MIT Course 8.033, Fall 2006, Relativistic Kinematics
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Topics

- Lorentz transformations toolbox
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  - inverse
  - composition (v addition)
  - boosts as rotations
  - the invariant
  - wave 4-vector
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- Implications
  - Time dilation
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  - Length contraction
  - c as universal speed limit
  - Rest length, proper time

Formula summary: transformation toolbox

- Lorentz transformation:

$$\mathbf{\Lambda}(\hat{\mathbf{x}}v) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma(x-\beta ct) \\ y \\ z \\ \gamma(ct-\beta x) \end{pmatrix}.$$

- This implies all the equations below, derived on the following pages:
- Inverse Lorentz transformation:

$$\mathbf{\Lambda}(\mathbf{v})^{-1} = \mathbf{\Lambda}(-\mathbf{v})$$

- Addition of parallel velocities:

$$\mathbf{\Lambda}(v_1)\mathbf{\Lambda}(v_2) = \mathbf{\Lambda}\left(\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}\right)$$

- Addition of arbitrary velocities:

$$\begin{aligned} u_x &= \frac{u'_x + v}{1+\frac{u'_x v}{c^2}} \\ u_y &= \frac{u'_y\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u'_x v}{c^2}} \\ u_z &= \frac{u'_z\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u'_x v}{c^2}} \end{aligned}$$

- Boosts as generalized rotations:

$$\mathbf{\Lambda}(-v) = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},$$

where $\eta \equiv \tanh^{-1}\beta$

- All Lorentz matrices $\mathbf{\Lambda}$ satisfy

$$\mathbf{\Lambda}^t \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta},$$

where the Minkowski metric is

$$\boldsymbol{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

- All Lorentz transforms leave the interval

$$\Delta s^2 \equiv \Delta \mathbf{x}^t \boldsymbol{\eta} \Delta x = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$

invariant

- Wave 4-vector

$$\mathbf{K} \equiv \gamma_u \begin{pmatrix} k_x \\ k_y \\ k_z \\ w/c \end{pmatrix},$$

- Velocity 4-vector

$$\mathbf{U} \equiv \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}, \quad \gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

- Aberration:

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

- Doppler effect:

$$\omega' = \omega\gamma(1 - \beta\cos\theta)$$

Formula summary: other

- Proper time interval:

$$\Delta\tau = \int_{t_A}^{t_B} \sqrt{1 - \frac{|\dot{\mathbf{r}}(t)|^2}{c^2}} dt$$

- Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0$$

Implications: time dilation

- In the frame S, a clock is at rest at the origin ticking at time intervals that are Δt =1 seconds long, so the two consecutive ticks at $t = 0$ and $t = \Delta t$ have coordinates

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix}.$$

- In the frame S', the coordinates are

$$\mathbf{x}'_1 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{x}'_2 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix} = \begin{pmatrix} -\gamma v\Delta t \\ 0 \\ 0 \\ \gamma c\Delta t \end{pmatrix}$$

- So in S', the clock appears to tick at intervals $\Delta t' = \gamma \Delta t > \Delta t$, *i.e.*, slower! (Draw Minkowski diagram.)

Time dilation, cont'd

- The light clock movie says it all:
  $http://www.anu.edu.au/Physics/qt/$
- Cosmic ray muon puzzle
  - Created about 10km above ground
  - Half life 1.56×10^{-6} second
  - In this time, light travels 0.47 km
  - So how can they reach the ground?
  - $v \approx 0.99c$ gives $\gamma \approx 7$
  - $v \approx 0.9999c$ gives $\gamma \approx 71$
- Leads to twin paradox

Consider two frames in relative motion. For $t = 0$, the Lorentz transformation gives $x' = \gamma x$, where $\gamma > 1$.

Question: How long does a yard stick at rest in the unprimed frame look in the primed frame?

