{dt}\,|\Psi,0\rangle \\
&=\langle\Psi,0|\,\big[\,\hat{A}_H(t)\,,\,H_H(t)\big]\,|\Psi,0\rangle
\end{aligned}
\qquad (3.24)
$$

We write this as

$$
\boxed{\, i\hbar\frac{d}{dt}\,\langle\,\hat{A}_H(t)\rangle \;=\; \big\langle\,\big[\,\hat{A}_H(t)\,,\,H_H(t)\big]\big\rangle\,.} \qquad (3.25)
$$

Notice that this equation takes exactly the same form in the Schrödinger picture (recall the comments below (3.16):

$$
\boxed{\, i\hbar\frac{d}{dt}\,\langle\,\hat{A}_S\,\rangle \;=\; \big\langle\,\big[\,\hat{A}_S\,,\,H_S\big]\big\rangle\,.} \qquad (3.26)
$$

3. A time-independent operator $\hat{A}_S$ is said to be **conserved** if it commutes with the Hamiltonian:

$$\text{Conserved operator } \hat{A}_S: \quad [\hat{A}_S\,, H_S\,] \;=\; 0\,. \tag{3.27}$$

It then follows that $[\,\hat{A}_H(t)\,, H_H(t)\,] \;= 0$, and using (3.23) that

$$\frac{d\hat{A}_H(t)}{dt} \;= 0\,. \tag{3.28}$$

The Heisenberg operator is plain constant. Thus the expectation value of the operator is also constant. This is consistent with comment 6 in the previous section: $\hat{A}_H$ is in fact equal to $\hat{A}_S$!

3.3 Three examples.

Example 1. Part of the Homework. We just discuss here a few facts. Consider the Hamiltonian

$$H \;=\; \frac{\hat{p}^2}{2m} + V(\hat{x})\,, \tag{3.29}$$

where $V(x)$ is a potential. You will show that

$$\begin{aligned} \frac{d}{dt}\langle \hat{x}\rangle \;&=\; \frac{1}{m}\langle \hat{p}\rangle\,, \\ \frac{d}{dt}\langle \hat{p}\rangle \;&=\; -\Big\langle \frac{\partial V}{\partial \hat{x}}\Big\rangle\,. \end{aligned} \tag{3.30}$$

These two equations combined give

$$m\,\frac{d^2}{dt^2}\langle \hat{x}\rangle \;=\; -\Big\langle \frac{\partial V}{\partial \hat{x}}\Big\rangle\,. \tag{3.31}$$

This is the quantum analog of the classical equation

$$m\frac{d^2}{dt^2}x(t) \;=\; -\frac{\partial V}{\partial x}\,, \tag{3.32}$$

which describes the classical motion of a particle of mass m in a potential $V(x)$. Note that the force is $F = -\frac{\partial V}{\partial x}$.

Example 2. Harmonic oscillator. The Schrödinger Hamiltonian is

$$H_S \;=\; \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,, \tag{3.33}$$

and is time independent. Using (3.10) we note that the Heisenberg Hamiltonian takes the form

$$H_H(t) \;=\; \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2(t)\,. \tag{3.34}$$

Consider now the Schrödinger operators $\hat{x}$ and $\hat{p}$. Using the Heisenberg equation of motion, we have for $\hat{x}$:

$$\begin{aligned} \frac{d}{dt}\,\hat{x}_H(t) &= \frac{1}{i\hbar}\big[\hat{x}_H(t), H_H(t)\big] = \frac{1}{i\hbar}\Big[\hat{x}_H(t), \frac{\hat{p}_H^2(t)}{2m}\Big] \\ &= \frac{1}{i\hbar}\,2\,\frac{\hat{p}_H(t)}{2m}\Big[\hat{x}_H(t), \hat{p}_H(t)\Big] = \frac{1}{i\hbar}\,\frac{\hat{p}_H(t)}{m}\,i\hbar = \frac{\hat{p}_H(t)}{m}\,, \end{aligned} \tag{3.35}$$

so that our first equation is

$$\boxed{\frac{d}{dt}\,\hat{x}_H(t) = \frac{\hat{p}_H(t)}{m}\,.} \tag{3.36}$$

For the momentum operator we get

$$\begin{aligned} \frac{d}{dt}\,\hat{p}_H(t) &= \frac{1}{i\hbar}\big[\hat{p}_H(t), H_H(t)\big] = \frac{1}{i\hbar}\Big[\hat{p}_H(t),\ \frac{1}{2}m\omega^2\,x_H^2(t)\Big] \\ &= \frac{1}{i\hbar}\,\frac{1}{2}m\omega^2\cdot 2(-i\hbar)\hat{x}_H(t) = -m\omega^2\,\hat{x}_H(t)\,, \end{aligned} \tag{3.37}$$

so our second equation is

$$\boxed{\frac{d}{dt}\,\hat{p}_H(t) = -m\omega^2\,\hat{x}_H(t)\,.} \tag{3.38}$$

Taking another time derivative of (3.36) and using (3.38) we get

$$\frac{d^2}{dt^2}\,\hat{x}_H(t) = -\omega^2\,\hat{x}_H(t)\,. \tag{3.39}$$

We now solve this differential equation. Being just an oscillator equation the solution is

$$\hat{x}_H(t) = \hat{A}\,\cos\omega t + \hat{B}\,\sin\omega t\,, \tag{3.40}$$

where $\hat{A}$ and $\hat{B}$ are time-independent operators to be determined by initial conditions. From (3.36) we can find the momentum operator

$$\hat{p}_H(t) = m\frac{d}{dt}\hat{x}_H(t) = -m\omega\hat{A}\sin\omega t + m\omega\hat{B}\cos\omega t\,. \tag{3.41}$$

At zero time the Heisenberg operators must equal the Schrödinger ones so,

$$\hat{x}_H(0) = \hat{A} = \hat{x}\,, \quad \hat{p}_H(0) = m\omega\hat{B} = \hat{p}\,. \tag{3.42}$$

We have thus found that

$$\hat{A} = \hat{x}\,, \qquad \hat{B} = \frac{1}{m\omega}\,\hat{p}\,. \tag{3.43}$$

Finally, back in (3.40) and (3.41) we have our full solution for the Heisenberg operators of the SHO:

$$\boxed{\begin{aligned} \hat{x}_H(t) &= \hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\,, \\ \hat{p}_H(t) &= \hat{p}\cos\omega t - m\omega\,\hat{x}\,\sin\omega t\,. \end{aligned}} \tag{3.44}$$

Let us do a couple of small computations. Consider the energy eigenstate $|n\rangle$ of the harmonic oscillator:

$$|\psi, 0\rangle \;=\; |n\rangle\,. \tag{3.45}$$

We ask: What is the time-dependent expectation value of the x operator in this state? We compute

$$\langle\hat{x}\rangle \;=\; \langle\psi, t|\hat{x}|\psi, t\rangle \;=\; \langle\psi, 0|\hat{x}_H(t)|\psi, 0\rangle \;=\; \langle n|\hat{x}_H(t)|n\rangle\,. \tag{3.46}$$

Now we use the expression for $\hat{x}_H(t)$:

$$\langle\hat{x}\rangle \;=\; \langle n|\big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\big)|n\rangle \;=\; \langle n|\hat{x}|n\rangle\,\cos\omega t + \;\langle n|\,\hat{p}\,|n\rangle\,\frac{1}{m\omega}\sin\omega t\,. \tag{3.47}$$

We now recall that $\langle n|\hat{x}|n\rangle = 0$ and $\langle n|\,\hat{p}\,|n\rangle = 0$. So as a result we find that on the energy eigenstate $|n\rangle$, the expectation value of x is zero at all times:

$$\langle\,\hat{x}\,\rangle \;=\; 0\,. \tag{3.48}$$

So energy eigenstates do not exhibit classical behavior (an oscillatory time-dependent $\langle\,\hat{x}\,\rangle$).

As a second calculation let us confirm that the Heisenberg Hamiltonian is time independent and in fact equal to the Schrödinger Hamiltonian. Starting with (3.34) and using (3.44) we have

$$\begin{aligned} H_H(t) =&\; \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2(t) \\ =&\; \frac{1}{2m}\big(\hat{p}\cos\omega t - m\omega\,\hat{x}\,\sin\omega t\big)^2 + \frac{1}{2}m\omega^2\Big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\Big)^2 \\ =&\; \frac{\cos^2\omega t}{2m}\hat{p}^2 + \frac{m^2\omega^2\sin^2\omega t}{2m}\,\hat{x}^2 - \frac{\omega}{2}\sin\omega t\cos\omega t(\hat{p}\hat{x} + \hat{x}\hat{p}) \\ &+ \frac{\sin^2\omega t}{2m}\hat{p}^2 + \frac{m\omega^2\cos^2\omega t}{2}\,\hat{x}^2 + \frac{\omega}{2}\cos\omega t\sin\omega t\,(\hat{x}\hat{p} + \hat{p}\hat{x}) \\ =&\; \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,. \end{aligned} \tag{3.49}$$

This is what we wanted to show.

Example 3. What are the Heisenberg operators corresponding to the simple harmonic oscillator creation and annihilation operators?

Given the Schrödinger operator $\hat{a}$, the Heisenberg operator would be denoted as $\hat{a}_H(t)$, but for simplicity we will just denote it as $\hat{a}(t)$. Since the harmonic oscillator Hamiltonian is time independent, we can use the definition

$$\hat{a}(t) \equiv e^{\frac{i}{\hbar}Ht}\, \hat{a}\, e^{-\frac{i}{\hbar}Ht} = e^{i\omega t\hat{N}}\, a\, e^{-i\omega t\hat{N}}\,, \tag{3.50}$$

where we wrote $H = \hbar\omega(\hat{N} + \frac{1}{2})$ and noted that the additive constant has no effect on the commutator. A simple way to evaluate $a(t)$ goes through a differential equation. We take the time derivative of the above to find

$$\begin{aligned} \frac{d}{dt}\,\hat{a}(t) &= e^{i\omega t\hat{N}}(i\omega\hat{N})\,\hat{a}\,e^{-i\omega t\hat{N}} - e^{i\omega t\hat{N}}\,\hat{a}\,(i\omega\hat{N})e^{-i\omega t\hat{N}}\,, \\ &= i\omega\, e^{i\omega t\hat{N}}\,\big[\,\hat{N}\,,\hat{a}\,\big]e^{-i\omega t\hat{N}} = -i\omega\, e^{i\omega t\hat{N}}\;a\,e^{-i\omega t\hat{N}}\,. \end{aligned} \tag{3.51}$$

we recognize in final right-hand side the operator $a(t)$ so we have obtained the differential equation

$$\frac{d}{dt}\,\hat{a}(t) = -i\omega t\,\hat{a}(t)\,. \tag{3.52}$$

Since $\hat{a}(t=0) = \hat{a}$, the solution is

$$\hat{a}(t) = e^{-i\omega t}\,\hat{a}\,. \tag{3.53}$$

Together with the adjoint of this formula we have

$$\boxed{\begin{aligned} \hat{a}(t) &= e^{-i\omega t}\,\hat{a}\,. \\ \hat{a}^\dagger(t) &= e^{i\omega t}\;\hat{a}^\dagger\,. \end{aligned}} \tag{3.54}$$

The two equations above are our answer. As a check we consider the operator equation

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger)\,, \tag{3.55}$$

whose Heisenberg version is

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}(t) + \hat{a}^\dagger(t)) = \sqrt{\frac{\hbar}{2m\omega}}\,(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger)\,. \tag{3.56}$$

Expanding the exponentials, we recognize,

$$\begin{aligned} \hat{x}_H(t) &= \sqrt{\frac{\hbar}{2m\omega}}\Big((\hat{a} + \hat{a}^\dagger)\cos\omega t + i(\hat{a}^\dagger - \hat{a})\sin\omega t\Big)\,, \\ &= \hat{x}\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\,, \end{aligned} \tag{3.57}$$

in agreement with (3.44).

