8.05, Quantum Physics II, Fall 2011
TEST
Wednesday October 26, 12:30-2:00pm
You have 90 minutes.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

There are five questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

Time management: I suggest half an hour for the first two questions and 20 minutes for each of the remaining three.

Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \qquad J(x,t) = \frac{\hbar}{2im}\left[\psi^* \frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \int dx\, \psi^*(x) H \psi(x)\,, \quad \text{for all } \psi(x) \text{ satisfying } \int dx \psi^*(x)\psi(x) = 1$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach :} \quad H = -\vec{\mu}\cdot\vec{B}\,, \qquad \vec{\mu} = g\,\frac{e\hbar}{2m}\,\frac{1}{\hbar}\vec{S} = \gamma\vec{S}$$

$$\mu_B = \frac{e\hbar}{2m_e}\,, \qquad \vec{\mu}_e = -2\,\mu_B\frac{\vec{S}}{\hbar}\,,$$

In the basis $|1\rangle \equiv |z;+\rangle = |+\rangle\,,\ |2\rangle \equiv |z;-\rangle = |-\rangle$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ;\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ ;\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \left[S_i\,,\, S_j\right] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp\big(i\vec{a}\cdot\vec{\sigma}\big) = \mathbf{1}\cos a + i\vec{\sigma}\cdot\left(\frac{\vec{a}}{a}\right)\sin a\,, \quad a = |\vec{a}|$$

$$\begin{aligned}
\exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta) \\
\exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\,.
\end{aligned}$$

$$S_{\vec{n}} = \vec{n}\cdot\vec{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\,\vec{n}\cdot\vec{\sigma}\,.$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad S_{\vec{n}}\,|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}\,|\vec{n};\pm\rangle$$

$$\begin{aligned}
|\vec{n};+\rangle &= \cos(\theta/2)|+\rangle + \sin(\theta/2)\exp(i\phi)|-\rangle \\
|\vec{n};-\rangle &= -\sin(\theta/2)\exp(-i\phi)|+\rangle + \cos(\theta/2)|-\rangle
\end{aligned}$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\mathcal{O}_{ij} = \langle i|\mathcal{O}|j\rangle \quad \leftrightarrow \quad \mathcal{O} = \sum_{i,j} \mathcal{O}_{ij}\ |i\rangle\langle j|$$

$$\langle i|A^\dagger|j\rangle = \langle j|A|i\rangle^*$$

hermitian operator: $\mathcal{O}^\dagger = \mathcal{O}$, unitary operator: $U^\dagger = U^{-1}$

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x-y)\,, \quad \mathbf{1} = \int dx\,|x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q-p)\,, \quad \mathbf{1} = \int dp\,|p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\big(\frac{ipx}{\hbar}\big)\,; \qquad \tilde{\psi}(p) = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\exp\big(-\frac{ipx}{\hbar}\big)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n\psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \Big(i\hbar\frac{d}{dp}\Big)^n\tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i}f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$$

$$(\Delta A)^2(\Delta B)^2 \geq \Big(\langle\Psi|\frac{1}{2i}[A,B]|\Psi\rangle\Big)^2$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \quad \text{and} \quad \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \quad \text{for a gaussian wavefuntion } \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

$$\int_{-\infty}^{+\infty} dx\exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\Delta H\Delta t \geq \frac{\hbar}{2}\,, \quad \Delta t \equiv \frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

- Commutator identities

$$
\begin{aligned}
[A, BC] &= [A, B]C + B[A, C]\,, \\
e^A B e^{-A} &= B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots\,, \\
e^A B e^{-A} &= B + [A, B]\,, \quad \text{if} \quad [[A, B], A] = 0\,, \\
[\,B\,,\, e^A\,] &= [\,B\,, A\,]e^A\,, \qquad \text{if} \quad [[A, B], A] = 0 \\
e^{A+B} \;&=\; e^A e^B e^{-\frac{1}{2}[A,B]} \;=\; e^B e^A e^{\frac{1}{2}[A,B]}\;, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B
\end{aligned}
$$

- Harmonic Oscillator

$$
\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}
$$

$$
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\;, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\;,
$$

$$
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\;, \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,
$$

$$
[\hat{x}, \hat{p}] = i\hbar\,, \quad [\hat{a}, \hat{a}^\dagger] = 1\;.
$$

$$
|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle
$$

$$
\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\big(n + \frac{1}{2}\big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,, \quad \langle m|n\rangle = \delta_{mn}
$$

$$
\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\;, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\;.
$$

$$
\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\;.
$$

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

   Consider a potential $V(x)$ such that $V(x) = 2V_0$ for $x < 0$, $V(x) = 0$ for $0 \leq x \leq a$, and $V(x) = V_0$ for $x > a$, with $V_0 > 0$. (We recommend a sketch!) The next three questions apply to this potential.

