Quantum Physics II (8.05) Fall 2013
Assignment 11

Massachusetts Institute of Technology
Physics Department
29 November 2013

Due Friday, December 6, 2013
3:00 pm

Problem Set 11

1. **Measurement of Angular Momentum for a particle with $\boldsymbol{\ell}$ =1** [10 points]

   The purpose of this problem is to generalize the analysis for Stern Gerlach experiments with a two-state spin-1/2 system to a three-state spin-1 system.

   A quantum particle is known to have total angular momentum one, *i.e.* $\ell = 1$.

   (a) Use the eigenstates of L^2 and L_3 (denote them as $|\ell, m\rangle = |1, m\rangle$) as a basis and find the *matrix* representation of the operators, L^2, L_+, L_-, L_1, L_2 and L_3 in this three dimensional subspace.

   [Hint: You know the action of the operators L_3, and $L_\pm$ on the states $|1, m\rangle$.]

   (b) Verify that the matrices in part a) satisfy the commutator $[L_1, L_2] = i\hbar L_3$.

   (c) Find expressions for the L_1 eigenstates $|1, m_1\rangle$ with $m_1 = 1, 0, -1$, as superpositions of L_3 eigenstates. [Hint: Consider the eigenvectors of the matrix representation for L_1 in the $|1, m\rangle$ basis.]

   (d) If a particle is in the state $|1, m_1 = 1\rangle$ and a measurement is made of the L_3 component of its angular momentum, what are the possible results and the assoicated probabilities?

   (e) A particle is in the state $|1, m_1 = 1\rangle$. The L_3 component of its angular momentum is measured and the result $m_3 = -1$ is obtained. Immediately afterwards, the L_1 component of angular momentum is measured. Explain what results are obtained and with what probability. Suppose you measured $m_1 = -1$, and now you decide to measure L_3 again. What are the possible outcomes and with what probability?

2. **A curious rewriting of the Hydrogen Hamiltonian.** [10 points]

   Consider the hydrogen atom Hamiltonian

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{e^2}{r}\,.$$

   We will write it as

$$H = \gamma + \frac{1}{2m}\sum_{k=1}^{3}\Big(\hat{p}_k + i\beta\,\frac{\hat{x}_k}{r}\Big)\Big(\hat{p}_k - i\beta\,\frac{\hat{x}_k}{r}\Big)\,,$$

   where $\hat{p}_k$ and $\hat{x}_k$ are, respectively, the Cartesian components of the momentum and position operators, and β and γ are real constants to be adjusted so that the two Hamiltonians are the same.

   (a) Calculate the constants β and γ. Express them in terms of e^2 the Bohr radius a_0 and other constants.

   (b) Explain why for any state $\langle H \rangle \geq \gamma$. Find the wavefunction of the state for which this energy inequality is saturated. This is the ground state of Hydrogen. Give the normalized wavefunction.

3. **β-Decay of Tritium** [5 points]

   An electron is in the ground state of tritium, for which the nucleus is made up of a proton and two neutrons. A weak interaction (β-decay) instantaneously changes the nucleus to ^{3}He, which has two protons and one neutron. What is the probability that the electron is in the ground state of ^{3}He immediately after the decay? (Ignore the tiny change in reduced mass.) What is the probability that it is in some state with $\ell = 0$? [No calculation should be required to answer the latter question.]

4. **The finite spherical well** [20 points]

A particle of mass m is in a potential $V(r)$ that represents a finite depth spherical well of radius a:

$$V(r) = \begin{cases} -V_0 & \text{for } r < a\,, \\ 0 & \text{for } r > a\,, \end{cases}$$

where V_0 is a positive constant with units of energy. Given the parameters m, a and $\hbar$ the natural energy is $\hbar^2/(2ma^2)$. Thus, for convenience we will write

$$V_0 \;=\; (v_0)^2\,\frac{\hbar^2}{2ma^2}\,,$$

where $v_0 > 0$ is a unit free number that tells us how much bigger is V_0 compared to the natural energy scale. The larger v_0 the deeper the potential.