4 Coherent states of the Harmonic oscillator

Coherent states are quantum states that exhibit some sort of classical behavior. We will introduce them and explore their properties. To begin our discussion we introduce translation operators.

4.1 Translation operator

Let us construct unitary *translation* operators T_{x_0} that acting on states moves them (or translates them) by a distance x_0, where x_0 is a real constant with units of length:

$$\boxed{\text{Translation operator:} \quad T_{x_0} \;\equiv\; e^{-\frac{i}{\hbar}\hat{p}\,x_0}\,.} \tag{4.1}$$

This operator is unitary because it is the exponential of an antihermitian operator ($\hat{p}$ is hermitian, and $i\hat{p}$ antihermitian). The multiplication of two such operators is simple:

$$T_{x_0}T_{y_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,x_0}e^{-\frac{i}{\hbar}\hat{p}\,y_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,(x_0+y_0)}\,, \tag{4.2}$$

since the exponents commute ($e^Ae^B = e^{A+B}$ if $[A,B]=0$). As a result

$$T_{x_0}T_{y_0} \;=\; T_{x_0+y_0}\,. \tag{4.3}$$

The translation operators form a group: the product of two translation is a translation. There is a unit element $T_0 = I$ corresponding to $x_0 = 0$, and each element T_{x_0} has an inverse T_{-x_0}. Note that the group multiplication rule is commutative.

It follows from the explicit definition of the translation operator that

$$(T_{x_0})^\dagger \;=\; e^{\frac{i}{\hbar}\hat{p}\,x_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,(-x_0)} \;=\; T_{-x_0} \;=\; (T_{x_0})^{-1}\,. \tag{4.4}$$

confirming again that the operator is unitary. In the following we denote $(T_{x_0})^\dagger$ simply by $T^\dagger_{x_0}$.

We say that T_{x_0} translates by x_0 because of its action[2] on the operator $\hat{x}$ is as follows:

$$T^\dagger_{x_0}\,\hat{x}\,T_{x_0} \;=\; e^{\frac{i}{\hbar}\hat{p}\,x_0}\,\hat{x}\,e^{-\frac{i}{\hbar}\hat{p}\,x_0} \;=\; \hat{x} + \frac{i}{\hbar}\,[\hat{p},\hat{x}]x_0 \;=\; \hat{x} + x_0\,, \tag{4.5}$$

where we used the formula $e^ABe^{-A} = B + [A,B] + \ldots$ and the dots vanish in this case because $[A,B]$ is a number (check that you understand this!).

To see physically why the above is consistent with intuition, consider a state $|\psi\rangle$ and the expectation value of $\hat{x}$ on this state

$$\langle\hat{x}\rangle_\psi \;=\; \langle\psi|\,\hat{x}\,|\psi\rangle \tag{4.6}$$

[2]The action of a unitary operator $\mathcal{U}$ on an operator $\mathcal{O}$ is defined as $\mathcal{O} \to \mathcal{U}^\dagger\mathcal{O}\mathcal{U}$.

Now we ask: What is the expectation value of $\hat{x}$ on the state $T_{x_0}|\psi\rangle$? We find

$$\langle \hat{x} \rangle_{T_{x_0}\psi} \;=\; \langle \psi | T^\dagger_{x_0}\, \hat{x}\, T_{x_0} | \psi \rangle \tag{4.7}$$

The right-hand side explains why $T^\dagger_{x_0}\, \hat{x}\, T_{x_0}$ is the natural thing to compute! Indeed using our result for this

$$\langle \hat{x} \rangle_{T_{x_0}\psi} \;=\; \langle \psi | (\hat{x} + x_0) | \psi \rangle \;=\; \langle \hat{x} \rangle_{\psi} + x_0\,. \tag{4.8}$$

The expectation value of $\hat{x}$ on the displaced state is indeed equal to the expectation value of $\hat{x}$ in the original state plus x_0, confirming that *we should view $T_{x_0}|\psi\rangle$ as the state $|\psi\rangle$ displaced a distance x_0.*

As an example we look at position states. We claim that on position states the translation operator does what we expect:

$$\boxed{\; T_{x_0} | x_1 \rangle \;=\; | x_1 + x_0 \rangle\,. \;} \tag{4.9}$$

We can prove (4.9) by acting on the above left-hand side an arbitrary momentum bra $\langle p|$:

$$\langle p | T_{x_0} | x_1 \rangle \;=\; \langle p | e^{-\frac{i}{\hbar}\hat{p}x_0} | x_1 \rangle \;=\; e^{-\frac{i}{\hbar}\,p x_0} \frac{e^{-\frac{i}{\hbar}p x_1}}{\sqrt{2\pi\hbar}} \;=\; \langle p | x_1 + x_0 \rangle\,, \tag{4.10}$$

proving the desired result, given that $\langle p|$ is arbitrary. It also follows from unitarity and (4.9) that

$$T^\dagger_{x_0} | x_1 \rangle \;=\; T_{-x_0} | x_1 \rangle \;=\; | x_1 - x_0 \rangle\,. \tag{4.11}$$

Taking the Hermitian conjugate we find

$$\boxed{\; \langle x_1 | T_{x_0} \;=\; \langle x_1 - x_0 |\,. \;} \tag{4.12}$$

In terms of arbitrary states $|\psi\rangle$, we can also discuss the action of the translation operator by introducing the wavefunction $\psi(x) = \langle x|\psi\rangle$. Then the "translated" state $T_{x_0}|\psi\rangle$ has a wavefunction

$$\langle x | T_{x_0} | \psi \rangle \;=\; \langle x - x_0 | \psi \rangle \;=\; \psi(x - x_0)\,. \tag{4.13}$$

Indeed, $\psi(x - x_0)$ is the function $\psi(x)$ translated by the distance $+x_0$. For example, the value that $\psi(x)$ takes at $x = 0$ is taken by the function $\psi(x - x_0)$ at $x = x_0$.

4.2 Definition and basic properties of coherent states

We now finally introduce a coherent state $|\tilde{x}_0\rangle$ of the simple harmonic oscillator. The state is labeled by x_0 and the tilde is there to remind you that it is *not* a position state.[3] Here is the definition

$$\boxed{\text{Coherent state:}\quad |\tilde{x}_0\rangle \;\equiv\; T_{x_0}|0\rangle \;= e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle\,,} \tag{4.14}$$

where $|0\rangle$ denotes the ground state of the oscillator. Do not confuse the coherent state with a position state. The coherent state is simply the translation of the ground state by a distance x_0. This state has no time dependence displayed, so it may be thought as the state of the system at $t = 0$. As t increases the state will evolve according to the Schrödinger equation, and we will be interested in this evolution, but not now. Note that the coherent state is well normalized

$$\langle\tilde{x}_0|\tilde{x}_0\rangle \;= \langle 0|T^\dagger_{x_0}T_{x_0}|0\rangle \;=\; \langle 0|0\rangle \;= 1\,. \tag{4.15}$$

This had to be so because T_{x_0} is unitary.

To begin with let us calculate the wavefunction associated to the state:

$$\psi_{x_0}(x) \;\equiv\; \langle x|\tilde{x}_0\rangle \;=\; \langle x|T_{x_0}|0\rangle \;=\; \langle x - x_0|0\rangle \;=\; \psi_0(x - x_0)\,, \tag{4.16}$$

where we used (4.12) and we denoted $\langle x|0\rangle = \psi_0(x)$, as the ground state wavefunction. So, as expected the wavefunction for the coherent state is just the ground state wavefunction displaced x_0 to the right. This is illustrated in Figure 2.

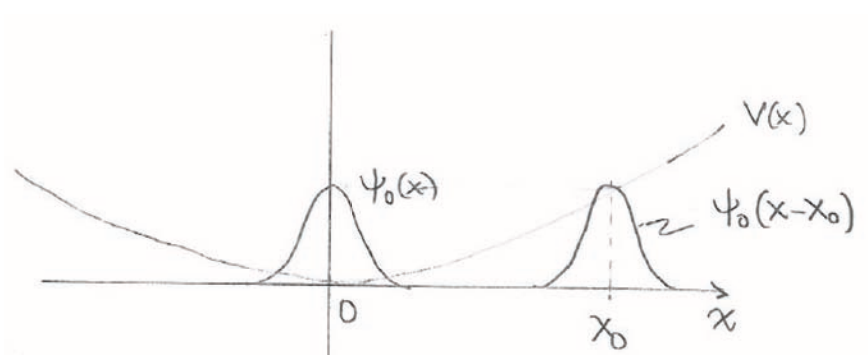

Figure 2: The ground state wavefunction $\psi_0(x)$ displaced to the right a distance x_0 is the wavefunction $\psi_0(x - x_0)$. The corresponding state, denoted as $|\tilde{x}_0\rangle$, is the simplest example of a coherent state.

Let us now do a few sample calculations to understand better these states.

[3]This is not great notation, but it is better than any alternative I have seen!

1. Calculate the expectation value of $\hat{x}$ in a coherent state.

$$\langle\tilde{x}_0|\hat{x}\,|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\,\hat{x}\,T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x}+x_0)|0\rangle\,, \tag{4.17}$$

where we used (4.5). Recalling now that $\langle 0|\hat{x}|0\rangle = 0$ we get

$$\langle\tilde{x}_0|\hat{x}\,|\tilde{x}_0\rangle \;=\; x_0\,. \tag{4.18}$$

Not that surprising! The position is essentially x_0.