1. Longer than one yard
2. Shorter than one yard
3. One yard

Implications: relativity of simultaneity

- Consider two events simultaneous in frame S:

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- In the frame S', they are

$$\mathbf{x}'_1 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}'_2 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L \\ 0 \\ 0 \\ -\gamma\beta L \end{pmatrix}$$

- So in S', the second event happened first!
- So S-clocks appear unsynchronized in S' - those with larger x run further ahead

Implications: length contraction

- Trickier than time dilation, opposite result (interval appears shorter, not longer)
- In the frame S, a yardstick of length L is at rest along the x-axis with its endpoints tracing out world lines with coordinates

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix}.$$

- In the frame S', these world lines are

$$\begin{aligned}
\mathbf{x}'_1 &= \begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \\ ct'_1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} -\gamma\beta ct \\ 0 \\ 0 \\ \gamma ct \end{pmatrix} \\
\mathbf{x}'_2 &= \begin{pmatrix} x'_2 \\ y'_2 \\ z'_2 \\ ct'_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} \gamma L - \gamma\beta ct \\ 0 \\ 0 \\ \gamma ct - \gamma\beta L \end{pmatrix}
\end{aligned}$$

- An observer in S' measures length as $x'_2 - x'_1$ at the same time t', - *not* at the same time t.
- Let's measure at $t' = 0$.
- $t'_1 = 0$ when $t = 0$ — at this time, $x'_1 = 0$
- $t'_2 = 0$ when $ct = \beta L$ - at this time, $\mathbf{x}'_2 = \gamma L - \gamma\beta^2 L = L/\gamma$
- So in S'-frame, measured length is $L' = L/\gamma$, *i.e.*, *shorter*
- Let's work out the new world lines of the yard stick endpoints
- $\mathbf{x}'_1 + \beta ct'_1 = 0$, so left endpoint world line is

$$x'_1 = -vt'_1$$

- $\mathbf{x}'_2 - \gamma L + \beta(ct'_2 + \gamma\beta L) = 0$, so right endpoint world line is

$$x'_2 = \gamma L - \beta(ct'_2 + \gamma\beta L) = \frac{L}{\gamma} - vt'_2$$

- Length in S' is

$$x'_2 - x'_1 = \frac{L}{\gamma} + v(t'_1 - t'_2) = \frac{L}{\gamma}$$

  since both endpoints measured at same time ($t'_1 = t'_2$)
- Draw Minkowski diagram of this

Superluminal communication?

- Velocity addition formula shows that it's impossibe to accelerate something past the speed of light
- But could there be another way, say a type of radiation that moves faster than light?
- Can an event A influence another event B at spacelike separation (hence transmitting information faster than the speed of light)?
- There is another frame where B happened before A! (PS3)
- Draw Minkowski diagram of this
- By inertial frame invariance, B can then send a signal that arrives back to A before she sent her initial signal, telling her not to send it.
- Implication: c isn't merely the speed of light, but the limiting speed for *anything*

"Everything is relative" — or is it?

- All observers agree on rest length
- All observers agree on proper time
- All observers (as we'll see later) agree on rest mass

Transformation toolbox: the inverse Lorentz transform

- Since $\mathbf{x}' = \mathbf{\Lambda}(v)\mathbf{x}$ and $\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}'$, we get the consistency requirement

  $$\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}' = \mathbf{\Lambda}(-v)\mathbf{\Lambda}(v)\mathbf{x}$$

  for any event $\mathbf{x}$, so we must have $\mathbf{\Lambda}(-v) = \mathbf{\Lambda}(v)^{-1}$, the matrix inverse of $\mathbf{\Lambda}(v)$.

- Is it?

  $$\mathbf{\Lambda}(-\mathbf{v})\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

  i.e., yes!

Transformation toolbox: velocity addition

- If the frame S' has velocity v_1 relative to S and the frame S'' has velocity v_2 relative to S' (both in the x-direction), then what is the speed v_3 of S'' relative to S?