(a) For the potential to have bound states it should be deep enough. Show that $v_0 = \frac{\pi}{2}$. Hint: Barely having a bound state means that the lowest $\ell = 0$ state must have a energy of essentially zero.

(b) Consider now the general problem of finding the $\ell = 0$ bound states when V_0 is deep enough to have them. For the energy eigenvalues E use the notation

$$E \;=\; -\,e^2\,\frac{\hbar^2}{2ma^2}\,,$$

where $e > 0$ are unit free constants that encode the energy (in terms of the natural energy). Show that the energy eigenvalues are determined by the equations

$$\begin{aligned} \eta^2 + e^2 &= v_0^2\,, \\ -\eta\cot\eta &= e\,, \end{aligned} \tag{1}$$

where $\eta > 0$ is a unit-free constant you will be led to introduce.

(c) Perform a graphical analysis of the above equations by plotting η along the horizontal axis and e along the vertical axis. Sketch the curves corresponding to the second equation and a few illustrative curves corresponding to the first equation, for a few values of v_0. Confirm your result of part (a), namely, no solutions for $v_0 < \pi/2$. Assume now that v_0 is a large number. Show that the lowest energy bound states, for low integers n take values

$$E \;=\; -V_0 + (n\pi)^2\,\frac{\hbar^2}{2ma^2}\,,$$

For the $n = 1$ find a little better approximation to the energy, including the first nontrivial correction that would vanish as $v_0 \to \infty$.

(d) Now consider the delta function potential

$$V(r) = -\frac{4\pi}{3}(V_0 L^3)\, \delta(x)\,,$$

where $V_0 > 0$ is a constant with units of energy and L is a constant with units of length that is needed for dimensional reasons. It is not easy to solve this problem so we will regulate it by replacing this potential by a potential $V_a(r)$ of the form

$$V_a(r) = \begin{cases} -V_0\left(\frac{L}{a}\right)^3 & \text{for } \; r < a\,, \\ 0 & \text{for } \; r > a\,, \end{cases}$$

The regulator parameter is a. It is not a parameter of the theory, we had to introduce it to represent the delta function. Confirm that $\int d^3x V(r) = \int d^3x V_a(r)$ which means that this is a good representation of the delta function as $a \to 0$.

A successful regulation would mean that the bound state energies are independent of the artificial regulator a in the limit as $a \to 0$. Show that this does *not* happen.

Another way to show that there is a serious problem is by simple dimensional analysis. The parameters of this theory are $\hbar$, m, and the quantity $V_0 L^3$ (not V_0 and L separately). Use dimensional analysis to construct the "natural" energy of the bound states. Then argue that this result is absurd!

5. **Qualitative behavior of the radial wavefunction** [10 points]

Consider a particle of mass m moving under the influence of an attractive Yukawa potential $V(r) = -ge^{-\alpha r}/r$.

(a) Write the Schrödinger equation for the radial wavefunction, $u(r)$. Define a dimensionless radial variable $x \equiv \alpha r$ and rewrite the radial equation in the form

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} - g'\frac{e^{-x}}{x}\right] u(x) = \lambda u(x).$$

What is the effective potential $V_{eff}(x)$ for the scaled equation? Relate λ and g' to the original parameters of the problem.

(b) Show graphically that the parameter g' can be chosen so the effective potential has a) no bound states, or b) many bound states (for any $\ell \neq 0$). Note that it is the interplay between the interaction $V(r)$ and the angular momentum barrier, $\hbar^2\ell(\ell+1)/2mr^2$ that determines whether and how many bound states occur.

(c) Suppose the parameters are such that there are four distinct bound energy levels with $\ell = 2$ for this problem. Sketch as accurately as you can the wavefunction of the second most tightly bound energy level with $\ell = 2$. Do not solve the Schrödinger equation numerically. Instead use your physical understanding of the solutions to the equation. You should include: the behavior as $x \to 0$, the behavior as $x \to \infty$, the correct number of nodes, the relative magnitude of the wavefunction at large, small, and intermediate x.