2. Calculate the expectation value of $\hat{p}$ in a coherent state. Since $\hat{p}$ commutes with T_{x_0} we have

$$\langle\tilde{x}_0|\hat{p}\,|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\,\hat{p}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,T^\dagger_{x_0}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,|0\rangle \;=\; 0\,, \tag{4.19}$$

The coherent state has no (initial) momentum. It has an initial position (as seen in 1. above)

3. Calculate the expectation value of the energy in a coherent state. Note that the coherent state is not an energy eigenstate (nor a position eigenstate, nor a momentum eigenstate!). With H the Hamiltonian we have

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \langle 0|T^\dagger_{x_0}HT_{x_0}|0\rangle\,. \tag{4.20}$$

We now compute

$$\begin{aligned} T^\dagger_{x_0}HT_{x_0} \;&=\; T^\dagger_{x_0}\Big(\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\Big)T_{x_0} \;=\; \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2(\hat{x}+x_0)^2 \\ &=\; H + m\omega^2 x_0\hat{x} + \frac{1}{2}m\omega^2 x_0^2\,. \end{aligned} \tag{4.21}$$

where we recall that T_{x_0} commutes with $\hat{p}$ and used eqn. (4.5). Back in (4.20) we have

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \langle 0|H|0\rangle + m\omega^2 x_0\langle 0|\hat{x}|0\rangle + \frac{1}{2}m\omega^2 x_0^2\,. \tag{4.22}$$

Recalling that the ground state energy is $\hbar\omega/2$ and that in the ground state $\hat{x}$ has no expectation value we finally get

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2 x_0^2\,. \tag{4.23}$$

This is reasonable: the total energy is the zero-point energy plus the potential energy of a particle at x_0. The coherent state $|\tilde{x}_0\rangle$ is the quantum version of a point particle on a spring held stretched to $x = x_0$.

4.3 Time evolution and uncertainties

Evolving the coherent states in time is a somewhat involved procedure that will be explained later. We can discuss time evolution quite easily using the Heisenberg picture, since we have already calculated in (3.44) the time-dependent Heisenberg operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$.

If we have at time equal zero the coherent state $|\tilde{x}_0\rangle$ then at time t we write the time-evolved state as $|\tilde{x}_0, t\rangle$. We now ask what is the (time-dependent) expectation value of $\hat{x}$ on this state:

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0, t|\, \hat{x}\, |\tilde{x}_0, t\rangle \;=\; \langle \tilde{x}_0|\, \hat{x}_H(t)\, |\tilde{x}_0\rangle\,. \tag{4.24}$$

Using (3.44) we get

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0| \Big(\hat{x}\, \cos\omega t + \frac{1}{m\omega}\, \hat{p}\, \sin\omega t\Big)\, |\tilde{x}_0\rangle\,. \tag{4.25}$$

Finally, using (4.18) and (4.19) we get

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{x}_H(t)\, |\tilde{x}_0\rangle \;=\; x_0 \cos\omega t\,. \tag{4.26}$$

The expectation value of $\hat{x}$ is performing oscillatory motion! This confirms the classical interpretation of the coherent state. For the momentum the calculation is quite similar,

$$\langle\, \hat{p}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{p}_H(t)\, |\tilde{x}_0\rangle \;=\; \langle \tilde{x}_0| \Big(\hat{p} \cos\omega t - m\omega\, \hat{x}\, \sin\omega t\Big)\, |\tilde{x}_0\rangle \tag{4.27}$$

and we thus find

$$\langle\, \hat{p}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{p}_H(t)\, |\tilde{x}_0\rangle \;=\; -m\omega\, x_0\, \sin\omega t\,, \tag{4.28}$$

which is the expected result as it is equal to $m\frac{d}{dt}\langle \hat{x}\rangle(t)$.

We have seen that the harmonic oscillator ground state is a minimum uncertainty state. We will now discuss the extension of this fact to coherent states. We begin by calculating the uncertainties Δx and Δp in a coherent state at $t = 0$. We will see that the coherent state has minimum uncertainty for the product. Then we will calculate uncertainties of the coherent state as a function of time!

We have

$$\langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\hat{x}^2 T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x} + x_0)^2|0\rangle \;=\; \langle 0|\hat{x}^2|0\rangle + x_0^2\,. \tag{4.29}$$

The first term on the right-hand side was calculated in (1.58). We thus find

$$\langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle \;=\; \frac{\hbar}{2m\omega} + x_0^2\,. \tag{4.30}$$

Since $\langle \tilde{x}_0|\hat{x}|\tilde{x}_0\rangle = x_0$ we find the uncertainty

$$(\Delta x)^2 \;=\; \langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle - (\langle \tilde{x}_0|\hat{x}|\tilde{x}_0\rangle)^2 = \frac{\hbar}{2m\omega} + x_0^2 - x_0^2$$

$$\rightarrow \quad (\Delta x)^2 \;=\; \frac{\hbar}{2m\omega}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.31}$$

For the momentum operator we have, using (1.58),

$$\langle\tilde{x}_0|\hat{p}^2|\tilde{x}_0\rangle \;=\; \langle 0|T_{x_0}^\dagger \hat{p}^2 T_{x_0}|0\rangle \;=\; \langle 0|\hat{p}^2|0\rangle \;=\; \frac{m\hbar\omega}{2}\,. \tag{4.32}$$

Since $\langle\tilde{x}_0|\hat{p}|\tilde{x}_0\rangle = 0$, we have

$$(\Delta p)^2 \;=\; \frac{m\hbar\omega}{2}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.33}$$

As a result,

$$\Delta x \Delta p \;=\; \frac{\hbar}{2}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.34}$$

We see that the coherent state has minimum $\Delta x \Delta p$ at time equal zero. This is not surprising because at this time the state is just a displaced ground state.

For the time dependent situation we have

$$\begin{aligned}(\Delta x)^2(t) \;&=\; \langle\tilde{x}_0, t|\hat{x}^2|\tilde{x}_0, t\rangle - \big(\langle\tilde{x}_0, t|\hat{x}|\tilde{x}_0, t\rangle\big)^2 \\ &=\; \langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle - \big(\langle\tilde{x}_0|\hat{x}_H(t)|\tilde{x}_0\rangle\big)^2 \\ &=\; \langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle - x_0^2\cos^2\omega t\,,\end{aligned} \tag{4.35}$$

where we used the result in (4.26). The computation of the first term takes a few steps:

$$\begin{aligned}\langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle \;&=\; \langle\tilde{x}_0|\Big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\Big)^2|\tilde{x}_0\rangle \\ &=\; \langle\tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle\cos^2\omega t + \langle\tilde{x}_0|\hat{p}^2|\tilde{x}_0\rangle\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos m\omega\sin m\omega}{m\omega}\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle \\ &=\; \Big(\frac{\hbar}{2m\omega} + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos m\omega\sin m\omega}{m\omega}\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle\,.\end{aligned}$$

We now show that the last expectation value vanishes:

$$\begin{aligned}\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle \;&=\; \langle 0|\big((\hat{x}+x_0)\hat{p} + \hat{p}(\hat{x}+x_0)\big)|0\rangle \\ &=\; \langle 0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|0\rangle \\ &=\; i\frac{\hbar}{2}\,\langle 0|\big((\hat{a}+\hat{a}^\dagger)(\hat{a}^\dagger-\hat{a}) + (\hat{a}^\dagger-\hat{a})(\hat{a}+\hat{a}^\dagger)\big)|0\rangle \\ &=\; i\frac{\hbar}{2}\,\langle 0|\big(\hat{a}\hat{a}^\dagger + (-\hat{a})\hat{a}^\dagger\big)|0\rangle \;=\; 0\,.\end{aligned} \tag{4.36}$$

As a result,

$$\begin{aligned}\langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle \;&=\; \Big(\frac{\hbar}{2m\omega} + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 \\ &=\; \frac{\hbar}{2m\omega} + x_0^2\cos^2\omega t\,.\end{aligned} \tag{4.37}$$

Therefore, finally, back in (4.35) we get

$$(\Delta x)^2(t) \;=\; \frac{\hbar}{2m\omega}\,. \tag{4.38}$$

The uncertainty Δx does not change in time as the state evolves! This suggests, but does not yet prove, that the state does not change shape[4]. It is therefore useful to calculate the time-dependent uncertainty in the momentum:

$$\begin{aligned}(\Delta p)^2(t) \;&=\; \langle \tilde{x}_0, t|\hat{p}^2|\tilde{x}_0, t\rangle - \big(\langle \tilde{x}_0, t|\hat{p}|\tilde{x}_0, t\rangle\big)^2\\ &=\; \langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle - \big(\langle \tilde{x}_0|\hat{p}_H(t)|\tilde{x}_0\rangle\big)^2\\ &=\; \langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle - m^2\omega^2\, x_0^2 \sin^2\omega t\,,\end{aligned} \tag{4.39}$$

where we used (4.28). The rest of the computation (recommended!) gives

$$\langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle \;=\; \frac{1}{2}\, m\hbar\omega \;+\; m^2\omega^2\, x_0^2 \sin^2\omega t\,, \tag{4.40}$$

so that we have

$$(\Delta p)^2(t) \;=\; \frac{m\hbar\omega}{2}\,. \tag{4.41}$$

This together with (4.38) gives

$$\Delta x(t)\Delta p(t) \;=\; \frac{\hbar}{2}\,, \qquad \text{on the state } |\tilde{x}_0, t\rangle\,. \tag{4.42}$$

The coherent state remains a minimum $\Delta x \Delta p$ packet for all times. Since only gaussians have such minimum uncertainty, the state remains a gaussian for all times! Since Δx is constant the gaussian does not change shape. Thus the name *coherent state*, the state does not spread out in time, it just moves "coherently" without changing its shape.

In the harmonic oscillator there is a quantum length scale d that can be constructed from $\hbar, m$, and ω. This length scale appears, for example, in the uncertainty Δx in (4.38). We thus define

$$d \;\equiv\; \sqrt{\frac{\hbar}{m\omega}}\,, \tag{4.43}$$

and note that

$$\Delta x(t) \;=\; \frac{d}{\sqrt{2}}\,. \tag{4.44}$$

The length d is typically very small for a macroscopic oscillator. A coherent state with a large x_0 –large compared to d– is classical in the sense that the position uncertainty $\sim d$, is much smaller than the typical excursion x_0. Similarly, the momentum uncertainty

$$\Delta p(t) \;=\; m\omega\,\frac{d}{\sqrt{2}}\,. \tag{4.45}$$

[4]By this we mean that the shape of $|\psi(x,t)|^2$ does not change: at different times $|\psi(x,t)|^2$ and $|\psi(x,t')|^2$ differ just by an overall displacement in x.

is much smaller than the typical momentum $m\omega x_0$, by just the same factor $\sim d/x_0$.

