

Quantum Physics II (8.05) Fall 2013

Assignment 9

Massachusetts Institute of Technology
Physics Department, November 8, 2013

Due Friday, November 15, 2013
3:00 pm

Problem Set 9

1. Arbitrary spin state [10 points]

Consider a spin state $|\mathbf{n}; +\rangle$ where $\mathbf{n} = (n_x, n_y, n_z)$ points in the direction defined by the angles (θ_0, ϕ_0) . Consider also the spin vector operator, defined as

$$\mathbf{S} \equiv \hat{S}_x \hat{\mathbf{x}} + \hat{S}_y \hat{\mathbf{y}} + \hat{S}_z \hat{\mathbf{z}},$$

Note that the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ along the coordinate axes are not operators on spin states. They can be moved across operators with impunity. Calculate the vector $\langle \mathbf{S} \rangle_{\mathbf{n}}$:

$$\langle \mathbf{S} \rangle_{\mathbf{n}} \equiv \langle S_x \rangle_{\mathbf{n}} \hat{\mathbf{x}} + \langle S_y \rangle_{\mathbf{n}} \hat{\mathbf{y}} + \langle S_z \rangle_{\mathbf{n}} \hat{\mathbf{z}},$$

where $\langle \dots \rangle_{\mathbf{n}}$ denotes expectation value in the spin state $|\mathbf{n}; +\rangle$. Your answer should be in terms of functions of the angles θ_0 and ϕ_0 . Relate the vector $\langle \mathbf{S} \rangle_{\mathbf{n}}$ to the vector \mathbf{n} .

Given another unit vector \mathbf{n}' , what is the value of $\langle \mathbf{S} \cdot \mathbf{n}' \rangle_{\mathbf{n}}$?

2. Rotating spin operators and spin states [10 points]

We have considered earlier the operator $R_\alpha(\mathbf{n})$ defined by

$$R_\alpha(\mathbf{n}) \equiv \exp\left(-\frac{i\alpha S_{\mathbf{n}}}{\hbar}\right) = \exp\left(-i\frac{\alpha}{2}\mathbf{n} \cdot \boldsymbol{\sigma}\right)$$

We have claimed that acting on the spin state $|\mathbf{n}'\rangle$ it gives another spin state $|\mathbf{n}''\rangle$ where \mathbf{n}'' is obtained from \mathbf{n}' by a rotation of angle α about the axis going along \mathbf{n} .

- (a) Assume $\alpha = \epsilon$, with ϵ an infinitesimal angle. Calculate the action of the rotation operator on the spin operator \mathbf{S} (defined in the previous problem) by completing the equation below

$$R_\epsilon^\dagger(\mathbf{n}) \mathbf{S} R_\epsilon(\mathbf{n}) = \mathbf{S} + \epsilon \dots + \mathcal{O}(\epsilon^2) \tag{1}$$

where you must find the $\mathcal{O}(\epsilon)$ term indicated by dots and $\mathcal{O}(\epsilon^2)$ terms can be neglected.

- (b) Use equation (1) and take the expectation in the spin state $|\mathbf{n}'; +\rangle$ to argue that the operator $R_\epsilon(\mathbf{n})$ does rotate spin states as claimed.

3. The Ammonia Molecule (15 points)

Consider an ammonia molecule placed in an electric field ε . The Hamiltonian, setting the average energy conventionally equal to zero, is given by

$$H = \begin{pmatrix} \mu\varepsilon & -\Delta \\ -\Delta & -\mu\varepsilon \end{pmatrix}.$$

Here μ is the electric dipole moment and $\Delta > 0$. We are using a basis where $|1\rangle = |\uparrow\rangle$ and $|2\rangle = |\downarrow\rangle$ are, respectively, those in which the nitrogen atom is above and below the plane of the hydrogens. These states are not energy eigenstates.

- (a) Show that the Hamiltonian may be written as

$$H = -K \mathbf{n} \cdot \boldsymbol{\sigma},$$

for some constant $K > 0$ and some unit vector \mathbf{n} defined by polar angles (θ, ϕ) that you should specify in terms of variables in the original Hamiltonian.

