MIT Course 8.033, Fall 2005, Supplement
Max Tegmark

Matrix Primer

- An $m \times n$ *matrix* is a rectangular array of numbers with m rows and n columns. Example of a 2×3 matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{pmatrix}.$$

- $\mathbf{A}_{ij}$ denotes the number on row i and column j — for example, $\mathbf{A}_{13} = 4$.

- The *transpose* of a matrix, denoted by a superscripted t, is a matrix with the rows and columns interchanged, *i.e.*, $\mathbf{A}^t_{ij} = \mathbf{A}_{ji}$. For example,

$$\begin{pmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{pmatrix}^t = \begin{pmatrix} 3 & 1 \\ 1 & 5 \\ 4 & 9 \end{pmatrix},$$

- Two matrices of identical shape can be added by adding their corresponding elements: If $\mathbf{C} = \mathbf{A}+\mathbf{B}$, then $\mathbf{C}_{ij} = \mathbf{A}_{ij}+\mathbf{B}_{ij}$. Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix} = \begin{pmatrix} 11 & 22 \\ 33 & 44 \end{pmatrix}$$

- A matrix can be multiplied by a number by multiplying all of its elements by that number: If $\mathbf{B} = a\mathbf{A}$, then $\mathbf{B}_{ij} = a\mathbf{A}_{ij}$. Example:

$$10 \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}$$

- The product $\mathbf{C} = \mathbf{AB}$ of an $l \times m$ matrix $\mathbf{A}$ and an $m \times n$ matrix $\mathbf{B}$ is defined as

$$\mathbf{C}_{ij} \equiv \sum_{k=1}^{m} \mathbf{A}_{ik} B_{kj}.$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix} = \begin{pmatrix} 1 \cdot 10 + 2 \cdot 30 & 1 \cdot 20 + 2 \cdot 40 \\ 3 \cdot 10 + 4 \cdot 30 & 3 \cdot 20 + 4 \cdot 40 \end{pmatrix} = \begin{pmatrix} 70 & 90 \\ 150 & 220 \end{pmatrix}$$

- An *identity matrix* is a square matrix with 1 on the diagonal and 0 everywhere else. It is denoted $\mathbf{I}$. Its acts like the number 1, since multiplying another matrix by it has no effect: $\mathbf{IA} = \mathbf{A}$ and $\mathbf{AI} = \mathbf{A}$ for any $\mathbf{A}$. Example: the 2×2 identity matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- A matrix $\mathbf{B}$ is said to be the *inverse* of a matrix $\mathbf{A}$ if $\mathbf{AB} = \mathbf{I}$. Example:

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

  since

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- 1×1 matrices are simply numbers, and it is easy to see that the above rules for addition, multiplication and inversion reduce to the familiar ones for this special case.

- Vectors are special cases of matrices and therefore obey the above rules for addition and multiplication.

- A matrix with only one column is called a *column vector*. All vectors in 8.033 are column vectors, usually referred to simply as vectors. Example:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- A matrix with only one row is called a *row vector*. Example:

$$\mathbf{a}^t = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

- In a linear algebra class, you typically learn more advanced aspects of matrices, such as their determinant, eigenvalues and eigenvectors.