Quantum Physics II (8.05) Fall 2013
Assignment 8

Massachusetts Institute of Technology
Physics Department
November 2, 2013

Due Friday, November 8, 2013
3:00 pm

Problem Set 8

1. **Exercises with position and momentum eigenstates in the harmonic oscillator** [10 points]

   (a) Use the expression for $|x\rangle$ to calculate the wavefunction for the excited state $|2\rangle = \frac{1}{\sqrt{2}} a^\dagger a^\dagger |0\rangle$.

   (b) Construct explicitly the *momentum* eigenstates $|p\rangle$ of the harmonic oscillator. These states satisfy the familiar property $\hat{p}|p\rangle = p|p\rangle$. You may consult the derivation of the position eigenstates in the lecture notes.

2. **Coherent States of the Harmonic Oscillator** [10 points]

   This problem reviews coherent states. Some of the questions are answered in the lecture notes. For maximum benefit try using the notes as little as possible!

   For any complex number α we define the coherent state $|\alpha\rangle$ by

   $$|\alpha\rangle \equiv e^{\alpha \hat{a}^\dagger - \alpha^* a}|0\rangle \, .$$

   (a) Show that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

   (b) Write the state $|\alpha\rangle$ as a superposition of the energy eigenstates $|n\rangle$. What is the probability that a particle in the state $|\alpha\rangle$ has energy E_n?

   (c) Show that any two different coherent states $|\alpha\rangle$ and $|\beta\rangle$ are *not* orthogonal by calculating $\langle\beta|\alpha\rangle$.

   (d) Evaluate $\Delta H/\langle H\rangle$ on the state $|\alpha\rangle$ and show that it decreases as $|\alpha|$ increases.

   (e) Evaluatee $\langle\alpha|\hat{x}|\alpha\rangle$ and $\langle\alpha|\hat{p}|\alpha\rangle$. Evaluate Δx and Δp for the state $|\alpha\rangle$ and show that this is a minimum uncertainty state.

   (f) Expand the $t = 0$ state $|\alpha\rangle$ as a superposition of energy eigenstates to write an expression for the state at all subsequent time $t > 0$. Show that the state continues to be a coherent state for some value of $\alpha(t)$ that you must determine.

   (g) Compute the time-dependent expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{H}\rangle$, Δx and Δp assuming that at $t = 0$ we have a coherent state $|\alpha_0\rangle$ with α_0 real.

3. **More general squeezed states** [15 points] (no hats on the a's here!)

Consider the normalized squeezed vacuum state $|0_\gamma\rangle$, with $\gamma \in \mathbb{R}$ given by

$$|0_\gamma\rangle \equiv S(\gamma)\,|0\rangle\,, \quad \text{with} \quad S(\gamma) = \exp\Big(-\frac{\gamma}{2}(a^\dagger a^\dagger - aa)\Big)\,.$$

For coherent states we used the operator $D(\alpha)$ acting on the vacuum

$$D(\alpha) = \exp\big(\alpha\, a^\dagger - \alpha^* a\big)\,, \quad |\alpha\rangle = D(\alpha)|0\rangle\,.$$

More general squeezed states $|\alpha, \gamma\rangle$ arise by first squeezing and then translating:

$$|\alpha, \gamma\rangle \equiv D(\alpha)S(\gamma)|0\rangle\,.$$

Note that $|0, \gamma\rangle = |0_\gamma\rangle$ and $|\alpha, 0\rangle = |\alpha\rangle$.

(a) Calculate the operator $a(\gamma)$ defined by the relation

$$a(\gamma) = S^\dagger(\gamma)\, a\, S(\gamma)\,.$$

The way to do this is to derive a (second-order) differential equation for $a(\gamma)$. The answer should express $a(\alpha)$ as a linear combination of a and $a^\dagger$. Write also the corresponding expression for $a^\dagger(\gamma) \equiv (a(\gamma))^\dagger$ and check the value of the commutator $[a(\gamma)\,,\, a^\dagger(\gamma)]$.