6. **Hyperfine splitting of hydrogen ground state in a magnetic field** [10 points]

For a magnetic field of magnitude B along the z-direction, the hydrogen atom Hamiltonian relevant to the ground state with hyperfine spitting has the additional terms terms

$$H' \;=\; \frac{2\epsilon}{\hbar}\,(\mathbf{S}_e)_z \;+\; \frac{4\epsilon'}{\hbar^2}\,\mathbf{S}_e\cdot\mathbf{S}_p\,, \tag{1}$$

where $\mathbf{S}_e$ and $\mathbf{S}_p$ are the electron and proton spins, respectively. Moreover, ϵ and ϵ' are positive constants with units of energy. In particular $\epsilon = \mu_B B$.

There are two natural basis in this problem. The uncoupled basis $|m_e, m_p\rangle$ where the first entry refers to the electron and the second entry refers to the proton:

$$\text{Uncoupled basis: } |1\rangle = |\uparrow\uparrow\rangle,\ \ |2\rangle = |\uparrow\downarrow\rangle,\ \ |3\rangle = |\downarrow\uparrow\rangle,\ \ |4\rangle = |\downarrow\downarrow\rangle\,. \tag{2}$$

There is the coupled basis $|jm\rangle$ of eigenstates of J^2 and J_z, where $\mathbf{J} = \mathbf{S}_e + \mathbf{S}_p$

$$\text{Coupled basis: } |1\rangle = |1,1\rangle,\ \ |2\rangle = |1,0\rangle,\ \ |3\rangle = |1,-1\rangle,\ \ |4\rangle = |0,0\rangle\,. \tag{3}$$

(a) Find the matrix elements of H' in the uncoupled basis. Calculate the energy eigenvalues and the eigenvectors.

(b) Find the matrix elements of H' in the coupled basis. Calculate the energy eigenvalues and the eigenvectors.

(c) Sketch the energy eigenvalues as a function of the magnetic field. What basis is more suitable for small magnetic fields and which basis is more suitable for large magnetic fields?

(d) Find the energy eigenvalues and eigenstates, correct to first order in the magnetic field B, when this magnetic field is small (the eigenstates need not be normalized).

PRACTICE PROBLEMS – DO NOT HAND IN

1. **Quantum conservation of the Runge-Lenz vector**. [10 points]

For the Hydrogen atom Hamiltonian

$$H \;=\; \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r}\,.$$

there is a conserved Runge-Lenz vector

$$\mathbf{R} \;\equiv\; \frac{1}{2m}\big(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\big) - \frac{e^2}{r}\mathbf{r}\,.$$

This is a Hermitian operator. In this problem we want to show that $\mathbf{R}$ is indeed conserved:

$$\big[\,\mathbf{R}\,, H\,\big] \;=\; 0\,. \tag{1}$$

To do the computation without it becoming a mess, and to gain perspective we will do some general computations that make (1) seem less accidental.

(a) Show the following commutator identity, valid for arbitrary function $f(r)$

$$\big[\,\mathbf{p}^2\,, f(r)\,\mathbf{r}\,\big] \;=\; \frac{\hbar}{i}\bigg(\,(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}\,\frac{f'(r)}{r} \;+\; \frac{f'(r)}{r}\,\mathbf{r}\,(\mathbf{r}\cdot\mathbf{p}) \;+\; \mathbf{p}\,f(r) + f(r)\,\mathbf{p}\,\bigg)\,. \tag{2}$$

Explain why the operator in parenthesis on the right-hand side is Hermitian. How is that consistent with the left-hand side being the commutator of two Hermitian operators?