- $\mathbf{x}' = \mathbf{\Lambda}(v_1)\mathbf{x}$ and $\mathbf{x}'' = \mathbf{\Lambda}(v_2)\mathbf{x}' = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)\mathbf{x}$, so

- $\mathbf{\Lambda}(\mathbf{v}_3) = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)$, *i.e.*

$$\begin{aligned} \begin{pmatrix} \gamma_3 & 0 & 0 & -\gamma_3\beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_3\beta_3 & 0 & 0 & \gamma_3 \end{pmatrix} &= \begin{pmatrix} \gamma_2 & 0 & 0 & -\gamma_2\beta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2\beta_2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{pmatrix} \\ &= \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_2 & 0 & 0 & -[\beta_1+\beta_2] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -[\beta_1+\beta_2] & 0 & 0 & 1+\beta_1\beta_2 \end{pmatrix} \end{aligned}$$

- Take ratio between $(1,4)$ and $(1,1)$ elements:

  $$\beta_3 = -\frac{\mathbf{\Lambda}(v_3)_{41}}{\mathbf{\Lambda}(v_3)_{11}} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}.$$

- In other words,

  $$v_3 = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

Transformation toolbox: perpendicular velocity addition

- Here's an alternative derivation of velocity addition that easily gives the non-parallel components too (but 4-vector method on next page is simpler)
- If the frame S' has velocity v in the x-direction relative to S and a particle has velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in S', then what is its velocity $\mathbf{u}$ in S?
- Applying the inverse Lorentz transformation

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + vx'/c^2) \end{aligned}$$

to two nearby points on the particle's world line and subtracting gives

$$\begin{aligned} dx &= \gamma(dx' + vdt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma(dt' + vdx'/c^2). \end{aligned}$$

$$\begin{aligned} dx &= \gamma(dx' + vdt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma(dt' + vdx'/c^2). \end{aligned}$$

- Answer:

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + \frac{vdx'}{c^2})} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \\ u_y &= \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{vdx'}{c^2})} = \frac{\gamma^{-1}\frac{dy'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \\ u_z &= \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + \frac{vdx'}{c^2})} = \frac{\gamma^{-1}\frac{dz'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \end{aligned}$$

Transformation toolbox: velocity as a 4-vector

- For a particle moving along its world-line, define its velocity 4-vector

$$\mathbf{U} \equiv \frac{d\mathbf{X}}{d\tau} = \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix},$$

  where

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

- This is the derivative of its 4-vector $\mathbf{x}$ w.r.t. its proper time τ, since $d\tau = dt/\gamma_u$

- $\mathbf{U}' = \mathbf{\Lambda U}$:

$$\mathbf{U}' = \frac{d\mathbf{X}'}{d\tau'} = \frac{d\mathbf{\Lambda X}}{d\tau} = \mathbf{\Lambda}\frac{d\mathbf{X}}{d\tau} = \mathbf{\Lambda U},$$

  since the proper time interval $d\tau$ is Lorentz-invariant

- This means that all velocity 4-vectors are normalized so that

$$\mathbf{U}^t \boldsymbol{\eta} \mathbf{U} = -c^2.$$

- This immediately gives the velocity addition formulas:

$$\begin{aligned} \mathbf{U}' &= \gamma_{u'} \begin{pmatrix} u'_x \\ u'_y \\ u'_z \\ c \end{pmatrix} = \mathbf{\Lambda}(-\mathbf{v})\mathbf{U} = \gamma_u \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} \\ &= \begin{pmatrix} \gamma_u \gamma [u_x + v] \\ \gamma_u u_y \\ \gamma_u y_z \\ \gamma_u \gamma [1 + \frac{u_x v}{c^2}] c \end{pmatrix} = \gamma_{u'} \begin{pmatrix} \frac{u_x + v}{1 + u_x v/c^2} \\ \frac{u_y/\gamma}{1 + u_x v/c^2} \\ \frac{u_z/\gamma}{1 + u_x v/c^2} \\ c \end{pmatrix}, \end{aligned}$$

  where $\gamma_{u'} = \gamma_u \gamma \left[1 + \frac{u_x v}{c^2}\right]$ — this last equation follows from the fact that the 4-vector normalization in Lorentz invariant, *i.e.*,$\mathbf{u}'^t \boldsymbol{\eta} \mathbf{u}' = \mathbf{u}^t \boldsymbol{\eta} \mathbf{u} = -1$.

