Quantum Physics II (8.05) Fall 2013
Assignment 10

Massachusetts Institute of Technology
Physics Department
17 November 2013

Due Tuesday, 26 November 2013
6:00 pm

Reading

- Griffiths §4.1, §4.2, and §4.3 on separation of variables and angular momentum.
- For further reading on angular momentum with more emphasis on operators than Griffiths, you should consult one or more of the following:

  Ohanian Ch.7.

  Shankar (Second Edition) §12.5. If you want to read more about the importance of angular momentum in the theory of rotations you can read the material in Shankar surrounding §12.5.

Problem Set 10

1. **Another view on Bell's inequalities.** [10 points]

   Let the singlet state stand for the entangled total-spin-zero state of two spins, and let $\mathbf{a}$ and $\mathbf{b}$ denote two unit vectors. Following the EPR logic consider the correlation coefficient $C(\mathbf{a}, \mathbf{b})$ that takes the average, over an 'ensemble' of singlet states, of the product of the measured spin of particle one along $\mathbf{a}$ and the measured spin of particle two along $\mathbf{b}$:

   $$C(\mathbf{a}, \mathbf{b}) \equiv \left[\frac{4}{\hbar^2}\mathbf{S}_{\mathbf{a}}^{(1)}\mathbf{S}_{\mathbf{b}}^{(2)}\right]_{av}.$$

   Since any value of measured spin can be only $\pm\hbar/2$, this ensemble average must range between minus one and plus one.

   Now consider three directions $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ and the following quantity to be measured

   $$g(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv -\frac{4}{\hbar^2}\mathbf{S}_{\mathbf{a}}^{(1)}\mathbf{S}_{\mathbf{b}}^{(1)}\left(1 - \frac{4}{\hbar^2}\mathbf{S}_{\mathbf{b}}^{(1)}\mathbf{S}_{\mathbf{c}}^{(1)}\right).$$

   Note that all superscripts refer to particle one. In the sense of EPR all of the $\mathbf{S}_{\mathbf{n}}^{(i)}$ are not operators but rather denote possible measured values. Thus for any unit vector $\mathbf{n}$ we have $\mathbf{S}_{\mathbf{n}}^{(1)}\mathbf{S}_{\mathbf{n}}^{(1)} = \hbar^2/4$ and the anticorrelations of singlets imply that one can use

   $$\mathbf{S}_{\mathbf{n}}^{(1)} = -\mathbf{S}_{\mathbf{n}}^{(2)}.$$

(a) Show that

$$\big[g(\mathbf{a},\mathbf{b},\mathbf{c})\big]_{av} \;=\; C(\mathbf{a},\mathbf{b}) - C(\mathbf{a},\mathbf{c})$$

(b) Using the inequality

$$\big|[g]_{av}\big| \leq \big[\,|g|\big]_{av}$$

where $|\ldots|$ denote absolute values, show that

$$\big|C(\mathbf{a},\mathbf{b}) - C(\mathbf{a},\mathbf{c})\big| - C(\mathbf{b},\mathbf{c}) \leq 1 \tag{1}$$

This is an inequality that must be obeyed by a good theory in the sense of EPR.

(c) Turning to quantum mechanics, $\mathbf{S}_{\mathbf{a}}^{(1)}$ and $\mathbf{S}_{\mathbf{b}}^{(2)}$ are spin operators of the first and second particle, respectively, along the $\mathbf{a}$ and $\mathbf{b}$ unit vectors. Moreover, the coefficient $C(\mathbf{a},\mathbf{b})$ is the expectation value of their operator product in the singlet state

$$C(\mathbf{a},\mathbf{b}) \;\equiv \left\langle \frac{4}{\hbar^2}\mathbf{S}_{\mathbf{a}}^{(1)}\mathbf{S}_{\mathbf{b}}^{(2)} \right\rangle .$$

Prove that

$$C(\mathbf{a},\mathbf{b}) \;=\; -\cos\theta_{ab}\,,$$

where θ_{ab} is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

(d) Let $\mathbf{a},\mathbf{b},\mathbf{c}$ be coplanar vectors with $\theta_{ab} = \theta_{bc} = \theta$ and $\theta_{ac} = 2\theta$. Plot the left-hand side of (1) as a function of θ and show that the inequality is violated for $\theta \leq \pi/2$.