(b) Now compute the commutator

$$\big[\,\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\;,\;f(r)\,\big] \;=\; \frac{\hbar}{i}\bigg(\;\ldots\qquad\ldots\qquad\bigg)\,. \tag{2}$$

Your goal as you compute this is to get a Hermitian operator inside the parenthesis. Its structure should end up to be analogous to that in (2).

(c) Attempt to make the problem more general by setting

$$H_f \;\equiv\; \frac{\mathbf{p}^2}{2m} - f(r)\,,\quad \mathbf{R}_f \;\equiv\; \frac{1}{2m}\big(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\big) \;-\; f(r)\,\mathbf{r}\,.$$

Compute $[H_f\,, \mathbf{R}_f]$ with the help of the results in parts (a) and (b). Show that one gets a vanishing commutator if and only if

$$r f'(r) \;=\; -f(r)\,.$$

Verify that the unique solution of this equation is $f(r) = c/r$ with c an arbitrary constant.

2. **Length-squared of the quantum Runge-Lenz vector**. [10 points]

   It is convenient to rescale the Runge-Lenz vector for it to have no units. Thus we take

   $$\mathbf{R} \equiv \frac{1}{2me^2}\big(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\big) - \frac{\mathbf{r}}{r}\,.$$

   Check that the vector can be alternatively written as

   $$\begin{aligned}\mathbf{R} &= \frac{1}{me^2}\big(\ \mathbf{p}\times\mathbf{L} - i\hbar\,\mathbf{p}\big) - \frac{\mathbf{r}}{r}\,,\\ &= \frac{1}{me^2}\big(-\mathbf{L}\times\mathbf{p} + i\hbar\,\mathbf{p}\big) - \frac{\mathbf{r}}{r}\,.\end{aligned}$$

   In this problem we want to calculate $\mathbf{R}^2$:

   $$\mathbf{R}^2 = \Big(\frac{1}{me^2}\big(-\mathbf{L}\times\mathbf{p} + i\hbar\,\mathbf{p}\big) - \frac{\mathbf{r}}{r}\Big)\cdot\Big(\frac{1}{me^2}\big(\ \mathbf{p}\times\mathbf{L} - i\hbar\,\mathbf{p}\big) - \frac{\mathbf{r}}{r}\Big)\,.$$

   To do the computation more easily first prove the following identities

   $$\begin{aligned}(\mathbf{L}\times\mathbf{p})\cdot\mathbf{r} &= -\mathbf{L}^2\\ i\hbar\Big(\mathbf{p}\cdot\frac{\mathbf{r}}{r} - \frac{\mathbf{r}}{r}\cdot\mathbf{p}\Big) &= \frac{2\hbar^2}{r}\,,\\ (\mathbf{L}\times\mathbf{p})\cdot(\mathbf{p}\times\mathbf{L}) &= -\mathbf{p}^2\,\mathbf{L}^2\end{aligned}$$

   Now show that

   $$\mathbf{R}^2 = 1 + \frac{2}{me^4}\,H\big(\mathbf{L}^2 + \hbar^2\big)\,.$$

3. **Clebsch-Gordan Coefficients for $\mathbf{1}\otimes\frac{\mathbf{1}}{\mathbf{2}}$**. [16 points]

   Consider a spin 1/2 particle in a state with orbital angular momentum $\ell = 1$. Construct states of definite total angular momentum from simultaneous eigenstates of orbital angular momentum and spin. Label the eigenstates in the "uncoupled basis" (ie eigenstates of L^2, S^2, L_z, and S_z) by $|\ell\, s\, m_\ell\, m_s\rangle$. Label the states in the "coupled basis" (ie eigenstates of J^2 and J_z) by $|j, m\rangle$.

   Note: All $|jm\rangle$ states were computed in lecture. If you want to use those, this exercise begins with part (g).

   (a) Find the state with maximum j and m $(= j_{max})$ in terms of the $|\ell\, s\, m_\ell\, m_s\rangle$ states.

   (b) Use $J_- = L_- + S_-$ to generate all the $|j_{max}, m\rangle$ states.

