

8.05, Quantum Physics II, Fall 2013  
TEST  
Wednesday October 23, 12:30-2:00pm  
You have 90 minutes.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are five questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

**TIME MANAGEMENT:** I suggest 45 minutes for the first three questions, 20 for question 4 and 25 for question 5.

# Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} J(x, t) = 0$$

$$\rho(x, t) = |\psi(x, t)|^2 ; \quad J(x, t) = \frac{\hbar}{2im} \left[ \psi^* \frac{\partial}{\partial x} \psi - \psi \frac{\partial}{\partial x} \psi^* \right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx \psi^*(x) H \psi(x)}{\int dx \psi^*(x) \psi(x)}, \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach : } H = -\vec{\mu} \cdot \vec{B}, \quad \vec{\mu} = g \frac{e\hbar}{2m} \frac{1}{\hbar} \vec{S} = \gamma \vec{S}$$

$$\mu_B = \frac{e\hbar}{2m_e}, \quad \vec{\mu}_e = -2 \mu_B \frac{\vec{S}}{\hbar},$$

$$\text{In the basis } |1\rangle \equiv |z; +\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle \equiv |z; -\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_i = \frac{\hbar}{2} \sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk} \sigma_k \quad \rightarrow \quad [S_i, S_j] = i\hbar \epsilon_{ijk} S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk} \sigma_k \quad \rightarrow \quad (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} I + i\vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1} \cos \theta + i\mathbf{M} \sin \theta, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp(i\vec{a} \cdot \vec{\sigma}) = \mathbf{1} \cos a + i\vec{\sigma} \cdot \left( \frac{\vec{a}}{a} \right) \sin a, \quad a = |\vec{a}|$$

$$\exp(i\theta\sigma_3) \sigma_1 \exp(-i\theta\sigma_3) = \sigma_1 \cos(2\theta) - \sigma_2 \sin(2\theta)$$

$$\exp(i\theta\sigma_3) \sigma_2 \exp(-i\theta\sigma_3) = \sigma_2 \cos(2\theta) + \sigma_1 \sin(2\theta).$$

$$S_{\vec{n}} = \vec{n} \cdot \vec{S} = n_x S_x + n_y S_y + n_z S_z = \frac{\hbar}{2} \vec{n} \cdot \vec{\sigma}.$$

$$(n_x, n_y, n_z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad S_{\vec{n}} |\vec{n}; \pm\rangle = \pm \frac{\hbar}{2} |\vec{n}; \pm\rangle$$

$$|\vec{n}; +\rangle = \cos(\theta/2) |+\rangle + \sin(\theta/2) \exp(i\phi) |-\rangle$$

$$|\vec{n}; -\rangle = -\sin(\theta/2) \exp(-i\phi) |+\rangle + \cos(\theta/2) |-\rangle$$

- Bras and kets: For an operator  $\Omega$  and a vector  $v$ , we write  $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint: } \langle u|\Omega^\dagger v\rangle = \langle \Omega u|v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \iff \langle \alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis  $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}, \quad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \iff \Omega = \sum_{i,j} \Omega_{ij} |i\rangle\langle j|$$

$$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$$

$$\Omega \text{ hermitian: } \Omega^\dagger = \Omega, \quad U \text{ unitary: } U^\dagger = U^{-1}$$

- Matrix  $M$  is normal ( $[M, M^\dagger] = 0$ )  $\iff$  unitarily diagonalizable.
- Position and momentum representations:  $\psi(x) = \langle x|\psi\rangle$ ;  $\tilde{\psi}(p) = \langle p|\psi\rangle$  ;