2. **Beating the odds using entangled states.** [10 points]

Consider the entangled state $|\Psi\rangle$ of two spins given by

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|+\rangle\otimes|+\rangle \;+\; |-\rangle\otimes|-\rangle\big) \tag{1}$$

Given a state $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ we define $|\bar\psi\rangle \equiv \alpha^*|+\rangle + \beta^*|-\rangle$.

(a) Consider a spin state $|v_a\rangle$ of the first particle and a spin state $|w_b\rangle$ of the second particle. Show that the probability $P(v_a, w_b)$ of getting those states upon measurement on the entangled state is

$$P(v_a, w_b) \;=\; \tfrac{1}{2}\big|\langle\bar v_a|w_b\rangle\big|^2 \tag{2}$$

We now consider a game in which Alice and Bob, who are far away from each other and incommunicado, play a game against Charlie. The game uses bits, which are variables that can take two values: zero or one. In a given round of the game, Charlie supplies a bit x to Alice and a bit y to Bob. Alice does not know what bit Bob got, and Bob does not know what bit Alice got. Alice must use the bit x to output a bit

$a(x)$ and Bob must use his bit y to output a bit $b(y)$. Alice and Bob win the round against Charlie if

$$a + b \equiv xy \pmod 2$$

Thus, for example, if $x = y = 1$ the right hand side is one and winning requires $a \neq b$ ($a = 0, b = 1$ or $a = 1, b = 0$). If either x or y is zero, the right-hand side is zero, and winning works for $a = b$ (either both one or both zero). A good strategy for winning is for Alice and Bob to output $a = b = 1$ (or zero) for all inputs x, y. The left-hand side is then zero and this will win for all cases except $x = y = 1$. Thus the probability of winning is then $3/4$.

(b) In a classical strategy Alice chooses a function $a(x)$ for her output bit. She has four choices that can be represented by the functions $\{0, 1, x, 1-x\}$. Explain why. Similarly, Bob's choice of a function $b(y)$ runs over the possibilities $\{0, 1, y, 1-y\}$. For any choice they make,

$$a(x) + b(y) \equiv c_0 + c_1 x + c_2 y \pmod 2 ,$$

with some constants c_0, c_1, and c_2 that can take values zero or one. Show that there is no choice of these constants for which the right-hand side above equals xy (mod 2), thus giving a strategy that wins always. Explain why this result implies that no (classical) strategy has a winning probability bigger than $3/4$.

Now Alice and Bob device a quantum strategy using their shared entangled pair (1). Alice will make use a basis

$$\begin{aligned} |v_0^x\rangle &\equiv \cos\tfrac{\alpha_x}{2}|+\rangle + \sin\tfrac{\alpha_x}{2}|-\rangle , \\ |v_1^x\rangle &\equiv -\sin\tfrac{\alpha_x}{2}|+\rangle + \cos\tfrac{\alpha_x}{2}|-\rangle . \end{aligned}$$

Since the bit x takes two values, Alice actually has a couple of bases: one for $x = 0$, comprised by the two states above with α_0 and another for $x = 1$, comprised by the two states above for α_1. Similarly, Bob will use

$$\begin{aligned} |w_0^y\rangle &\equiv \cos\tfrac{\beta_y}{2}|+\rangle + \sin\tfrac{\beta_y}{2}|-\rangle , \\ |w_1^y\rangle &\equiv -\sin\tfrac{\beta_y}{2}|+\rangle + \cos\tfrac{\beta_y}{2}|-\rangle . \end{aligned}$$

Since y takes two values, Bob also has a couple of bases: one for $y = 0$, comprised by the two states above with β_0 and another for $y = 1$, comprised by the two states above for β_1.