   (c) Use orthonormality to find the state $|j_{max}-1, j_{max}-1\rangle$. (To facilitate comparison with the tables, use the phase convention discussed in lecture.)

   (d) Use J_- to generate all the states $|j_{max} - 1, m\rangle$.

(e) Repeat steps (c) and (d) for smaller j's as many times as necessary.

(f) Check you results with the table at the end of this problem set (or Table 4.8 of Griffiths).

(g) What is the expectation value of L_z in the state with $j = 1/2$, $m = 1/2$? What is the expectation value of S_z in this state?

(h) Suppose that this particle moves in an external magnetic field in the z-direction, $\vec{B} = B\vec{e}_z$. Assume the particle is an electron, and take $g = 2$. The Hamiltonian describing the interaction of the electron with the field is

$$H_B = \frac{\mu_B}{\hbar}\vec{B}\cdot(\mathbf{L} + 2\mathbf{S}) \, .$$

What is $\langle H_B \rangle$ in each of the eigenstates $|j, m\rangle$?

(i) For the eigenstate $j = 1/2$, $m = 1/2$, what are the possible values of the magnetic energy and what are their probabilities?

4. **Addition of angular momentum** [10 points]

Consider the addition of angular momentum for two particles each of angular momentum j. We write $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ for the total angular momentum in terms of the angular momentum of the first and second particles. As you know the result is

$$j \otimes j = (2j) \oplus (2j-1) \oplus \ldots \oplus 0 \, .$$

(a) Construct the (normalized) states of highest and second highest J_z for total angular momentum $2j$.

(b) Construct the (normalized) states of highest and second highest J_z for total angular momentum $2j - 1$.

(c) Consider the states in (a). Are they symmetric, antisymmetric, or neither, under the exchange of the two particles? Answer the same question for the states in (b).

(d) Do you expect all states in the $2j$ multiplet and all states in the $2j-1$ multiplet to have the same exchange property? Explain.

5. **General addition of L and S**

Consider two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ and states $j_1 = \ell$ tensored with $j_2 = 1/2$. We define $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$. Derive a formula for the 'coupled' basis state

$$\left| j = \ell + \tfrac{1}{2}\,, m = M + \tfrac{1}{2} \right\rangle ,$$

where M is an integer in the range $-\ell - 1 \leq M \leq \ell$, in terms of suitable superpositions of uncoupled states $|j_1 j_2; m_1, m_2\rangle$. Your general formula should reduce to familiar results when $M = \ell$ and $M = -\ell - 1$. Since M is arbitrary, the strategy of using lowering or raising operators is not suitable. Hint: use $\mathbf{J}^2$. You can also consult Shankar p.414 for a solution.

6. **Hamiltonian for three spin-1 particles** [10 points]

   Consider 3 distinguishable **spin-1** particles, called 1,2, and 3, with spin operators $\mathbf{S}_1, \mathbf{S}_2$ and $\mathbf{S}_3$, respectively. The spins are placed along a circle and the interactions are between nearest neighbors. The Hamiltonian takes the form

   $$H = \frac{\Delta}{\hbar^2}\Big(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1\Big)\,,$$

   with $\Delta > 0$ a constant with units of energy. For this problem it is useful to consider the total spin operator $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$.

   (a) What is the dimensionality of the state space of the three combined particles. Write the Hamiltonian in terms of squares of spin operators.

   (b) Determine the energy eigenvalues for H and the degeneracies of these eigenvalues.

   (c) Calculate the ground state, expressing it as a superposition of states of the form

   $$|m_1, m_2, m_3\rangle \;\equiv\; |1, m_1\rangle \otimes |1, m_2\rangle \otimes |1, m_3\rangle\,,$$

   where $\hbar m_i$ is the eigenvalue of $(S_z)_i$ and applying some suitable constraint. [Hint: The general superposition with arbitrary coefficients has 7 candidate states. Show that the coefficient of $|0,0,0\rangle$ is zero and determine all others. Write your answer as a normalized state.]