Problem. Prove that

$$\frac{\Delta p(t)}{\sqrt{\overline{\langle \hat{p}^2\rangle(t)}}} = \frac{\Delta x(t)}{\sqrt{\overline{\langle \hat{x}^2\rangle(t)}}} = \frac{1}{\sqrt{1+\frac{x_0^2}{d^2}}} \tag{4.46}$$

where the overlines on the expectation values denote time average.

4.4 Coherent states in the energy basis

We can get an interesting expression for the coherent state $|\tilde{x}_0\rangle$ by rewriting the momentum operator in terms of creation and annihilation operators. From (1.18) we have

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(\hat{a}^\dagger - \hat{a}) = i\,\frac{\hbar}{\sqrt{2}\,d}\,(\hat{a}^\dagger - \hat{a})\,. \tag{4.47}$$

The final form is also nice to see that units work. We now have that the coherent state (4.14) is given by

$$|\tilde{x}_0\rangle = \exp\Bigl(-\frac{i}{\hbar}\,\hat{p}\,x_0\Bigr)|0\rangle = \exp\Bigl(\frac{x_0}{\sqrt{2}d}\,(\hat{a}^\dagger - \hat{a})\Bigr)|0\rangle\,. \tag{4.48}$$

Since $\hat{a}|0\rangle = 0$ the above formula admits simplification: we should be able to get rid of all the $\hat{a}$'s! We could do this if we could split the exponential into two exponentials, one with the $\hat{a}^\dagger$'s to the *left* of another one with the $\hat{a}$'s. The exponential with the $\hat{a}$'s would stand near the vacuum and give no contribution, as we will see below. For this purpose we recall the commutator identity

$$e^{X+Y} = e^X\,e^Y\,e^{-\frac{1}{2}[X,Y]}\,, \quad \text{if } [X,Y] \text{ commutes with } X \text{ and with } Y. \tag{4.49}$$

Think of the term we are interested in as it appears in (4.48), and identify X and Y as

$$\exp\Bigl(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Bigr) \quad\to\quad X = \frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\,,\quad Y = -\frac{x_0}{\sqrt{2}d}\hat{a} \tag{4.50}$$

Then

$$[X,Y] = -\frac{x_0^2}{2d^2}\,[\,\hat{a}^\dagger\,,\hat{a}] = \frac{x_0^2}{2d^2} \tag{4.51}$$

and we find

$$\exp\Bigl(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Bigr) = \exp\Bigl(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\Bigr)\,\exp\Bigl(-\frac{x_0}{\sqrt{2}d}\hat{a}\Bigr)\,\exp\Bigl(-\frac{1}{4}\frac{x_0^2}{d^2}\Bigr) \tag{4.52}$$

Since the last exponential is just a number, and $\exp(\gamma\hat{a})|0\rangle = |0\rangle$, for any γ, we have

$$\exp\Bigl(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Bigr)|0\rangle = \exp\Bigl(-\frac{1}{4}\frac{x_0^2}{d^2}\Bigr)\exp\Bigl(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\Bigr)|0\rangle\,. \tag{4.53}$$

As a result, our coherent state in (4.48) becomes

$$|\tilde{x}_0\rangle \;=\; \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle \;=\; \exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\Big)|0\rangle\,. \tag{4.54}$$

While this form is quite nice to produce an expansion in energy eigenstates, the unit normalization of the state is not manifest anymore. Expanding the exponential with creation operators we get

$$\begin{aligned}|\tilde{x}_0\rangle \;&=\; \sum_{n=0}^{\infty}\exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{n!}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n(\hat{a}^\dagger)^n|0\rangle \\ &=\; \sum_{n=0}^{\infty}\exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n|n\rangle\end{aligned} \tag{4.55}$$

We thus have the desired expansion:

$$|\tilde{x}_0\rangle \;=\; \sum_{n=0}^{\infty}c_n|n\rangle\,, \quad \text{with}\;\; c_n \;=\; \exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n. \tag{4.56}$$

Since the probability to find the energy E_n is equal to c_n^2, we note that

$$c_n^2 \;=\; \exp\Big(-\frac{x_0^2}{2d^2}\Big)\cdot\frac{1}{n!}\Big(\frac{x_0^2}{2d^2}\Big)^n \tag{4.57}$$

If we define the quantity $\lambda(x_0, d)$ as

$$\lambda \;\equiv\; \frac{x_0^2}{2d^2}\,, \tag{4.58}$$

we can then see that

$$c_n^2 \;=\; \frac{\lambda^n}{n!}\,e^{-\lambda}\,. \tag{4.59}$$

The probability to measure an energy $E_n = \hbar\omega(n+\frac{1}{2})$ in the coherent state is c_n^2, so the c_n^2's must define a probability distribution for $n \in \mathbb{Z}$, parameterized by λ. This is in fact the familiar *Poisson distribution*. It is straightforward to verify that

$$\sum_{n=0}^{\infty}c_n^2 \;=\; e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} \;=\; e^{-\lambda}e^{\lambda} \;=\; 1\,, \tag{4.60}$$

as it should be. The physical interpretation of λ can be obtained by computing the expectation value of n:[5]

$$\langle n\rangle \;\equiv\; \sum_{n=0}^{\infty} n c_n^2 \;=\; e^{-\lambda}\sum_{n=0}^{\infty} n\,\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\sum_{n=0}^{\infty}\lambda\frac{d}{d\lambda}\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\,\lambda\frac{d}{d\lambda}e^{\lambda} \;=\lambda\,. \tag{4.61}$$

Therefore λ equals the expected value $\langle n\rangle$. Note that $\langle n\rangle$ is just the expected value of the number operator $\hat{N}$ on the coherent state. Indeed,

$$\langle\tilde{x}_0|\hat{N}|\tilde{x}_0\rangle \;=\; \sum_{n,m} c_m c_n\langle m|\hat{N}|n\rangle \;=\; \sum_{n,m} c_m c_n\, n\delta_{m,n} \;=\; \sum_{n} n\, c_n^2 \;=\; \langle n\rangle\,. \tag{4.62}$$

It is also easy to verify (do it!) that

$$\langle n^2\rangle \;\equiv\; \sum_{n=0}^{\infty} n^2\, c_n^2 \;=\; \lambda^2+\lambda\,. \tag{4.63}$$

It then follows that

$$(\Delta n)^2 \;=\; \langle n^2\rangle - \langle n\rangle^2 \;=\; \lambda \qquad\rightarrow\qquad \Delta n \;=\; \sqrt{\lambda}\,. \tag{4.64}$$

In terms of energy we have $E=\hbar\omega(n+\frac{1}{2})$ so that

$$\langle E\rangle \;=\; \hbar\omega\Big(\langle n\rangle+\frac{1}{2}\Big) \;=\; \hbar\omega\Big(\lambda+\frac{1}{2}\Big)\,. \tag{4.65}$$

Similarly,

$$\langle E^2\rangle \;=\; \hbar^2\omega^2\Big\langle\Big(n+\frac{1}{2}\Big)^2\Big\rangle \;=\; \hbar^2\omega^2\Big\langle n^2+n+\frac{1}{4}\Big\rangle \;=\; \hbar^2\omega^2\Big(\lambda^2+\lambda+\lambda+\frac{1}{4}\Big)\,, \tag{4.66}$$

so that

$$\langle E^2\rangle \;=\; \hbar^2\omega^2\Big(\lambda^2+2\lambda+\frac{1}{4}\Big) \tag{4.67}$$

The energy uncertainty is thus obtained as

$$(\Delta E)^2 \;=\; \langle E^2\rangle-\langle E\rangle^2 \;=\; \hbar^2\omega^2\Big(\lambda^2+2\lambda+\frac{1}{4}-\Big(\lambda+\frac{1}{2}\Big)^2\Big) \;=\; \hbar^2\omega^2\,\lambda\,, \tag{4.68}$$

so that

$$\Delta E \;=\; \hbar\omega\sqrt{\lambda} \;=\; \hbar\omega\,\frac{x_0}{\sqrt{2}d}\,. \tag{4.69}$$

Note now the fundamental inequality, holding for $x_0/d \gg 1$,

$$\hbar\omega \quad\ll\quad \Delta E=\hbar\omega\,\frac{x_0}{\sqrt{2}d} \quad\ll\quad \langle\, E\,\rangle=\hbar\omega\Big(\frac{x_0}{\sqrt{2}d}\Big)^2+\frac{1}{2}\hbar\omega\,. \tag{4.70}$$

[5]Here we are thinking of n as a random variable of the probability distribution. In the quantum viewpoint $\langle n\rangle$ is simply the expectation value of the number operator.

We see that the uncertainty ΔE is big enough to contain about $\frac{x_0}{\sqrt{2}d}$ levels of the harmonic oscillator –a lot of levels. But even then, ΔE is about a factor $\frac{x_0}{\sqrt{2}d}$ smaller than the expected value $E\rangle$ of the energy. So, alternatively,

$$\frac{\Delta E}{\hbar\omega} \simeq \frac{x_0}{\sqrt{2}d} \simeq \frac{\langle E\rangle}{\Delta E}. \tag{4.71}$$

This is part of the semi-classical nature of coherent states.

Example of Poisson distribution. Consider a sample of radioactive material with $N_0 \gg 1$ atoms at $t = 0$. Assume that the half-lifetime of the material is τ_0, which means that the number $N(t)$ of atoms that have not yet decayed after time $t > 0$ is given by

$$N(t) = N_0 \exp(-t/\tau_0) \quad \rightarrow \quad \frac{dN}{dt}(t=0) = \frac{N_0}{\tau_0}.$$

It follows that in the time interval $t \in [0, \Delta t]$, with $\Delta t \ll \tau_0$ we expect a number of decays

$$\frac{N_0 \Delta t}{\tau_0} \equiv \lambda.$$

One can then show that the probability p_n to observe n decays during that same time interval Δt is (approximately) given by the Poisson distribution: $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$.