The quantum strategy is now as follows. For any given x Alice measures her entangled spin along the basis $(|v_0^x\rangle, |v_1^x\rangle)$. If she finds the spin along the first basis vector she outputs $a = 0$, if she finds the spin along the second basis vector she outputs $a = 1$. For a given y Bob measures his entangled spin along the basis $(|w_0^y\rangle, |w_1^y\rangle)$. If he finds the spin along the first basis vector he outputs $b = 0$, if he finds the spin along the second basis vector she outputs $b = 1$. The strategy requires fixing $\alpha_0, \alpha_1, \beta_0$, and β_1.

(c) Show that with the above strategy the probability $P[\, a = b \,|\, x, y \,]$ that a is equal to b for input values x, y is given by

$$P[\, a = b \,|\, x, y \,] \;=\; \cos^2 \tfrac{1}{2}(\alpha_x - \beta_y)\,.$$

(d) Find a formula for the probability P of winning in terms of the four unknown angles $\alpha_0, \alpha_1, \beta_0$, and β_1. If you get the right answer, you should find that

$$P = \cos^2 \tfrac{\pi}{8} = \tfrac{1}{2} + \tfrac{1}{2\sqrt{2}} \;\simeq\; 0.85355\,.$$

for $\alpha_0 = 0, \alpha_1 = \pi/2$, and $\beta_0 = \pi/4, \beta_1 = -\pi/4$. This gives about 10% better chances of winning than classically. Note that in the quantum strategy the outputs a, b for a given x, y are not deterministic.

3. **Properties of the Angular Momentum Operators** [20 points]

   Solve exercises one through seven in the Angular Momentum notes (Chapter 9).

4. **Orbital angular momentum does not have half-integer eigenvalues** [10 points] Griffiths Problem 4.57, p. 197.

5. **A three-dimensional angular momentum in the two-dimensional oscillator** [15 points]

   In an earlier p-set we considered an isotropic two-dimensional harmonic oscillator with operators $(a_x, a_x^\dagger)$ and $(a_y, a_y^\dagger)$. By introducing linear combinations

$$a_L = \frac{1}{\sqrt{2}}(a_x + i a_y)\,, \qquad a_R = \frac{1}{\sqrt{2}}(a_x - i a_y)\,,$$

as well as the definitions

$$N_L \;=\; a_L^\dagger a_L\,, \quad N_R \;=\; a_R^\dagger a_R\,,$$

we found that the Hamiltonian H and the angular momentum $L_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ take the form

$$\begin{aligned} H &= \hbar\omega\big(N_R + N_L + 1\big) \;=\; \hbar\omega\big(a_R^\dagger a_R + a_L^\dagger a_L + 1\big)\,, \\ L_z &= \hbar\big(N_R - N_L\big) \;=\; \hbar\big(a_R^\dagger a_R - a_L^\dagger a_L\big)\,. \end{aligned}$$

We are going to show that there is a "hidden" three-dimensional algebra of angular momentum here. The operators are going to be

$$\begin{aligned} J_z &= \alpha\, L_z \;=\; \alpha\,\hbar\big(N_R - N_L\big)\,, \\ J_+ &= \beta\,\hbar\, a_R^\dagger a_L\,, \\ J_- &= \beta\,\hbar\, a_L^\dagger a_R\,, \end{aligned}$$

where α and β are (real) constants to be determined. Note that, as required, $J_+^\dagger = J_-$.

(a) Determine the constants α and β by the condition that the operators above obey the algebra of angular momentum.

(b) Show that all J_i's commute with the Hamiltonian (they are conserved!).

(c) Associated to this angular momentum there are states $|j, m\rangle$ with $J^2 = \hbar^2 j(j+1)$ and $J_z = \hbar m$. Show that the state $(a_R^\dagger)^n|0\rangle$ is a state $|j, m\rangle$ with $j = m = n/2$.

(d) Describe the full spectrum of states of the two-dimensional harmonic oscillator in terms of representations of the 'hidden' angular momentum.

(e) Comment why, in retrospect, the constant α could not have been equal to one. This shows that the hidden angular momentum is really well hidden!

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