(b) Calculate the expectation value $\langle N\rangle$ of the number operator $N = a^\dagger a$ and its uncertainty ΔN on the squeezed vacuum state $|0_\gamma\rangle$. The answers should just be functions of γ. Find the ratio $\Delta N/\langle N\rangle$ and plot it as a function of γ. Can this ratio be made small?

(c) Calculate the expectation value $\langle N\rangle$ on the generalized state $|\alpha, \gamma\rangle$. Here α can be complex.

(d) In class we showed that the electric field operator at some point in space for a single-mode electromagnetic field of frequency ω takes the form

$$\widehat{E}(t) = \mathcal{E}_0\Big(ae^{-i\omega t} + a^\dagger e^{i\omega t}\Big)\,,$$

with $\mathcal{E}_0$ a real constant. Calculate the expectation value $\langle \widehat{E}(t)\rangle$ and the uncertainty $\Delta E(t)$ on the state $|\alpha, \gamma\rangle$.

4. **Maxwell's equations and photon states** [10 points]

   Consider the electromagnetic fields on the cavity as given in eqn. (5.38) of the updated lecture notes (Section 5.3). Note that we are working with conventions in which E and cB have the same units. Write the set of four Maxwell equations (in the appropriate units!) and derive the conditions on $q(t)$ and $p(t)$ that they imply.

   Derive the equations of motion for the Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$ using (5.40) and show that they are in fact the same conditions arising from Maxwell's equations.

   This shows that the Hamiltonian $H(\hat{q}, \hat{p})$ provides a good quantum theory of the classical EM mode in the cavity.

5. **Time Evolution in a Two-State Problem** [10 points] (based on Sakurai's 2.9.)

   A box containing a particle is divided into right and left compartments by a partition. If the particle is known to be on the left side with certainty, we call the state $|L\rangle$; if on the right, we call the state $|R\rangle$. (Of all possible $|L\rangle$ and $|R\rangle$ states, we consider only the ones with the lowest energy, thus we have a two-state problem.) Assume that the box is symmetric and shift the zero of energy so that $\langle L|\hat{H}|L\rangle = \langle R|\hat{H}|R\rangle = 0$. The Hamiltonian does not vanish, however, because the particle can tunnel through the partition, as allowed by the coupling

   $$\hat{H} = \Delta(\,|L\rangle\langle R| + |R\rangle\langle L|\,)$$

   where Δ is a real number with the dimension of energy.

   (a) Find normalized energy eigenstates and the corresponding energy eigenvalues.

   (b) Suppose that at time $t = 0$ the state vector is given by

   $$|\psi, 0\rangle = c_L|L\rangle + c_R|R\rangle\ .$$

   Find $|\psi, t\rangle$ by applying the time evolution operator.

   (c) Suppose that at time $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

   (d) Suppose someone made an error and wrote $\hat{H}$ as

   $$\hat{H} = \Delta|L\rangle\langle R|\ .$$

   What's wrong with this Hamiltonian. Evolving with $\exp(-iHt/\hbar)$, as you did in part (b), show that probability is not conserved.

6. **Altering an oscillation** [10 points]

   Consider a two-state system with basis states $|1\rangle$ and $|2\rangle$ and a Hamiltonian

   $$H = \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix} = -\Delta\,\sigma_1\,, \quad \text{with} \quad \Delta > 0\,.$$

   As you know, if the system is initially in the state $|1\rangle$ then the probability that the state is in $|1\rangle$ varies periodically between one and zero as a function of time.

   Add to H a time-independent term along σ_3 so that, instead, the probability that the system is in state $|1\rangle$ varies periodically between one and some minimum value $p_{min} > 0$. Write the new term in the Hamiltonian in terms of Δ and p_{min}. Hint: It might be useful to write the Hamiltonian as the dot product of a vector with the spin operator and to think about time evolution as precession.

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