4.5 General coherent states and time evolution

We wrote earlier coherent states using creation and annihilation operators:

$$|\tilde{x}_0\rangle = \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle = \exp\Big(\frac{x_0}{\sqrt{2}d}\,(\hat{a}^\dagger - \hat{a})\Big)|0\rangle. \tag{4.72}$$

Such coherent states can be written as

$$|\alpha\rangle \equiv e^{\alpha\,(\hat{a}^\dagger - \hat{a})}|0\rangle, \quad \text{with} \quad \alpha = \frac{x_0}{\sqrt{2}d}. \tag{4.73}$$

This notation is not free of ambiguity: the label α in the coherent state above is now the coefficient of the factor $\hat{a}^\dagger - \hat{a}$ in the exponential. An obvious generalization is to take α to be a complex number: $\alpha \in \mathbb{C}$. This must be done with a little care, since the key property of the operator in the exponential (4.73) is that it is antihermitian (thus the exponential is unitary, as desired). We thus define

$$\boxed{|\alpha\rangle \equiv D(\alpha)|0\rangle \equiv \exp\big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\big)|0\rangle, \quad \text{with} \quad \alpha \in \mathbb{C}.} \tag{4.74}$$

In this definition we introduced the unitary 'displacement' operator

$$D(\alpha) \equiv \exp\Big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\Big)\,. \tag{4.75}$$

Since $D(\alpha)$ is unitary it is clear that $\langle\alpha|\alpha\rangle = 1$.

The action of the annihilation operator on the states $|\alpha\rangle$ is quite interesting,

$$\begin{aligned} \hat{a}|\alpha\rangle &= \hat{a}\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \big[\,\hat{a}\,,\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\big]|0\rangle \\ &= \big[\,\hat{a}\,,\, \alpha\hat{a}^\dagger - \alpha^*\hat{a}\,\big]e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \alpha e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle\,, \end{aligned} \tag{4.76}$$

so that we conclude that

$$\boxed{\hat{a}\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle\,.} \tag{4.77}$$

This result is kind of shocking: we have found eigenstates of the *non-hermitian* operator $\hat{a}$. Because $\hat{a}$ is not hermitian, our theorems about eigenstates and eigenvectors of hermitian operators do not apply. Thus, for example, the eigenvalues need not be real (they are not, in general $\alpha \in \mathbb{C}$), eigenvectors of different eigenvalue need not be orthogonal (they are not!) and the set of eigenvectors need not form a complete basis (coherent states actually give an overcomplete basis!).

Ordering the exponential in the state $|\alpha\rangle$ in (4.74) we find

$$|\alpha\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle\,. \tag{4.78}$$

Exercise. Show that

$$\langle\beta|\alpha\rangle \;=\; \exp\Big(-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \beta^*\alpha\Big)\,. \tag{4.79}$$

Hint: You may find it helpful to evaluate $e^{\beta^*\hat{a}+\alpha\hat{a}^\dagger}$ in two different ways using (4.49).

To find the physical interpretation of the complex number α we first note that when real, as in (4.73), α encodes the initial position x_0 of the coherent state (more precisely, it encodes the expectation value of $\hat{x}$ in the state at $t = 0$). For complex α, its real part is still related to the initial position:

$$\langle\alpha|\hat{x}|\alpha\rangle \;=\; \frac{d}{\sqrt{2}}\,\langle\alpha|(\hat{a}+\hat{a}^\dagger)|\alpha\rangle \;=\; \frac{d}{\sqrt{2}}(\alpha+\alpha^*) \;=\; d\sqrt{2}\,\mathrm{Re}(\alpha)\,, \tag{4.80}$$

where we used (1.18) and (4.77) both on bras and on kets. We have thus learned that

$$\mathrm{Re}(\alpha) \;=\; \frac{\langle\hat{x}\rangle}{\sqrt{2}\,d}\,. \tag{4.81}$$

It is natural to conjecture that the imaginary part of α is related to the momentum expectation value on the initial state. So we explore

$$\langle\alpha|\hat{p}|\alpha\rangle \;=\; \frac{i\hbar}{\sqrt{2}d}\,\langle\alpha|(\hat{a}^\dagger-\hat{a})|\alpha\rangle \;=\; -\frac{i\hbar}{\sqrt{2}d}(\alpha-\alpha^*) \;=\; -\frac{i\hbar}{\sqrt{2}d}(2i\mathrm{Im}(\alpha)) \;=\; \frac{\hbar\sqrt{2}}{d}\,\mathrm{Im}(\alpha)\,, \tag{4.82}$$

and learn that

$$\text{Im}(\alpha) \;=\; \frac{\langle \hat{p}\rangle\, d}{\sqrt{2}\,\hbar}\,. \tag{4.83}$$

The identification of α in terms of expectation values of $\hat{x}$ and $\hat{p}$ is now complete:

$$\boxed{\alpha \;=\; \frac{\langle \hat{x}\rangle}{\sqrt{2}\,d} \;+\; i\,\frac{\langle \hat{p}\rangle\, d}{\sqrt{2}\,\hbar}\,.} \tag{4.84}$$

A calculation in the problem set shows that

$$\alpha\, \hat{a}^\dagger - \alpha^* \hat{a} \;=\; -\frac{i}{\hbar}\big(\hat{p}\,\langle x\rangle - \langle \hat{p}\rangle\, \hat{x}\big)\,, \tag{4.85}$$

affording yet another rewriting of the general coherent state (4.74), valid when α is defined as in (4.84):

$$\boxed{|\alpha\rangle \;=\; \exp\Big(-\frac{i\hat{p}\,\langle x\rangle}{\hbar} + \frac{i\langle \hat{p}\rangle\, \hat{x}}{\hbar}\Big)|0\rangle\,.} \tag{4.86}$$

In order to find the time evolution of the coherent state we can use a trick from the Heisenberg picture. We have using (4.74)

$$|\alpha, t\rangle \;\equiv\; e^{-i\frac{Ht}{\hbar}}|\alpha\rangle \;=\; e^{-i\frac{Ht}{\hbar}}e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \Big(e^{-i\frac{Ht}{\hbar}}e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}e^{i\frac{Ht}{\hbar}}\Big)e^{-i\frac{Ht}{\hbar}}|0\rangle \tag{4.87}$$

For a time independent Hamiltonian (as that of the SHO) and a Schrödinger operator $\mathcal{O}$, we have

$$\mathcal{O}_H(t) \;=\; e^{iHt/\hbar}\,\mathcal{O}\,e^{-iHt/\hbar} \tag{4.88}$$

and therefore with the opposite signs for the exponentials we get

$$e^{-iHt/\hbar}\,\mathcal{O}\,e^{iHt/\hbar} \;=\; \mathcal{O}_H(-t)\,. \tag{4.89}$$

Such a relation is also valid for any function of an operator:

$$e^{-iHt/\hbar}\,F(\mathcal{O})\,e^{iHt/\hbar} \;=\; F(\mathcal{O}_H(-t))\,. \tag{4.90}$$

as you can convince yourself is the case whenever $F(x)$ has a good Taylor expansion in powers of x. It then follows that back in (4.87) we have

$$|\alpha, t\rangle \;=\; \exp\Big(\alpha\hat{a}^\dagger(-t) - \alpha^*\hat{a}(-t)\Big)e^{-i\omega t/2}|0\rangle\,. \tag{4.91}$$

Recalling ((3.53)) that $\hat{a}(t) \;=\; e^{-i\omega t}\,\hat{a}$, and thus $\hat{a}^\dagger(t) = e^{i\omega t}\,\hat{a}^\dagger$, we find

$$|\alpha, t\rangle \;=\; e^{-i\omega t/2}\exp\Big(\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{i\omega t}\hat{a}\Big)|0\rangle\,. \tag{4.92}$$

Looking at the exponential we see that it is in fact the displacement operator with $\alpha \to \alpha e^{-i\omega t}$. As a result we have shown that

$$\boxed{|\alpha, t\rangle \;=\; e^{-i\omega t/2} |e^{-i\omega t}\alpha\rangle \,.} \tag{4.93}$$

This is how a coherent state $|\alpha\rangle$ evolves in time: up to an irrelevant phase, the state remains a coherent state with a time-varying parameter $e^{-i\omega t}\alpha$. In the complex α plane the state is represented by a vector that rotates in the clockwise direction with angular velocity ω. The α plane can be viewed as having a real axis that gives $\langle x\rangle$ (up to a proportionality constant) and an imaginary axis that gives $\langle \hat{p}\rangle$ (up to a proportionality constant). It is a phase space and the evolution of any state is represented by a circle. This is illustrated in Figure 3.

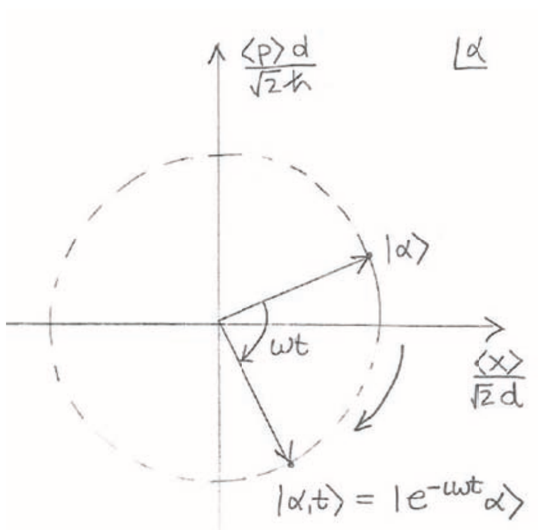

Figure 3: Time evolution of the coherent state $|\alpha\rangle$. The real and imaginary parts of α determine the expectation values $\langle x\rangle$ and $\langle p\rangle$ respectively. As time goes by the α parameter of the coherent state rotates clockwise with angular velocity ω.

An alternative, conventional, calculation of the time evolution begins by expanding the exponential in (4.78) to find:

$$|\alpha\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \alpha^n |n\rangle \,. \tag{4.94}$$

The time-evolved state is then given by the action of $\exp(-iHt/\hbar)$:

$$\begin{aligned} |\alpha,t\rangle \;\equiv\; e^{-i\frac{Ht}{\hbar}}|\alpha\rangle \;&=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,\alpha^n e^{-i\hbar\omega(n+\frac{1}{2})t/\hbar}|n\rangle \\ &=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,\alpha^n e^{-i\omega t n}e^{-i\omega t/2}|n\rangle \\ &=\; e^{-i\omega t/2}\,\underbrace{e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,(e^{-i\omega t}\alpha)^n|n\rangle}\;. \end{aligned} \tag{4.95}$$

Using (4.94) and noting that $|e^{-i\omega t}\alpha|^2 = |\alpha|^2$, we identify the terms under the brace as a coherent state $|\alpha e^{-i\omega t}\rangle$. This gives the earlier result (4.93).