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x|y\rangle = \delta(x-y), \quad \mathbf{1} = \int dx |x\rangle\langle x|, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle, \quad \langle q|p\rangle = \delta(q-p), \quad \mathbf{1} = \int dp |p\rangle\langle p|, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right); \quad \tilde{\psi}(p) = \int dx \langle p|x\rangle \langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\left(-\frac{ipx}{\hbar}\right) \psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \psi(x); \quad \langle p|\hat{x}^n|\psi\rangle = \left(i\hbar \frac{d}{dp}\right)^n \tilde{\psi}(p); \quad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

$$(\Delta A)^2 (\Delta B)^2 \geq \left( \langle \Psi | \frac{1}{2i} [A, B] | \Psi \rangle \right)^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \text{ and } \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \text{ for a gaussian wavefunction } \psi \sim \exp\left(-\frac{1}{2} \frac{x^2}{\Delta^2}\right)$$

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

Time independent operator  $Q$  :  $\frac{d}{dt}\langle Q \rangle = \frac{i}{\hbar}\langle [H, Q] \rangle$

$$\Delta H \Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{\left| \frac{d\langle Q \rangle}{dt} \right|}$$

- Commutator identities

$$[A, BC] = [A, B]C + B[A, C],$$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots,$$

$$e^A B e^{-A} = B + [A, B], \quad \text{if } [[A, B], A] = 0,$$

$$[B, e^A] = [B, A]e^A, \quad \text{if } [[A, B], A] = 0$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} = e^B e^A e^{\frac{1}{2}[A, B]}, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega \left( \hat{N} + \frac{1}{2} \right), \quad \hat{N} = \hat{a}^\dagger \hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i\hat{p}}{m\omega} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i\hat{p}}{m\omega} \right),$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a}),$$

$$[\hat{x}, \hat{p}] = i\hbar, \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{N}, \hat{a}] = -\hat{a}, \quad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger.$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad \hat{N}|n\rangle = n|n\rangle, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle.$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

- Unitary time evolution

$$|\Psi, t\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\Psi, 0\rangle, \quad \text{for time-independent } H$$

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

- (1) A state for which the uncertainty  $\Delta Q$  of an observable  $Q$  vanishes must be an eigenstate of  $Q$ .
- (2) The uncertainty  $\Delta S$  of a time-independent operator  $S$  is time independent.
- (3) For any integer  $N \geq 2$ , there exist  $N \times N$  matrices  $X$  and  $P$  such that  $[X, P] = i\hbar \mathbf{1}$ .
- (4) When two hermitian operators  $Q_1$  and  $Q_2$  commute, any eigenvector of  $Q_1$  is an eigenvector of  $Q_2$ .

The following questions assume a finite-dimensional complex vector space  $V$ .

- (5) The matrix representation of the identity operator is basis dependent.
- (6) The eigenvalues of an operator are basis independent.
- (7) The matrices for the operator  $Q$  and its adjoint  $Q^\dagger$  are related by complex conjugation and transposition in all bases.
- (8) An orthogonal projector (to a lower dimensional subspace) is neither injective nor surjective in  $V$ .
- (9) Any set of orthonormal vectors is linearly independent.
- (10) An operator that is both unitary and hermitian must square to the identity.

2. **A scaling operator** [15 points]

Consider the operator

$$U_\lambda \equiv \exp\left[-\frac{i\lambda}{2\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x})\right].$$

- (a) What is the condition on the constant  $\lambda$  given that  $U_\lambda$  is in fact unitary? Explain.
- (b) Determine the action of  $U_\lambda$  on the operator  $\hat{x}$ , namely, give the right-hand side of the equation

$$U_\lambda^\dagger \hat{x} U_\lambda = \dots$$

- (c) Give an expression for  $U_\lambda^\dagger \hat{p} U_\lambda$  consistent with your above result and the commutator  $[\hat{x}, \hat{p}] = i\hbar$ . Does  $U_\lambda$  commute with the Hamiltonian of a free particle in one dimension?

3. **A property of complex vector spaces** [15 points]

Consider two vectors  $|u\rangle$  and  $|v\rangle$  in a complex vector space  $\mathbb{V}$

- (a) Simplify the following expression

$$\frac{1}{4} \left[ \langle u+v, u+v \rangle - \langle u-v, u-v \rangle + i \langle u-iv, u-iv \rangle - i \langle u+iv, u+iv \rangle \right].$$

Use your answer to state a formula for the inner product of any two vectors in terms of norms of related vectors. Recall that  $|v|^2 \equiv \langle v, v \rangle$ . Explain why your result ensures that a linear operator  $S$  that preserves the norm of any vector, will preserve the inner-product of any two vectors  $\langle Su, Sv \rangle = \langle u, v \rangle$ .

