

8.05 Quantum Physics II, Fall 2013

FINAL EXAM

Tuesday December 17, 9:00 am-12:00 noon

You have 3 hours.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

**THIS TEST MUST BE RETURNED
WITH YOUR WHITE BOOK(S)**

YOUR NAME:

The formula sheet need not be returned

There are six questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

TIME MANAGEMENT:

55 minutes to answer for 1,2, and 3.

30 minutes for problem 4.

45 minutes for problem 5

50 minutes for problem 6.

The last questions in problems 5 and 6
may be challenging

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

- (1) The operators $\sigma_1 \otimes \sigma_1$ and $\sigma_3 \otimes \sigma_1$ commute.
- (2) The operators $\sigma_1 \otimes \sigma_1$ and $\sigma_3 \otimes \sigma_3$ commute.
- (3) Let $T \otimes S$ be a linear operator on $V \otimes W$. Then $(T \otimes S)^\dagger = S^\dagger \otimes T^\dagger$.
- (4) A linear operator on a finite-dimensional vector space is invertible if it is injective.
- (5) Angular momentum conservation prevents a particle with spin 1/2 from decaying into two spin-1/2 particles.
- (6) $[\mathbf{L}^2, \hat{x}_i] = 0$. Here \hat{x}_i is the position operator in any of the three directions.
- (7) $\mathbf{r} \cdot \mathbf{p} = \mathbf{p} \cdot \mathbf{r} - 3i\hbar$.
- (8) $\mathbf{A} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{A}$, when \mathbf{A} is a vector under rotations.
- (9) In the hydrogen atom the Runge-Lenz (RL) vector \mathbf{R} satisfies the algebra of angular momentum.
- (10) Both classically and quantum mechanically the RL vector \mathbf{R} satisfies $\mathbf{R} \cdot \mathbf{L} = 0$.

2. **Expectation value on a generalized squeezed state** [10 points]

Consider the general squeezed state $|\alpha, \gamma\rangle = D(\alpha)S(\gamma)|0\rangle$ of the harmonic oscillator at time equal zero (here $\alpha \in \mathbb{C}$, $\gamma \in \mathbb{R}$). Find the expectation value of the number operator \hat{N} in this state. As we let time change, does this expectation value exhibit time dependence?

3. **3D bound state** [10 points]

A particle of mass m is in a potential $V(r)$ that represents a finite depth spherical well of radius a :

$$V(r) = \begin{cases} -V_0 & \text{for } r < a, \\ 0 & \text{for } r > a, \end{cases}$$

Here $V_0 > 0$. For the potential to have bound states it should be deep enough. *Derive* the inequality that V_0 must satisfy in order that the potential have a bound state.

4. **Spectrum-generating angular momentum** [15 points]

Here we discuss an angular momentum algebra using the *one-dimensional* harmonic oscillator. Throughout this question we will set $\hbar = 1$ (to save ink).

Consider a pair of creation and annihilation operators \hat{a}^\dagger and \hat{a} satisfying $[\hat{a}, \hat{a}^\dagger] = 1$ and a number basis of states

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle, \quad \hat{N}|n\rangle \equiv \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle.$$

(a) Now define, with j a positive integer, the operators

$$J_z = -j + \hat{N}, \quad J_+ = \hat{a}^\dagger\sqrt{2j - \hat{N}}, \quad J_- = J_+^\dagger = \sqrt{2j - \hat{N}} \hat{a}.$$

Verify that these operators define an angular momentum algebra. Note: You should not have to expand the square roots in your manipulations. These operators are defined only on states for which the argument of the square root is not negative.

- (b) Construct the corresponding J^2 operator. Does it satisfy the proper commutation relations with the J_i operators?
- (c) Find a subset of the harmonic oscillator basis states $|n\rangle$ with $n = 0, 1, 2, \dots$ that forms a multiplet of \mathbf{J} with angular momentum j . Write explicitly your $|j, m\rangle$ states in terms of the oscillator basis states. Confirm that J_z and J_+ act as expected on these states.
- (d) Do the operators in part (a) commute with the harmonic oscillator Hamiltonian, $H = \hat{N} + \frac{1}{2}$? Explain your answer.

5. **A Four-Dimensional Harmonic Oscillator** [20 points]

Consider a particle of mass m in a four-dimensional isotropic harmonic oscillator:

$$\hat{H} = \hbar\omega(\hat{N}_1 + \hat{N}_2 + \hat{N}_3 + \hat{N}_4 + 2) = \hbar\omega(\hat{N} + 2),$$

$$\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1, \quad \hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2, \quad \hat{N}_3 = \hat{a}_3^\dagger \hat{a}_3, \quad \hat{N}_4 = \hat{a}_4^\dagger \hat{a}_4, \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2, 3, 4.$$

As usual \hat{a}_i^\dagger and \hat{a}_i are raising and lowering operators, \hat{N}_i are number operators and \hat{N} is the *total* number.

(a) Write out and count the states at the ground state and the first two excited levels.