In the coherent state $|\alpha\rangle$ the expectation value of $\hat{N}$ is easily calculated

$$\langle\hat{N}\rangle_\alpha \;=\; \langle\alpha|\hat{N}|\alpha\rangle \;=\; \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; \langle\alpha|\alpha^*\alpha|\alpha\rangle = |\alpha|^2\,. \tag{4.96}$$

To find the uncertainty ΔN we also compute

$$\begin{aligned} \langle\hat{N}^2\rangle_\alpha \;&=\; \langle\alpha|\,\hat{a}^\dagger\hat{a}\;\hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; |\alpha|^2\langle\alpha|\,\hat{a}\;\hat{a}^\dagger|\alpha\rangle \\ &=\; |\alpha|^2\langle\alpha|\,[\hat{a}\,,\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; |\alpha|^2\big(1+|\alpha|^2\big)\,. \end{aligned} \tag{4.97}$$

From these results we get

$$(\Delta N)^2 \;=\; \langle\hat{N}^2\rangle_\alpha - \langle\hat{N}\rangle_\alpha^2 \;=\; |\alpha|^2 + |\alpha|^4 - |\alpha|^4 = |\alpha|^2 \tag{4.98}$$

so that

$$\boxed{\Delta N \;=\; |\alpha|\,.} \tag{4.99}$$

in Figure 3 the magnitude of the rotating phasor is ΔN and the square of the magnitude is the expectation value $\langle\hat{N}\rangle_\alpha$.

We will soon discuss electromagnetic fields and waves as coherent states of photons. For such waves a number/phase uncertainty exists. A rough argument goes as follows. For a wave with N photons with frequency ω, the energy is $E = N\hbar\omega$ and the phase ϕ of the wave goes like $\phi = \omega t$. It follows that $\Delta E \sim \Delta N\hbar\omega$ and $\Delta\phi = \omega\Delta t$ (with the admittedly ambiguous meaning of Δt). Therefore

$$\Delta E\Delta t \geq \frac{\hbar}{2} \quad\rightarrow\quad \Delta N\hbar\omega\,\frac{\Delta\phi}{\omega} \geq \frac{\hbar}{2} \quad\rightarrow\quad \Delta N\Delta\phi \geq \frac{1}{2}\,. \tag{4.100}$$

A better intuition for this result follows from our coherent state $|\alpha\rangle$ for which we know that $\Delta N = |\alpha|$. The position and momentum uncertainties are the same as for the ground state:

$$\Delta x \;=\; \frac{d}{\sqrt{2}}\,,\quad \Delta p \;=\; \frac{\hbar}{d\sqrt{2}} \tag{4.101}$$

When we measure x on the state $|\alpha\rangle$ we expect to get a good fraction of values in a range Δx about the expected value $\langle x\rangle$ of x. This is, of course, just a rough estimate.

$$\text{Representative range for measured } x \;=\; \left[\langle x\rangle - \tfrac{1}{2}\Delta x\,,\, \langle x\rangle + \tfrac{1}{2}\Delta x\right] \tag{4.102}$$

Dividing by $\sqrt{2}d$ we have

$$\text{Representative range for measured } \frac{x}{\sqrt{2}d} \;=\; \left[\frac{\langle x\rangle}{\sqrt{2}d} - \tfrac{1}{2}\,,\, \frac{\langle x\rangle}{\sqrt{2}d} + \tfrac{1}{2}\right] \tag{4.103}$$

It follows that the position measurements, indicated on the horizontal axis of Figure 3, spread over a representative range of width one. Similarly, for momentum we have

$$\text{Representative range for measured } p \;=\; \left[\langle p\rangle - \tfrac{1}{2}\Delta p\,,\, \langle p\rangle + \tfrac{1}{2}\Delta p\right] \tag{4.104}$$

Multiplying by $d/(\sqrt{2}\hbar)$, we have

$$\text{Representative range for measured } \frac{p\,d}{\sqrt{2}\hbar} \;=\; \left[\frac{\langle p\rangle\,d}{\sqrt{2}\hbar} - \tfrac{1}{2}\,,\, \frac{\langle p\rangle\,d}{\sqrt{2}\hbar} + \tfrac{1}{2}\right] \tag{4.105}$$

It follows that the momentum measurements, indicated on the vertical axis of Figure 3, spread

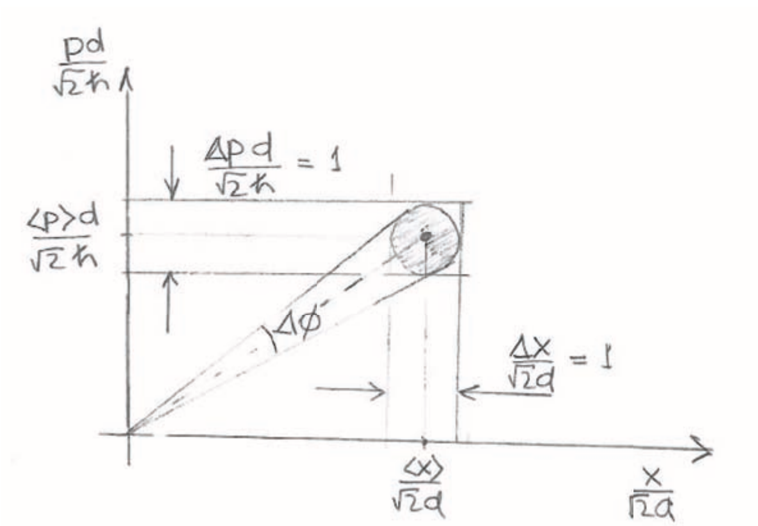

Figure 4: When doing measurements on $|\alpha\rangle$ the uncertainties on the value of α can be represented by a blob of unit diameter centered at α. The projections of this blob on the axes are, up to scale, the uncertainties Δx and Δp.

over a representative range of width one. We can thus reconsider the plot, this time indicating the ranges of values expected on the horizontal and vertical axes. Those ranges can be viewed as some kind of uncertainty in the value of α that we could find by measurements on the state

$|\alpha\rangle$. We draw a blob of unit diameter centered at α whose projections down along the axes reproduce the uncertainty ranges. This is shown in Figure 4. In the spirit of the discussion on time dependence, this blob must be imagined rotating with angular frequency ω. In such picture we have a phase ambiguity $\Delta\phi$, represented in the picture as the angle subtended by the uncertainty blob. Since the blob has diameter one and is centered at α, which is a distance $|\alpha|$ from the origin, we have

$$\Delta\phi \simeq \frac{1}{|\alpha|} \tag{4.106}$$

Recalling that $\Delta N = |\alpha|$ we finally obtain that for our coherent state

$$\Delta N \Delta\phi \simeq 1\,. \tag{4.107}$$

This is a familiar relation for coherent states of light. It then relates the uncertainty in the number of photons to the uncertainty in the phase of the wave.

5 Squeezed states

Squeezed states of the harmonic oscillator are states that are obtained by acting on the ground state with an exponential that includes terms quadratic in creation operators. They are the most general states for which $\Delta x\Delta p = \hbar/2$, thus achieving saturation of the uncertainty bound.

5.1 Squeezed vacuum states

One useful way to motivate the introduction of squeezed states is to consider the ground state of a harmonic oscillator Hamiltonian with mass and frequency parameters m_1 and ω_1, respectively:

$$H_1 = \frac{p^2}{2m_1} + \frac{1}{2}\, m_1\omega_1^2 x^2\,. \tag{5.1}$$

Such ground state has uncertaintites Δx and Δp that follow from (1.58) :

$$\begin{aligned} \Delta x &= \sqrt{\frac{\hbar}{2m_1\omega_1}}\,, \\ \Delta p &= \sqrt{\frac{\hbar m_1\omega_1}{2}}\,. \end{aligned} \tag{5.2}$$

Note that the product of uncertainties saturates the lower bound:

$$\Delta x\Delta p = \frac{\hbar}{2}\,. \tag{5.3}$$

Now we consider the following situation: suppose at time $t = 0^-$ the wavefunction is indeed that of the ground state of the oscillator. At $t = 0$, however, the oscillator parameters *change* *instantaneously* from (m_1, ω_1) to some (m_2, ω_2) that define a second, different Hamiltonian:

$$H_2 \;=\; \frac{p^2}{2m_2} + \frac{1}{2}\, m_2\omega_2^2 x^2\,. \tag{5.4}$$

During this change the wavefunction is assumed not to change, so at $t = 0^+$ the wavefunction is still the same – the ground state of H_1. Since the Hamiltonian changed, however, the state of the system is no longer an energy eigenstate: the gaussian wavefunction that is a ground state for H_1 is *not* a ground state of H_2. In fact it is not an energy eigenstate of H_2 and its time evolution will be nontrivial. We will see that the ground state of H_1 is indeed a squeezed state of H_2.

Since the wavefunction does not change, at $t = 0^+$ the uncertainties in (5.2) do not change, and we can rewrite

$$\begin{aligned}\Delta x \;&=\; \sqrt{\frac{m_2\omega_2}{m_1\omega_1}}\,\sqrt{\frac{\hbar}{2m_2\omega_2}} \;=\; e^{-\gamma}\sqrt{\frac{\hbar}{2m_2\omega_2}}\;,\\ \Delta p \;&=\; \sqrt{\frac{m_1\omega_1}{m_2\omega_2}}\,\sqrt{\frac{\hbar m_2\omega_2}{2}} \;=\; e^{\gamma}\,\sqrt{\frac{\hbar m_2\omega_2}{2}}\;,\end{aligned} \tag{5.5}$$

where we defined the real constant γ by

$$e^{\gamma} \;\equiv\; \sqrt{\frac{m_1\omega_1}{m_2\omega_2}}\;. \tag{5.6}$$

We learn from (5.5) that at $t = 0^+$ the uncertainties, from the viewpoint of the second Hamiltonian, have been squeezed from the values they would take on the H_2 ground state: if $\gamma > 0$, the position uncertainty is reduced and the momentum uncertainty increased. Of course, the product still saturates the bound.