   (1) There are two energy eigenstates for each energy E greater than V_0 but smaller than $2V_0$.

   (2) There is one normalizable state for each E greater than zero but smaller than V_0.

   (3) There are no energy eigenstates for $E < 0$.

   (4) The state of a spin-1/2 particle whose spin points along the $\hat{z}$ direction is orthogonal to the state in which the spin points along the $\hat{x}$ direction.

   (5) The spin operators S_x^2 and S_y^2 commute.

   (6) The product of two unitary operators is a unitary operator.

   (7) A projector operator can have an eigenstate with eigenvalue minus one.

   (8) If two arbitrary operators can be diagonalized simultaneously then they commute.

   (9) The exponential of a hermitian operator is a unitary operator.

   (10) The commutator of two hermitian operators is hermitian.

2. **Two short problems** [20 points]

   (a) The trace of an operator M is defined by

   $$\mathrm{Tr}(M) = \sum_i \langle a_i | M | a_i \rangle \,,$$

   where the set $\{|a_i\rangle\}$ is an orthonormal basis. Consider a different orthonormal basis $\{|b_i\rangle\}$. Show that $\mathrm{Tr}(M)$ calculated using this other basis gives the same result.

   (b) Consider two hermitian matrices A_1 and A_2 that commute:

   $$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \,, \quad A_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \,.$$

   The matrix A_1 has eigenvalues and orthonormal eigenvectors

   $$\lambda_1 = 2, |u_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;; \quad \lambda_2 = 0, |u_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \;; \quad \lambda_3 = 0, \, |u_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,.$$

In the basis $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ the matrix A_2 takes the form

$$\begin{pmatrix} 3 & * & * \\ 0 & * & -\sqrt{2} \\ 0 & * & 0 \end{pmatrix} . \qquad (1)$$

Determine the missing entries (denoted by $*$) in the above matrix. Use your result to find the eigenvalues of A_2.

For the following two problems you will find the following result useful:

$$\text{Useful fact:} \;\; \text{Min}_{x>0}\Big(\frac{A}{x^p} + Bx^q\Big) = \Big(1+\frac{q}{p}\Big)\Big(\frac{p}{q}A\Big)^{\frac{q}{p+q}} B^{\frac{p}{p+q}}, \quad p, q > 0 .$$

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3. **Variational principle** [20 points]

Consider a particle of mass m in the potential

$$V(x) = V_0|x| ,$$

where V_0 is a positive constant with units of energy per unit length.

Find an *upper* bound E_0 on the ground state energy using a gaussian trial wavefunction

$$\psi(x) = A\exp\Big(-\frac{1}{2}b\,x^2\Big) ,$$

where b is a parameter to be adjusted to obtain the best bound. Simplify your answer for the bound E_0 until it takes the form $E_0 = c_0 M^s$, where c_0 is a pure number (no units) M is an expression involving (powers of) $V_0, \hbar, m$, and s is a pure number.

Possibly useful integrals: $\int_{-\infty}^{\infty} x^2 e^{-bx^2} dx = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$, $\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$.

4. **Uncertainty Principle** [20 points]

Consider a particle of mass m in the *ground state* of the Schrödinger Hamiltonian with potential

$$V(x) = \gamma\, x^8 , \quad \text{with} \quad \gamma > 0 .$$

(a) Find an inequality relating the ground state expectation value $\langle x^8\rangle_{\text{gs}}$ to some suitable power of the ground state uncertainty $(\Delta x)_{\text{gs}}$. Explain your steps explicitly.

(b) Use the uncertainty principle to calculate a *lower* bound E_0 on the ground state energy. Simplify E_0 as in the previous problem.

5. **Spin state in a magnetic field** [20 points]

At $t = 0$ a spin points in the direction defined by the angles $\theta = \theta_0$ and $\phi = 0$. A magnetic field along the y-direction interacts with the magnetic dipole of the spin, resulting in a Hamiltonian

$$H = -\gamma B\, \hat{S}_y\,,$$

where γ is a constant and B is the magnitude of the magnetic field. Since this Hamiltonian is time independent, time evolution is generated by the action of $\exp(-iHt/\hbar)$ on states at time equal zero.

(a) Write down the state $|\Psi, 0\rangle$ that represents the spin-1/2 particle at $t = 0$ in terms of the basis states $|+\rangle$ and $|-\rangle$ along the z-direction.

(b) Calculate the state $|\Psi, t\rangle$. Write your answer in terms of the basis states $|+\rangle$ and $|-\rangle$ along the z-direction.

(c) Describe the time-dependent spin orientation in terms of functions $\theta(t)$ and $\phi(t)$.

Addition formulas for trigonometric functions are easily derived (if you forgot them) from

$$e^{i(A\pm B)} \;=\; e^{iA}e^{\pm iB}$$

by expanding the exponentials using $e^{i\theta} = \cos\theta + i\sin\theta$.