**MIT Course 8.033, Fall 2005, General Relativity**
**Max Tegmark**
**Last revised November 7 2006**

# Topics

- Key concept summary
- Summary of useful metrics

## Special relativity concept summary

- Space and time unified into 4D spacetime.
- Analogous unification for other 4-vectors (momentum+energy, *etc.*).
- Lorentz transform relates 4-vectors in different inertial frames. Example: fast moving clocks are slower, shorter and heavier.
- $E = mc^2$. Example: nuclear power.

## General relativity concept summary

- Spacetime is not static but dynamic, globally expanding and locally curving and contracting to form black holes *etc.*
- Matter curves spacetime so that things moving "straight" (along geodesics) through curved spacetime appear deflected/accelerated (gravity).

## Summary of useful metrics

- Minkowski metric:

$$d\tau^2 = d(ct)^2 - dx^2 - dy^2 - dz^2$$

- Newtonian metric:

$$d\tau^2 = \left(1 + \frac{2\phi}{c^2}\right) d(ct)^2 - dx^2 - dy^2 - dz^2$$

- Minkowski metric in polar coordinates:

$$d\tau^2 = d(ct)^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

- Friedman-Robertson-Walker (FRW) metric:

$$d\tau^2 = d(ct)^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

- Schwartzschild metric:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) d(ct)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2,$$

  where the *Schwartzschild radius* is defined as

$$r_s \equiv \frac{2MG}{c^2}.$$

- In GR, it's convenient to use units where $c = G = 1$, simplifying these metrics:

- Minkowski metric:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

- Newtonian metric:

$$d\tau^2 = (1 + 2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

- Minkowski metric in polar coordinates:

$$d\tau^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

- Friedman-Robertson-Walker (FRW) metric:

$$d\tau^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

- Schwartzschild metric ($r_s = 2M$):

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2,$$

# Summary of how to work with metrics

- Computing the ageing $\Delta\tau$ along a curve $\mathbf{r}(t)$ though spacetime:

$$\Delta\tau = \int d\tau.$$

- Example: for Minkowski metric,

$$\begin{aligned}\Delta\tau &= \int d\tau = \int\sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}dt \\ &= \int\sqrt{1-u^2}dt = \int\frac{dt}{\gamma}.\end{aligned}$$

- Example: for Newtonian metric in limit $|\phi| \ll 1$ and $u \ll 1$,

$$\begin{aligned}\Delta\tau &= \int d\tau = \int\sqrt{(1+2\phi)dt^2 - dx^2 - dy^2 - dz^2} = \int\sqrt{1+2\phi-u^2}dt \\ &\approx \int\left(1+\phi-\frac{1}{2}u^2\right)dt.\end{aligned}$$

  Interpretation: you age slower if you go faster but age faster if you go higher up in the gravitational potential.

- Computing the trajectory of a massive particle from event A to event B: find the geodesic between A and B. This is the path of maximal aging, *i.e.*, the path that maximizes $\Delta\tau$ from above. This is a variational calculus problem and can be zapped with the Euler-Lagrange equation.

- Computing the trajectory of a photon from event A to event B: photons move along geodesics that have $\Delta = \tau = 0$, so-called *null geodesics*. Many of our applications will involve only one space-dimension (say, motion in the $r$-direction, in which case you don't need to use variational calculus and can simply solve the equation $d\tau = 0$.

## Newtonian gravity

The "gravitational field" $\mathbf{g}$ is minus the gradient $\nabla\phi$ of the Newtonian gravitational potential $\phi$. Units: $\phi/c^2$ is dimensionless.

- How matter affects the gravitational field:

  $$\nabla^2\phi = 4\pi G\rho$$

  Implication: the gravitational potential from a single point mass $M$ at the origin is

  $$\phi = -\frac{GM}{r},$$

  and fields from different masses simply add.

- How the gravitational field affects matter:

  $$F = m\mathbf{g} = -m\nabla\phi.$$

## Equivalence principle (1911)

- General relativity (GR) consists of two parts: how matter (particles, electromagnetic fields, *etc.*) affects spacetime and how spacetime affect matter. The second part is specified by the strong equivalence principle.
- **Weak equivalence principle:** No local experiment can distinguish between a uniform gravitational field $\mathbf{g}$ and a frame accelerated with $\mathbf{a} = \mathbf{g}$.
- **Strong equivalence principle:** The laws of physics take on their special-relativistic form in any locally inertial frame frame.
- A freely falling elevator is a locally inertial frame (if the elevator is small enough and our experiment short enough), so the strong version says that special relativity applies in all such elevators anywhere and anytime in the universe, *i.e.*, independently of the spacetime position and velocity of the elevator.
- Where did this idea come from? Combining

  $$F = ma$$

  with

  $$F = \frac{GmM}{r^2}$$

  shows that the gravitational acceleration

  $$a = \frac{GM}{r^2}$$

  is mass-independent as long as

  "inertial mass" = "gravitational mass".