To work out the details of the state at $t = 0^+$ we need to relate the creation and annihilation operators of the two Hamiltonians. We note that the operators x and p have not been changed, we are not speaking about two oscillating particles, but rather a single one, with coordinate measured by the operator x and momentum measured by the operator p. We thus use the expression for x and p in terms of $a, a^\dagger$ (equation (1.18) to write

$$\begin{aligned}x \;&=\; \sqrt{\frac{\hbar}{2m_1\omega_1}}\;(\hat a_1 + \hat a_1^\dagger) \;=\; \sqrt{\frac{\hbar}{2m_2\omega_2}}\;(\hat a_2 + \hat a_2^\dagger)\\ p \;&=\; -\,i\sqrt{\frac{m_1\omega_1\hbar}{2}}(\hat a_1 - \hat a_1^\dagger) \;=\; -i\sqrt{\frac{m_2\omega_2\hbar}{2}}(\hat a_2 - \hat a_2^\dagger)\end{aligned} \tag{5.7}$$

Using the definition of e^{γ} we then have

$$\begin{aligned}\hat a_1 + \hat a_1^\dagger \;&=\; e^{\gamma}\;(\hat a_2 + \hat a_2^\dagger)\,,\\ \hat a_1 - \hat a_1^\dagger \;&=\; e^{-\gamma}\,(\hat a_2 - \hat a_2^\dagger)\,.\end{aligned} \tag{5.8}$$

Solving these equations for $(\hat{a}_1, \hat{a}_1^\dagger)$ in terms of $(\hat{a}_2, \hat{a}_2^\dagger)$ we find

$$\begin{aligned}\hat{a}_1 &= \hat{a}_2 \cosh\gamma \;+\; \hat{a}_2^\dagger \sinh\gamma\,,\\ \hat{a}_1^\dagger &= \hat{a}_2 \sinh\gamma \;+\; \hat{a}_2^\dagger \cosh\gamma\,.\end{aligned} \tag{5.9}$$

Note that the second equation is simply the adjoint of the first equation. The above relations are called *Bogoliubov* transformations. Notice that they preserve the commutation algebra. You can check that $[\hat{a}_1, \hat{a}_1^\dagger] = 1$ using (5.9) and the commutation relation of the $\hat{a}_2$ and $\hat{a}_2^\dagger$ operators. We can also obtain the second set of operators in terms of the first set by changing γ into $-\gamma$, as implied by the relations (5.8) :

$$\begin{aligned}\hat{a}_2 &= \quad \hat{a}_1 \cosh\gamma \;-\; \hat{a}_1^\dagger \sinh\gamma\,,\\ \hat{a}_2^\dagger &= -\,\hat{a}_1 \sinh\gamma \;+\; \hat{a}_1^\dagger \cosh\gamma\,.\end{aligned} \tag{5.10}$$

We can now examine explicitly the question of the ground state. The initial state is the ground state of H_1 denoted as $|0\rangle_{(1)}$. Its defining property is that it is killed by a_1:

$$\hat{a}_1\,|0\rangle_{(1)} \;=\; 0\,. \tag{5.11}$$

Using equation (5.9) we have

$$\big(\hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma\big)|0\rangle_{(1)} \;=\; 0\,. \tag{5.12}$$

Solving this equation means finding some expression for $|0\rangle_{(1)}$ in terms of some combination of $a_2^\dagger$ operators acting on $|0\rangle_{(2)}$. We should be able to write the original ground-state wavefunction in terms of eigenfunctions of the second Hamiltonian, or equivalently, write the original state as a superposition of energy eigenstates of the second Hamiltonian. Since the original wavefunction is even in x, only states with even number of creation operators should enter in such an expansion. We thus expect a solution of the form

$$|0\rangle_{(1)} \;=\; c_0|0\rangle_{(2)} + c_2\,\hat{a}_2^\dagger\hat{a}_2^\dagger|0\rangle_{(2)} + c_4\,\hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_2^\dagger|0\rangle_{(2)} + \ldots\,, \tag{5.13}$$

where the c_n's are coefficients to be determined. While we could proceed recursively, it is in fact possible to write an ansatz for the state and solve the problem directly.

We write an educated guess that uses the exponential of an expression quadratic in $\hat{a}_2^\dagger$:

$$|0\rangle_{(1)} \;=\; \mathcal{N}(\gamma)\,\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)}\,. \tag{5.14}$$

In here the functions $f(\gamma)$ and $\mathcal{N}(\gamma)$ are to be determined. Equation (5.12) gives

$$\big(\hat{a}_2 \cosh\gamma + \hat{a}_2^\dagger \sinh\gamma\big)\,\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)} \;=\; 0\,. \tag{5.15}$$

The action of $\hat{a}_2$ can be replaced by a commutator since it kills the vacuum $|0\rangle_{(2)}$:

$$\cosh\gamma\Big[\hat{a}_2\,,\,\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\Big]\,|0\rangle_{(2)} + \hat{a}_2^\dagger\sinh\gamma\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)} \;=\; 0\,. \tag{5.16}$$

We can now apply the familiar $[A, e^B] = [A, B]e^B$ (if $[[A, B], B] = 0$) to the evaluation of the commutator

$$\Big(\cosh\gamma\Big[\hat{a}_2\,,\,-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big]\; + \hat{a}_2^\dagger\sinh\gamma\Big)\,\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)} \;=\; 0\,. \tag{5.17}$$

Evaluating the remaining commutator gives

$$\Big(-\cosh\gamma\;f(\gamma)\;\;+\sinh\gamma\Big)\,\hat{a}_2^\dagger\exp\Big(-\frac{1}{2}\,f(\gamma)\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)} \;=\; 0\,. \tag{5.18}$$

Since no annihilation operators remain, the equality requires that the pre factor in parenthesis be zero. This determines the function $f(\gamma)$:

$$f(\gamma) = \tanh\gamma\,, \tag{5.19}$$

and we therefore have

$$|0\rangle_{(1)} \;=\; \mathcal{N}(\gamma)\,\exp\Big(-\frac{1}{2}\,\tanh\gamma\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)}\,. \tag{5.20}$$

The normalization $\mathcal{N}$ is not determined by the above calculation. It is could be determined, for example, by the demand that the state on the right-hand side above have unit normalization, just like $|0\rangle_{(1)}$ does. This is not a simple calculation. A simpler way uses the overlap of the two sides of the above equation with ${}_{(2)}\langle 0|$. We find

$${}_{(2)}\langle 0|0\rangle_{(1)} \;=\; \mathcal{N}(\gamma)\,, \tag{5.21}$$

because on the right hand side we can expand the exponential and all oscillators give zero on account of ${}_{(2)}\langle 0|\hat{a}_2^\dagger = 0$. Introducing a complete set of position states we get:

$$\mathcal{N}(\gamma) \;=\; \int_{-\infty}^{\infty} dx\,{}_{(2)}\langle 0|x\rangle\langle x|0\rangle_{(1)} \;=\; \int_{-\infty}^{\infty} dx\;(\psi_0^{(2)}(x))^*\,\psi_0^{(1)}(x)\,. \tag{5.22}$$

Using the expression (1.39) for the ground state wavefunctions

$$\begin{aligned}\mathcal{N}(\gamma) \;&=\; \Big(\frac{m_1\omega_1}{\pi\hbar}\Big)^{1/4}\Big(\frac{m_2\omega_2}{\pi\hbar}\Big)^{1/4}\int_{-\infty}^{\infty} dx\exp\Big(-\Big[\frac{m_1\omega_1+m_2\omega_2}{2\hbar}\Big]x^2\Big)\,,\\ &=\; \Big[\frac{\sqrt{m_1\omega_1 m_2\omega_2}}{\pi\hbar}\Big]^{1/2}\frac{\sqrt{2\pi\hbar}}{\sqrt{m_1\omega_1+m_2\omega_2}} \;=\; \Big(\frac{1}{2}\,\frac{m_1\omega_1+m_2\omega_2}{\sqrt{m_1\omega_1 m_2\omega_2}}\Big)^{-1/2}\\ &=\; \Big(\frac{1}{2}\Big[\sqrt{\frac{m_1\omega_1}{m_2\omega_2}}+\sqrt{\frac{m_2\omega_2}{m_1\omega_1}}\;\Big]\Big)^{-1/2} \;=\; \Big(\frac{1}{2}\,\big[\,e^{\gamma}+e^{-\gamma}\big]\Big)^{-1/2}\,,\end{aligned} \tag{5.23}$$

so that we finally have

$$\mathcal{N}(\gamma) \;=\; \frac{1}{\sqrt{\cosh\gamma}}\,. \tag{5.24}$$

All in all

$$\boxed{|0\rangle_{(1)} \;=\; \frac{1}{\sqrt{\cosh\gamma}}\,\exp\Big(-\frac{1}{2}\,\tanh\gamma\,\hat{a}_2^\dagger\hat{a}_2^\dagger\Big)\,|0\rangle_{(2)}\,.} \tag{5.25}$$

The state on the above right-hand side takes the form of an exponential of something quadratic in oscillators. It is a squeezed vacuum state of the second Hamiltonian.

Inspired by the discussion above we introduce squeezed states for an arbitrary harmonic oscillator Hamiltonian H with vacuum $|0\rangle$, parameters (m,ω) and operators $(a,a^\dagger)$. A a normalized squeezed vacuum state, denoted as $|0_\gamma\rangle$, thus takes the form

$$|0_\gamma\rangle \;\equiv\; \frac{1}{\sqrt{\cosh\gamma}}\,\exp\Big(-\frac{1}{2}\,\tanh\gamma\,\hat{a}^\dagger\hat{a}^\dagger\Big)\,|0\rangle\,. \tag{5.26}$$

For this state we have

$$\big(\hat{a}\,\cosh\gamma + \hat{a}^\dagger\,\sinh\gamma\big)|0_\gamma\rangle \;=\; 0\,. \tag{5.27}$$

For this squeezed vacuum state the x uncertainty follows directly from (5.5):

$$\Delta x \;=\; e^{-\gamma}\sqrt{\frac{\hbar}{2m\omega}}\,. \tag{5.28}$$

The above squeezed vacuum state can in fact be expressed in terms of a unitary operator $S(\gamma)$ acting on the vacuum. We claim that $|0_\gamma\rangle$ defined above is actually

$$|0_\gamma\rangle \;=\; S(\gamma)|0\rangle\,, \quad \text{with} \quad S(\gamma) \;=\; \exp\Big(-\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})\Big)\,. \tag{5.29}$$

This claim implies that the following nontrivial identity holds:

$$\exp\Big(-\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})\Big)|0\rangle \;=\; \frac{1}{\sqrt{\cosh\gamma}}\,\exp\Big(-\frac{1}{2}\,\tanh\gamma\,\hat{a}^\dagger\hat{a}^\dagger\Big)\,|0\rangle\,. \tag{5.30}$$

This equation takes a little effort to prove, but it is true.