- (b) Assume that we have a vector space where the inner product arises from a norm, as found in (a). The norm, as usual, is assumed to satisfy  $|\alpha u| = |\alpha||u|$ , for any complex constant  $\alpha$ . Show that the familiar property  $\langle v, u \rangle = \langle u, v \rangle^*$  holds.

4. **Variational principle** [25 points]

This problem refers to a Hamiltonian  $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$  for a one-dimensional system and taking the form

$$\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + \hbar\omega - \frac{m\omega^2}{2}\hat{x}^2 + \frac{m^2\omega^3}{16\hbar}\hat{x}^4 \\ &= \frac{\hat{p}^2}{2m} + \frac{m^2\omega^3}{16\hbar}\left(\hat{x}^2 - \frac{4\hbar}{m\omega}\right)^2, \quad \omega > 0.\end{aligned}$$

Note here the natural length scale  $d_0$  given by

$$d_0^2 = \frac{\hbar}{m\omega}.$$

- (a) Sketch the potential  $V(x)$  for this Hamiltonian indicating precisely the position and values at local maxima and minima.
- (b) For the variational trial wavefunction

$$\psi(x) = x e^{-\frac{x^2}{2\lambda^2}}$$

the expectation value of the energy in this state is given by

$$E(\lambda) = \hbar\omega - \frac{3}{4}m\omega^2\lambda^2 + A\lambda^4 + \frac{3\hbar^2}{4m}\frac{1}{\lambda^2}.$$

Complete the calculation by computing the constant  $A$  in terms of  $\hbar$ ,  $m$ , and  $\omega$ . We computed for you the expectation value of the kinetic energy (last term), and part of the expectation value of the potential (don't recalculate them!)

- (c) To clean up units, set  $\lambda^2 = d_0^2\eta^2$ , where  $\eta$  is unit-free and  $d_0$  was defined above. Write now  $E$  as a function of  $\eta$  by including the missing term indicated by dots:

$$E(\eta) = \hbar\omega\left(1 - \frac{3}{4}\eta^2 + \dots + \frac{3}{4}\frac{1}{\eta^2}\right).$$

Find the value of  $\eta$  that minimizes  $E(\eta)$ , *ignoring the last term*. Use this value to find an upper bound  $E_*$  on the energy of the ground state. Explain your logic.

- (d) The energy  $E_*$  in fact provides a more stringent bound on the energy of another energy eigenstate of the system. Which state and why?

The following integrals may be useful:

$$\int_{-\infty}^{\infty} u^{2n} e^{-u^2} du = c_n \sqrt{\pi}, \quad \text{with } c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{4}, \quad c_3 = \frac{15}{8}.$$

5. **Spin states and uncertainty** [25 points]

- (a) Derive an uncertainty relation for the  $\hat{S}_x$  and  $\hat{S}_y$  operators. That is, given a state  $|\psi\rangle$ , demonstrate an inequality of the form

$$(\Delta\hat{S}_x)(\Delta\hat{S}_y) \geq \dots,$$

where the right-hand side is a function of  $|\psi\rangle$ .

- (b) What is the minimum possible value of the product  $\Delta\hat{S}_x\Delta\hat{S}_y$ ? For which states  $|\psi\rangle$  is this minimum achieved?
- (c) What is the maximum possible value of  $\Delta\hat{S}_i$ , for any  $i = 1, 2, 3$ ? Explain.
- (d) Consider a general spin state  $|n; +\rangle$  described in terms of the angles  $\theta$  and  $\phi$ . Calculate  $\Delta\hat{S}_x$ ,  $\Delta\hat{S}_y$  and  $\Delta\hat{S}_z$ . Based on your result, what are the states that maximize the product  $\Delta\hat{S}_x\Delta\hat{S}_y$ ? Explain.

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