- The 1st 3 components give the velocity addition equations we derived previously.

Transformation toolbox: boosts as generalized rotations

- A "boost" is a Lorentz transformation with no rotation
- A rotation around the z-axis by angle θ is given by the transformation
$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- We can think of a boost in the x-direction as a rotation by an imaginary angle in the (x, ct)-plane:
$$\mathbf{\Lambda}(-v) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},$$
where $\eta \equiv \tanh^{-1}\beta$ is called the *rapidity*.
- Proof: use hyperbolic trig identities on next page
- Implication: for multiple boosts in same direction, rapidities add and hence the order doesn't matter

Hyperbolic trig reminders

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \\
\tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)
\end{aligned}$$

$$\cosh\tanh^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\sinh\tanh^{-1} x = \frac{x}{\sqrt{1-x^2}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

The Lorentz invariant

- The Minkowski metric

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  is left invariant by all Lorentz matrices $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta}$$

  (indeed, this equation is often used to define the set of Lorentz matrices — for comparison, $\boldsymbol{\Lambda}^t \mathbf{I} \boldsymbol{\Lambda} = \mathbf{I}$ would define rotation matrices)

- Proof: Show that works for boost along x-axis. Show that works for rotation along y-axis or z-axis. General case is equivalent to applying such transformations in succession.

- All Lorentz transforms leave the quantity

$$\mathbf{x}^t \boldsymbol{\eta} \mathbf{x} = x^2 + y^2 + z^2 - (ct)^2$$

  invariant

- Proof:

$$\mathbf{x}'^t \boldsymbol{\eta} \mathbf{x}' = (\boldsymbol{\Lambda}\mathbf{x})^t \boldsymbol{\eta} (\boldsymbol{\Lambda}\mathbf{x}) = \mathbf{x}^t (\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda}) \mathbf{x} = \mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$$

- (More generally, the same calculation shows that $\mathbf{x}^t \boldsymbol{\eta} \mathbf{y}$ is invariant)

- So just as the usual Euclidean squared length $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = \mathbf{r}^t \mathbf{r} = \mathbf{r}^t \mathbf{I} \mathbf{r}$ of a 3-vector is rotaionally invariant, the generalized "length" $\mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$ of a 4-vector is Lorentz-invariant.

- It can be positive or negative

- For events $\mathbf{x}_1$ and $\mathbf{x}_2$, their Lorentz-invariant separation is defined as

$$\Delta\sigma^2 \equiv \Delta\mathbf{x}^t \boldsymbol{\eta} \Delta\mathbf{x} = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$

- A separation $\Delta\sigma^2 = 0$ is called *null*

- A separation $\Delta\sigma^2 > 0$ is called *spacelike*, and

$$\Delta\sigma \equiv \sqrt{\Delta\sigma^2}$$

  is called the *proper distance* (the distance measured in a frame where the events are simultaneous)

- A separation $\Delta\sigma^2 < 0$ is called *timelike*, and

$$\Delta\tau \equiv \sqrt{-\Delta\sigma^2}$$

  is called the *proper time interval* (the time interval measured in a frame where the events are at the same place)

- More generally, any 4-vector is either null, spacelike of timelike.

- The velocity 4-vector $\mathbf{U}$ is always timelike.