5.2 More general squeezed states

In the limit $\gamma \to +\infty$ the state in (5.26) is completely squeezed in the x coordinate. It takes the form

$$|0_\infty\rangle \;\sim\; \exp\Big(-\frac{1}{2}\,\hat{a}^\dagger\hat{a}^\dagger\Big)\,|0\rangle\,, \tag{5.31}$$

where we have dropped the normalization constant, which is actually going to zero. We see that $(\hat{a}+\hat{a}^\dagger)|0_\infty\rangle = 0$ by direct (quick) computation or by consideration of (5.27). This means that the $\hat{x}$ operator kills this state. We conclude that the state must have a wavefunction proportional to $\delta(x)$. Alternatively, for $\gamma \to -\infty$ we have a state

$$|0_{-\infty}\rangle \;\sim\; \exp\Big(\frac{1}{2}\,\hat{a}^\dagger\hat{a}^\dagger\Big)\,|0\rangle\,, \tag{5.32}$$

This state is annihilated by $(\hat{a}-\hat{a}^\dagger)$, or equivalently, by the momentum operator. So it must be a state whose wavefunction in momentum space is $\delta(p)$ and in coordinate space is a constant! The right-hand side constructs the constant by superposition of Hermite polynomials times gaussians.

The above suggest that position states $|x\rangle$ (and momentum states $|p\rangle$) are squeezed states of the harmonic oscillator. Indeed, we can introduce the more general squeezed states

$$|x\rangle \;=\; \mathcal{N}\exp\Big(\sqrt{\frac{2m\omega}{\hbar}}\;x\,\hat{a}^\dagger \;-\; \frac{1}{2}\,\hat{a}^\dagger\hat{a}^\dagger\Big)|0\rangle\,. \tag{5.33}$$

A short calculation (do it!) shows that, indeed,

$$\hat{x}|x\rangle \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)|x\rangle = x|x\rangle\,. \tag{5.34}$$

The normalization constant is x dependent and is quickly determined by contracting (5.33) with the ground state

$$\langle 0|x\rangle \;=\; \mathcal{N}\langle 0|\exp\Big(\sqrt{\frac{2m\omega}{\hbar}}\;x\,\hat{a}^\dagger \;-\; \frac{1}{2}\,\hat{a}^\dagger\hat{a}^\dagger\Big)|0\rangle \;=\; \mathcal{N}. \tag{5.35}$$

We thus conclude that the normalization constant is just the ground state wavefunction: $\mathcal{N} = \psi_0(x)$. Using (1.42) we finally have

$$|x\rangle \;=\; \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4}\exp\Big(-\frac{m\omega}{2\hbar}x^2\Big)\exp\Big(\sqrt{\frac{2m\omega}{\hbar}}\;x\,\hat{a}^\dagger \;-\; \frac{1}{2}\,\hat{a}^\dagger\hat{a}^\dagger\Big)|0\rangle\,. \tag{5.36}$$

Rather general squeezed states are obtained as follows. Recall that for coherent states we used the operator $D(\alpha)$ (D for displacement!) acting on the vacuum

$$D(\alpha) \;=\; \exp\big(\alpha\,\hat{a}^\dagger - \alpha^*\hat{a}\big)\,, \qquad |\alpha\rangle \;=\; D(\alpha)|0\rangle\,.$$

We can now introduce more general squeezed states $|\alpha,\gamma\rangle$ by first squeezing and then translating:

$$|\alpha,\gamma\rangle \;\equiv\; D(\alpha)S(\gamma)|0\rangle\,.$$

Note that $|0,\gamma\rangle = |0_\gamma\rangle$ and $|\alpha,0\rangle = |\alpha\rangle$.

5.3 Photon states

For a classical electromagnetic field the energy E is obtained by adding the contributions of the electric and magnetic field:

$$E \;=\; \frac{1}{2}\int d^3x\,\epsilon_0\,\Big[\vec{E}^2(\vec{r},t) + c^2\vec{B}^2(\vec{r},t)\Big]\,. \tag{5.37}$$

We consider a rectangular cavity of volume V with a single mode of the electromagnetic field, namely, a single frequency ω and corresponding wavenumber $k = \omega/c$. The electromagnetic fields form a standing wave in which electric and magnetic fields are out of phase. They can take the form

$$\begin{aligned} E_x(z,t) \;&=\; \sqrt{\frac{2}{V\epsilon_0}}\;\omega\,q(t)\sin kz\,,\\ cB_y(z,t) \;&=\; \sqrt{\frac{2}{V\epsilon_0}}\;\,p(t)\,\cos kz\,, \end{aligned} \tag{5.38}$$

The classical time-dependent functions $q(t)$ and $p(t)$ are to become in the quantum theory Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$ with commutation relations $[\hat{q}(t)\,,\hat{p}(t)\,] \;=\; i\hbar$. A calculation of the energy E in (5.37) with the fields above gives[6]

$$E \;=\; \frac{1}{2}\big(p^2(t) + \omega^2\,q^2(t)\big) \tag{5.39}$$

There is some funny business here with units. The variables $q(t)$ and $p(t)$ do not have their familiar units, as you can see from the expression for the energy. Indeed one is missing a quantity with units of mass that divides the p^2 contribution and multiplies the q^2 contribution. One can see that p has units of $\sqrt{E}$ and q has units of $T\sqrt{E}$. Still, the product of q and p has the units of $\hbar$, which is useful. Since photons are massless particles there is no quantity with units of mass that we can use. Note that the dynamical variable $q(t)$ is not a position, it is essentially the electric field. The dynamical variable $p(t)$ is not a momentum, it is essentially the magnetic field.

The quantum theory of this EM field uses the structure implied by the above classical results. From the energy above we are let to *postulate* a Hamiltonian

$$H \;=\; \frac{1}{2}\big(\hat{p}^2 + \omega^2\,\hat{q}^2\big)\,, \tag{5.40}$$

with Schrödinger operators $\hat{q}$ and $\hat{p}$ (and associated Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$) that satisfy $[\hat{q},\hat{p}] = i\hbar$. As soon as we declare that the classical variables $q(t)$ and $p(t)$ are to become operators, we have the implication that the electric and magnetic fields in (5.38) will become

[6] If you wish to do the computation just recall that over the volume the average of $\sin^2 kz$ or $\cos^2 kz$ is $1/2$.

field operators, that is to say, space and time-dependent operators (more below!). This oscillator is our familiar SHO, but with m set equal to one, which is allowed given the unusual units of $\hat{q}$ and $\hat{p}$. With the familiar (1.17) and $m = 1$ we have

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \ , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}) \ , \qquad [\hat{a}\,,\,\hat{a}^\dagger] \ = \ 1\,. \tag{5.41}$$

It follows that

$$\hbar\omega\,\hat{a}^\dagger\hat{a} \ = \ \frac{1}{2}(\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p}) \ = \ \frac{1}{2}\big(\hat{p}^2 + \omega^2\hat{q}^2 + i\omega[\hat{q},\hat{p}]\big) \ = \ \frac{1}{2}\big(\hat{p}^2 + \omega^2\hat{q}^2 - \hbar\omega\big) \tag{5.42}$$

and comparing with (5.40) this gives the Hamiltonian

$$H \ = \ \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) \ = \ \hbar\omega\big(\hat{N} + \frac{1}{2}\big)\,. \tag{5.43}$$

This was the expected answer as this formula does not depend on m and thus our setting it to one should have no import. At this point we got photons! A quantum state of the electromagnetic field is a photon state, which is just a state of the harmonic oscillator Hamiltonian above. In the number basis the state $|n\rangle$ with number eigenvalue n, has energy $\hbar\omega(n + \frac{1}{2})$ which is, up to the zero-point energy $\hbar\omega/2$, the energy of n photons each of energy $\hbar\omega$.

For more intuition we now consider the electromagnetic field operator, focusing on the electric field operator. For this we first note that

$$\hat{q} \ = \ \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a} + \hat{a}^\dagger)\,, \tag{5.44}$$

and the corresponding Heisenberg operator is, using (3.53) and (3.54),

$$\hat{q}(t) \ = \ \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\,. \tag{5.45}$$

In quantum field theory –which is what we are doing here– the electric field becomes a Hermitian operator. Its form is obtained by substituting (5.45) into (5.38):

$$\boxed{\hat{E}_x(z,t) \ = \ \mathcal{E}_0\,\big(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\big)\sin kz\,, \qquad \mathcal{E}_0 \ = \ \sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\,.} \tag{5.46}$$

This is a field operator in the sense that it is an operator that depends on time and on space (z in this case). The constant $\mathcal{E}_0$ is sometimes called the electric field of a photon.

A classical electric field can be identified as the expectation value of the electric field operator in the given photon state. We immediately see that in the energy eigenstate $|n\rangle$ the expectation value of $\hat{E}_x$ takes the form

$$\langle\hat{E}_x(z,t)\rangle \ = \ \mathcal{E}_0\,\big(\langle n|\hat{a}|n\rangle e^{-i\omega t} + \langle n|\hat{a}^\dagger|n\rangle e^{i\omega t}\big)\sin kz \ = \ 0\,, \tag{5.47}$$

since the matrix elements on the right hand side are zero. Thus the energy eigenstates of the photon field do not correspond to classical electromagnetic fields. Consider now the expectation value of the field in a coherent state $|\alpha\rangle$ with $\alpha \in \mathbb{C}$. This time we get

$$\hat{E}_x(z,t)\rangle \;=\; \mathcal{E}_0 \quad \alpha|\hat{a}|\alpha\rangle e^{-i\omega t} + \;\alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\big) \sin kz\,. \tag{5.48}$$

Since $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ we get

$$\hat{E}_x(z,t)\rangle \;=\; \mathcal{E}_0 \quad \alpha\, e^{-i\omega t} + \alpha^*\, e^{i\omega t}\big) \sin kz\,. \tag{5.49}$$

This now looks like a familiar standing wave! If we set $\alpha = |\alpha| e^{i\theta}$, we have

$$\hat{E}_x(z,t)\rangle \;=\; 2\mathcal{E}_0\, \mathrm{Re}(\alpha e^{-i\omega t}) \sin kz \;=\; 2\mathcal{E}_0\, |\alpha|\, \cos(\omega t - \theta) \sin kz. \tag{5.50}$$

The coherent photon states are the ones that have a nice classical limit with classical electric fields. The standing wave in (5.50) corresponds to a state $|\alpha\rangle$ where the expectation value of the number operator $\hat{N}$ is $|\alpha|^2$. This is the expected number of photons in the state. It follows that the expectation value of the energy is

$$H\rangle_\alpha \;=\; \hbar\omega\, |\alpha|^2 + \frac{1}{2}\hbar\omega\,. \tag{5.51}$$

Up to the zero-point energy, the expected value of the energy is equal to the number of photons times $\hbar\omega$.