We think of this oscillator as the tensor product of two 2-dimensional-SHO's, one for the first and second coordinates and the other for the third and fourth coordinates. We thus consider redefined creation and annihilation operators

$$\begin{aligned} \hat{a}_L &\equiv \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2), & \hat{a}_R &\equiv \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2), & [\hat{a}_L, \hat{a}_L^\dagger] &= [\hat{a}_R, \hat{a}_R^\dagger] = 1, \\ \hat{b}_L &\equiv \frac{1}{\sqrt{2}}(\hat{a}_3 + i\hat{a}_4), & \hat{b}_R &\equiv \frac{1}{\sqrt{2}}(\hat{a}_3 - i\hat{a}_4), & [\hat{b}_L, \hat{b}_L^\dagger] &= [\hat{b}_R, \hat{b}_R^\dagger] = 1, \\ \hat{N}_L^{(1)} &= \hat{a}_L^\dagger \hat{a}_L, & \hat{N}_R^{(1)} &= \hat{a}_R^\dagger \hat{a}_R, & \hat{N}_L^{(2)} &= \hat{b}_L^\dagger \hat{b}_L, & \hat{N}_R^{(2)} &= \hat{b}_R^\dagger \hat{b}_R. \end{aligned}$$

Associated with each 2D SHO we have an algebra of angular momenta: $\mathbf{J}^{(1)}$ for the first and $\mathbf{J}^{(2)}$ for the second. The states of each 2D SHO fall into representations of the corresponding angular momentum as described in the formula sheet.

(b) Consider the ground state and the next *three* excited levels of the full system. These states can be grouped into multiplets labeled by $j_1 \otimes j_2$, with j_1 and j_2 corresponding to $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$, respectively. Organize this information in a table that includes columns for the total number N , $j_1 \otimes j_2$ and the corresponding number of states. (You do not need to list the states within each multiplet, nor construct them using the raising operators.)

Consider now the perturbed Hamiltonian \hat{H}' defined by

$$\hat{H}' = \hat{H} + \frac{2E_0}{\hbar^2} \mathbf{J}^{(1)} \cdot \mathbf{J}^{(2)},$$

with $E_0 \ll \hbar\omega$, a very small positive energy.

(c) Calculate the effect of the above perturbation on the ground state and on the degeneracies at the first three excited levels. Give your results using an energy-level diagram showing the original energy levels and the splittings, indicating numbers of states and representations. *Make a large, full-page diagram, since this is where your results should be available for us to grade.*

6. **Entanglement generation and the tensor product** [25 points]

A pair of spin-1/2 particles interact via the Hamiltonian

$$H = \hbar\omega \sigma_3 \otimes \sigma_3,$$

where ω is a scalar with units of frequency. In this problem we will work with two different bases for the two-particle states. For a single spin-1/2 particle, let $|+\rangle \equiv |z; +\rangle$ and $|-\rangle \equiv |z; -\rangle$ as usual and define $|x; \pm\rangle$ to be the ± 1 eigenstates of σ_1 with the phase convention:

$$|x; +\rangle \equiv \frac{|+\rangle + |-\rangle}{\sqrt{2}} \quad \text{and} \quad |x; -\rangle \equiv \frac{|+\rangle - |-\rangle}{\sqrt{2}}.$$

For the two-particle states define the Z basis to be

$$|1\rangle = |++\rangle, |2\rangle = |+-\rangle, |3\rangle = |-+\rangle, |4\rangle = |--\rangle,$$

where $|++\rangle \equiv |+\rangle \otimes |+\rangle$, etc.

- (a) Write down the matrices H and $\exp(-iHt/\hbar)$ in the Z basis.
- (b) Calculate $e^{-iHt/\hbar}|x; +\rangle \otimes |x; +\rangle$ in the Z basis. For what values of t is this state *not* entangled?
- (c) The following expression

$$e^{i(K \otimes \mathbf{1} + \mathbf{1} \otimes K)}$$

can be written as a tensor product of two operators. Find those two operators.

- (d) Suppose we are given a state

$$|\psi\rangle \equiv \alpha |x; +\rangle \otimes |x; +\rangle + \beta |x; -\rangle \otimes |x; -\rangle, \quad \text{with } \alpha, \beta \in \mathbb{C}.$$

Suppose we would like to transform $|\psi\rangle$ into the state

$$|\psi'\rangle \equiv \alpha |++\rangle + \beta |--\rangle,$$

using a Hamiltonian H' of the form

$$H' = H_0 \otimes \mathbf{1} + \mathbf{1} \otimes H_0.$$

Specifically we would like to ensure that $e^{-iH't'/\hbar}|\psi\rangle = |\psi'\rangle$ for some H_0 and t' . To simplify notation, define the operator $K \equiv -H_0 t'/\hbar$. Show that K should thus satisfy

$$e^{i(K \otimes \mathbf{1} + \mathbf{1} \otimes K)}|\psi\rangle = |\psi'\rangle.$$

Find K and justify your answer.

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