- (b) Find the energy eigenvalues and eigenstates. Write the eigenstates using the $|\uparrow\downarrow\rangle$ basis and the angles θ and ϕ .
- (c) Consider the cases $\mu\varepsilon \ll \Delta$, or effectively $\mu\varepsilon = 0$, and $\mu\varepsilon \gg \Delta$, or effectively $\Delta = 0$. In each case determine the values of θ , the energy eigenvalues, and eigenstates. Comment on your results.
- (d) Make a plot of the two energy eigenvalues as a function of ε . Include on your plot the limits you considered above.
- (e) Initially the electric field is turned off, $\varepsilon = 0$, and the system starts out in the state $|\psi(0)\rangle = |\downarrow\rangle$. Describe the evolution of the state viewed as a spin state rotating in time. At some later time we turn on a very strong electric field, so that $\mu\varepsilon \gg \Delta$. What is the further evolution of the state? What does turning on this strong electric field do to the probability of finding the system in the state $|\uparrow\rangle$?

4. Time Evolution in a Three-State System [15 points]

Carbon dioxide is a linear molecule (OCO) that can pick up an extra electron and become a negatively charged ion. Suppose that the electron would have energy E_O if it were attached to either oxygen atom, or energy E_C if it were attached to the carbon atom in the middle. Call these states $|L\rangle$, $|C\rangle$ and $|R\rangle$, for left oxygen, carbon, and right oxygen. The energy eigenstates need not, however, have either energy E_O or E_C because there is some probability that the electron may hop between an oxygen atom and the carbon atom. (Assume that the probability of jumping directly from oxygen to oxygen can be neglected.)

- (a) Write down a model Hamiltonian to describe the physics of the electron outlined above. You will need to introduce a new parameter, Δ , with units of energy, that characterizes the tunneling between C and O . Find the energy eigenvalues of this three-state system in terms of E_C , E_O , and Δ .
- (b) In this part and the next, assume $E_C = E_O$. Find the energy eigenstates.
- (c) Assume that at time $t = 0$ the electron is in state $|L\rangle$. That is, it is localized on the left oxygen atom. This is not a stationary state. What is the probability that at some later time t , the electron will be in state $|L\rangle$? In state $|C\rangle$? In state $|R\rangle$? Plot these three probabilities as functions of time.

5. **The evolution of the spin in NMR** [5 points]

An atom with spin $\frac{1}{2}$ and magnetic moment $\boldsymbol{\mu} = \gamma\mathbf{S}$ is placed in a time dependent magnetic field, tuned for resonance,

$$\mathbf{B}(t) = B_0\mathbf{z} + B_1(\mathbf{x} \cos \omega_0 t - \mathbf{y} \sin \omega_0 t), \quad \omega_0 = \gamma B_0.$$

Use the general solution for the evolution of a spin state, as given in lecture, to answer the following questions.

- (a) Assuming that initially the spin points in the positive z -direction, $|\psi(0)\rangle = |+\rangle$, give $|\psi(t)\rangle$ and describe it as a spin pointing in a time-dependent direction $\mathbf{n}(t)$ defined by time-dependent angles $\theta(t)$ and $\phi(t)$ that you should determine.
- (b) Use your answer above to eliminate t and write ϕ as a function of θ . Where does the spin point as it lies, for the first time, along the (x, y) -plane.

6. **Spin states for two particles** [10 points]

We examine states of two spin one-half particles. For this we have a tensor product vector space with basis states given by

$$|+\rangle \otimes |+\rangle, \quad |+\rangle \otimes |-\rangle, \quad |-\rangle \otimes |+\rangle, \quad |-\rangle \otimes |-\rangle$$

Consider now the entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle).$$

We want to establish that this is a rotationally invariant state. For this consider some rotation operator R with the property that $R|+\rangle = |\mathbf{n}; +\rangle$, where \mathbf{n} is some unit vector.

- (a) Why should it be that $R|-\rangle$ is (up to a phase) equal to $|\mathbf{n}; -\rangle$?
- (b) In the tensor product the rotation operator is $R \otimes R$. Show that $(R \otimes R)|\Psi\rangle$ is in fact equal to $|\Psi\rangle$ up to a phase. This is the statement of rotational invariance.

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