  Is it?
- Galileo's Pisa experiment showed it with low precision.
- Eötvös (1890) and later others showed with high precision that $a$ independent of both mass and composition (density, atomic element, matter/antimatter, etc). Coincidence? Einstein thought that no, it was telling us something.
- In other words, if you know the direction of the worldline of an object freely floating through a spacetime event (*i.e.*, the direction of the velocity 4-vector), then the continuation of the worldline under the influence of gravity is the same regardless of the mass and composition of the object. This suggested to Einstein that gravity was a purely geometric effect.

## Gravitational redshift

- Implied by equivalence principle. When light travels a distance $h$ from the floor to the ceiling of an elevator free-falling downward in a uniform gravitational field $g$, it will have no redshift according to an observer in the elevator. If the elevator was at rest in the lab frame when the light was emitted, then when it reaches the ceiling, the ceiling (and the locally inertial frame where the light is not redshifted) is moving downward with

$$u \approx gt \approx \frac{gh}{c}.$$

  Lorentz transforming to the lab frame and using the Doppler shift formula thus gives a gravitational redshift

$$\frac{\nu'}{\nu} \approx 1 - \frac{v}{c} \approx 1 - \frac{gh}{c^2}.$$

- Also implied by energy conservation and $E = mc^2$. If a photon travels upward in a gravitational field, the increase in potential energy $mgh = \frac{E}{c^2}gh$ must be offset by a reduction in the photon energy $h\nu$, *i.e.*, be a lowering of the frequency. This again gives the result

$$\frac{\nu'}{\nu} \approx 1 - \frac{gh}{c^2}.$$

- Both of these calculations are only approximate, correct to first order (in the small quantities $v/c$ and $\phi/c^2$).
- The gravitational redshift is simply given by the Newtonian gravitational potential $\phi$: Adding up many infinitesimal contributions from the above formula gives

$$\frac{\Delta\nu}{\nu} \int \frac{1}{c^2}\mathbf{g} \cdot d\mathbf{r} = -\frac{1}{c^2} \int \nabla\phi \cdot dr = -\frac{\phi}{c^2}.$$

- **Implication:** Time runs slower further down in the gravitational potential:

$$\frac{d\tau}{dt} = 1 + \frac{\phi}{c^2}.$$

- **Implication:** Taking $c = 1$, this means that we need a factor $(1+\phi)^2 \approx 1 + 2\phi$ multiplying the $dt^2$-factor in the metric:

$$d\tau^2 = (1+2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

- This simple metric reproduces all of Newtonian gravity! The Euler-Lagrange equation shows that time-like geodesics in this metric obey the Newtonian result

$$\ddot{r} = -\nabla\phi.$$

- Tidbit for the curious: Adding terms multipling elsewhere in the metric (say, for the $dy^2$-term or the off-diagonal $dtdx$-term) has no effect in the Newtonian limit, since they would get suppressed by a factor $v/c$ or $(v/c)^2$ in the Euler-Lagrange equation.

## Spherical coordinates

- Spherical coordinates $(r, \theta, \varphi)$ are defined by

$$
\begin{aligned}
x &= r \sin\theta \cos\varphi, \\
y &= r \sin\theta \sin\varphi, \\
z &= r \cos\theta.
\end{aligned}
$$

- This implies

$$
\begin{aligned}
dx &= \sin\theta \cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi, \\
dy &= \sin\theta \sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi, \\
dz &= \cos\theta dr - r\sin\theta d\theta.
\end{aligned}
$$

- This let's us reexpress the Minkowski metric in spherical coordinates:

$$
\begin{aligned}
d\tau^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\
&= dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2.
\end{aligned}
$$

(To get the second line, we simply plugged in the expressions for $dx$, $dy$ and $dz$ and simplified the result.)

## General covariance

- The analogous procedure is used to transform *any* metric into *any* coordinate system.
- **Key concept:** this means that we can do our calculations with metrics and geodesics in *any* system of space and time coordinates we like. In Minkowski space, inertial frames are just a special class of coordinate systems (the standard spacetime coordinates $(x, y, z, ct)$ and Lorentz transforms thereof), so we're *not* limited to working in inertial frames in GR.
- Einstein insisted that not only the metric but indeed all laws of physics should be expressible using *any* coordinate system. This requirement is called *general covariance*.
- This is why GR is called *General* relativity, special relativity being merely the special case where you were allowed to start with an inertial frame and make a Lorentz transformation (a particular linear coordinate transformation).
- If you think of Lorentz transformations as coordinate transformations, they are simply the ones that have the property

  $$d\tau^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 = d(ct')^2 - dx'^2 - dy'^2 - dz'^2,$$

  since we previously proved that $d\tau$ is Lorentz invariant.
- General covariance *defines* the second part of General relativity (how the metric affects matter): to compute the evolution of something near a certain spacetime event (the electromagnetic field, the position of a particle, *etc.*.), simply change to coordinates that correspond to a free-falling frame locally (in the spacetime region surrounding that event) and apply the equations special relativity. In particular, a particle not subjected to any non-gravitational forces moves along a geodesic.
- Note that it's not at all obvious just from staring at a metric that someone writes down whether it's really just Minkowski space in disguise, expressed in some funny coordinates.