Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \qquad J(x,t) = \frac{\hbar}{2im}\left[\psi^* \frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx\, \psi^*(x) H \psi(x)}{\int dx \psi^*(x)\psi(x)}\,, \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach}: \quad H = -\vec{\mu}\cdot\vec{B}\,, \qquad \vec{\mu} = g\,\frac{e\hbar}{2m}\,\frac{1}{\hbar}\vec{S} = \gamma\vec{S}$$

$$\mu_{\scriptscriptstyle B} = \frac{e\hbar}{2m_e}\,, \qquad \vec{\mu}_e = -2\,\mu_{\scriptscriptstyle B}\frac{\vec{S}}{\hbar}\,,$$

$$\text{In the basis } |1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,,\; |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\;;\; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\;;\; \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \big[S_i\,,\,S_j\big] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp\big(i\vec{a}\cdot\vec{\sigma}\big) = \mathbf{1}\cos a + i\vec{\sigma}\cdot\left(\frac{\vec{a}}{a}\right)\,\sin a\,, \quad a = |\vec{a}|$$

$$\begin{aligned}\exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)\\ \exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\,.\end{aligned}$$

$$S_{\vec{n}} = \vec{n}\cdot\vec{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\,\vec{n}\cdot\vec{\sigma}\,.$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad S_{\vec{n}}\,|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}\,|\vec{n};\pm\rangle$$

$$\begin{aligned}|\vec{n};+\rangle &= \cos(\theta/2)|+\rangle + \sin(\theta/2)\exp(i\phi)|-\rangle\\ |\vec{n};-\rangle &= -\sin(\theta/2)\exp(-i\phi)|+\rangle + \cos(\theta/2)|-\rangle\end{aligned}$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint:} \quad \langle u|\Omega^\dagger v\rangle \;=\; \langle \Omega u|v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \quad \longleftrightarrow \quad \langle \alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \quad \leftrightarrow \quad \Omega = \sum_{i,j} \Omega_{ij}\; |i\rangle\langle j|$$

$$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$$

$$\Omega \;\; \text{hermitian:} \;\; \Omega^\dagger = \Omega, \qquad U \text{ unitary:} \;\; U^\dagger = U^{-1}$$

- Matrix M is normal ($[M, M^\dagger] = 0$) $\longleftrightarrow$ unitarily diagonalizable.
- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x-y)\,, \quad \mathbf{1} = \int dx\; |x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q-p)\,, \quad \mathbf{1} = \int dp\; |p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\Big(\frac{ipx}{\hbar}\Big)\,; \qquad \tilde{\psi}(p) = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\Big(-\frac{ipx}{\hbar}\Big)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n \psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \Big(i\hbar\frac{d}{dp}\Big)^n \tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$$

$$(\Delta A)^2(\Delta B)^2 \geq \Big(\langle \Psi|\frac{1}{2i}[A,B]|\Psi\rangle\Big)^2$$

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \;\; \text{and} \;\; \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \;\; \text{for a gaussian wavefuntion } \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\text{Time independent operator } Q: \quad \frac{d}{dt}\langle Q\rangle \;=\; \frac{i}{\hbar}\langle\, [H, Q]\rangle$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}\,, \quad \Delta t \equiv \frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

- Commutator identities

$$[A,BC] = [A,B]C + B[A,C]\,,$$
$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots\,,$$
$$e^A B e^{-A} = B + [A,B]\,, \quad \text{if} \quad [[A,B],A] = 0\,,$$
$$[\,B\,,\,e^A\,] = [\,B\,,A\,]e^A\,, \quad \text{if} \quad [[A,B],A] = 0$$
$$e^{A+B} \;=\; e^A e^B e^{-\frac{1}{2}[A,B]} \;=\; e^B e^A e^{\frac{1}{2}[A,B]}\;, \quad \text{if } [A,B] \text{ commutes with } A \text{ and with } B$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\;, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\;, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\;,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\;, \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x},\hat{p}] = i\hbar\,, \quad [\hat{a},\hat{a}^\dagger] = 1\;, \qquad [\hat{N},\hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,,\hat{a}^\dagger\,] \;=\; \hat{a}^\dagger\,.$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\big(n + \frac{1}{2}\big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\;, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\;.$$
$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\;.$$

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