Transforming a wave vector

- A plane wave

$$E(\mathbf{x}) = \sin(k_x x + k_y y + k_z z - \omega t) \tag{1}$$

is defined by the four numbers

$$\mathbf{K} \equiv \begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega/c \end{pmatrix}.$$

- If the wave propagates with the speed of light c (like for an electromagnetic or gravitational wave), then the frequency is determined by the 3D wave vector (k_x, k_y, k_z) through the relation $\omega/c = k$, where $k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$
- How does the 4-vector $\mathbf{K}$ transform under Lorentz transformations? Let's see.
- Using the Minkowski matrix, we can rewrite equation (1) as

$$E(\mathbf{X}) = \sin(\mathbf{K}^t \boldsymbol{\eta} \mathbf{X}).$$

- Let's Lorentz transform this: $\mathbf{X} \to \mathbf{X}'$, $\mathbf{K} \to \mathbf{K}'$. Using that $\mathbf{X}' = \mathbf{\Lambda X}$, let's determine $\mathbf{K}'$.

$$E' = \sin(\mathbf{K}'^t \boldsymbol{\eta} \mathbf{X}') = \sin(\mathbf{K}'^t \boldsymbol{\eta} \mathbf{\Lambda X}) = \sin[(\mathbf{\Lambda}^{-1}\mathbf{K}')^t (\mathbf{\Lambda}^t \boldsymbol{\eta} \mathbf{\Lambda})\mathbf{X}] = \sin[(\mathbf{\Lambda}^{-1}\mathbf{K}')^t \boldsymbol{\eta} \mathbf{X}].$$

- This equals E if $\mathbf{\Lambda}^{-1}\mathbf{K}' = \mathbf{K}$, *i.e.*, if the wave 4-vector transforms just as a normal 4-vector:

$$\mathbf{K}' = \mathbf{\Lambda K}$$

- This argument assumed that $E' = E$. Later we'll see that the electric and magnetic fields *do* in fact change under Lorentz transforms, but not in a way that spoils the above derivation (in short, the phase of the wave, $\mathbf{K}^t \boldsymbol{\eta} \mathbf{X}$, must be Lorentz invariant)
- So a plane wave $\mathbf{K}$ in S is also a plane wave in S', and the wave 4-vector transforms in exactly the same way as $\mathbf{X}$ does.

Aberration and Doppler effects

- Consider a plane wave propagating with speed c in the frame S:

$$\mathbf{K} = k \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \\ 1 \end{pmatrix},$$

  where ck is the wave frequency and the angles θ and ϕ give the propagation direction in polar coordinates.

- Let's Lorentz transform this into a frame S' moving with speed v relative to S in the z-direction: $\mathbf{k}' = \mathbf{\Lambda k}$, *i.e.*,

$$\begin{aligned} \mathbf{K}' &= k' \begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\gamma\beta \\ 0 & 0 & -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \\ 1 \end{pmatrix} \\ &= k \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \gamma(\cos\theta - \beta) \\ \gamma(1 - \beta\cos\theta) \end{pmatrix}, \end{aligned}$$

  so

$$\begin{aligned} \phi' &= \phi \\ \cos\theta' &= \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \\ k' &= k\gamma(1 - \beta\cos\theta) \end{aligned}$$

- This matches equations (1)-(4) in the Weiskopf et al ray tracing handout

- The change in the angle θ is known as *aberration*

- The change in frequency ck is known as the Doppler shift — note that since $k = 2\pi/\lambda$, we have $\lambda'/\lambda = k/k'$.

- If we instead take the ratio $\sqrt{k'^2_x + k'^2_y}/k'_z$ above, we obtain the mathematically equivalent form of the aberration formula given by Resnick (2-27b):

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

- Examine classical limits
- Transverse Doppler effect: $\cos\theta = 0$ gives $\omega' = \omega\gamma$, *i.e.*, simple time dilation (classically, $\omega' = \omega$, *i.e.*, no transverse effect)
- Longitudinal doppler effect: $\cos\theta = 1$ gives

$$\frac{\omega'}{\omega} = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}.$$

- For comparison, classical physics, moving observer:

$$\frac{\omega'}{\omega} = 1-\beta.$$

- For comparison, classical physics, moving source:

$$\frac{\omega'}{\omega} = \frac{1}{1+\beta}$$

Accelerated motion & proper time

- Consider a clock moving along a curve $\mathbf{r}(t)$ though spacetime, as measured in a frame S. During an infinitesimal time interval between t and $t+dt$, it moves with velocity $\mathbf{u}(t) = \dot{\mathbf{r}}(t)$ and measures a proper time interval

$$d\tau = \frac{dt}{\gamma_u} = \sqrt{1 - \frac{|\dot{\mathbf{r}}(t)|^2}{c^2}}dt.$$

- The proper time interval (a.k.a. wristwatch time) measured by the clock as it moves from event A to event B along this path is

$$\Delta\tau = \int_{t_A}^{t_B} d\tau = \int_{t_A}^{t_B} \sqrt{1 - \frac{|\dot{\mathbf{r}}(t)|^2}{c^2}}dt$$

- If the two events are at the same position in S, *i.e.*, if $\mathbf{r}(t_A) = \mathbf{r}(t_B)$, then the path $\mathbf{r}(t)$ between the two events that maximizes $\Delta\tau$ is clearly the straight line $\mathbf{r}(t) = \mathbf{r}(t_A)$ where the clock never moves, giving $\mathbf{u} = \mathbf{0}$ and $\Delta\tau = \Delta t = t_B - t_A$.

- For any two events with timelike separation, the proper time is again maximized when the path between the two points is a straight line though spacetime.
  Proof: Lorentz transform to a frame S' where A and B are at the same position, conclude the the path is a straight line in S' and use the fact that the Lorentz transform of a straight line through spacetime is always a straight line through spacetime.

- One can also deduce this with calculus of variations, which is overkill for this simple case.

Calculus of variations

- The much more general optimization problem of finding the path $x(t)$ that minimizes or maximizes a quantity

$$S[x] \equiv \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)]dt$$

  subject to the constraints that $x(t_0) = x_0$ and $x(t_1) = x_1$ reduces to solving the differential equation known as the Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0.$$

- Here the meaning of $\frac{\partial f}{\partial \dot{x}}$ is simply the partial derivative of f with respect to its third argument, *i.e.*, just treat $\dot{x}$ as a variable totally independent of x when evaluating this derivative.

Metrics and geodesics

- In an n-dimensional space, the *metric* is a (usually position-dependent) $n \times n$ symmetric matrix $\mathbf{g}$ that defines the way distances are measured. The length of a curve is $\int d\sigma$, where

$$d\sigma^2 = d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r},$$

  and $\mathbf{r}$ are whatever coordinates you're using in the space. If you change coordinates, the metric is transformed so that $d\sigma$ stays the same ($d\sigma$ is invariant under all coordinate transformations).

- **Example:** 2D Euclidean space in Cartesian coordinates.

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$d\sigma^2 = d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r} = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2,$$

$$\int d\sigma = \int \sqrt{d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r}} = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2} dx.$$

  Applying the Euler-Lagrange equation to this shows that the shortest path between any two points is a straight line.

- **Example:** 4D Minkowski space in Cartesian coordinates ($c = 1$ for simplicity)

$$\mathbf{g} = \boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{aligned} d\tau^2 &= -d\sigma^2 = d\mathbf{x}^t \, \mathbf{g} \, d\mathbf{x} = \\ &= \begin{pmatrix} dx & dy & dz & dt \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} \\ &= dt^2 - dx^2 - xy^2 - dz^2, \end{aligned}$$

$$\begin{aligned} \Delta\tau &= \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} dt \\ &= \int \sqrt{1 - u^2} dt = \int \frac{dt}{\gamma}. \end{aligned}$$

  Applying the Euler-Lagrange equation to this shows that the extremal interval between any two events is a straight line though spacetime.