**MIT Course 8.033, Fall 2005, Electromagnetism**
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**Last revised November 3 2005**

# Key formula summary

- Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B})$$

- Lorentz transforming the electromagnetic field:

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - \beta B_z) \\ E'_z &= \gamma(E_z + \beta B_y) \\ B'_x &= B_x \\ B'_y &= \gamma(B_y + \beta E_z) \\ B'_z &= \gamma(B_z - \beta E_y). \end{aligned}$$

- Current 4-vector:

$$\mathbb{J} \equiv \begin{pmatrix} J_x \\ J_y \\ J_z \\ \rho c \end{pmatrix} = \rho_0 \mathbf{U},$$

where the proper charge density $\rho_0$ is the local charge density in a frame where $\mathbf{J} = 0$.

- Electric field from stationary charge $q$ (Coulomb's law):

$$\mathbf{E} = \frac{q}{r^2}\hat{\mathbf{r}} = \frac{q}{\mathbf{x}^2 + y^2 + z^2}\hat{\mathbf{r}}$$

- Electric field from charge $q$ moving in $x$-direction:

$$\mathbf{E}' = \frac{\gamma q r'}{\left(\gamma^2 x'^2 + y'^2 + z'^2\right)^{3/2}}\hat{\mathbf{r}}'$$

## How relativity and electricity implies magnetism

- We know that the force $\mathbf{F}$ on charged particle of charge $q$ in an electric field $\mathbf{E}$ is
$$\mathbf{F} = q\mathbf{E},$$
independent of the velocity $\mathbf{u}$ of the particle.

- We can rewrite this equation in a mathematically equivalent way using 4-vectors:
$$\mathbb{F} = \frac{q}{c}\mathbf{M}\mathbf{U}, \tag{1}$$
where $\mathbb{F}$ is the force 4-vector, $\mathbf{U}$ is the velocity 4-vector and $\mathbf{M}$ is the $4 \times 4$ matrix
$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & E_x \\ 0 & 0 & 0 & E_y \\ 0 & 0 & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}.$$
The 4th component of this equation reads $P = q\mathbf{E}\cdot\mathbf{u}$, so $P = \mathbf{F}\cdot\mathbf{u}$ as should be.

- Let's Lorentz transform to a frame $S'$ moving with with velocity $v = \beta c$ in the $x$-direction:
$$\mathbb{F}' \equiv \mathbf{\Lambda}\mathbb{F} = \frac{q}{c}\mathbf{\Lambda}\mathbf{M}\mathbf{U} = \frac{q}{c}\mathbf{\Lambda}\mathbf{M}\mathbf{\Lambda}^{-1}\mathbf{\Lambda}\mathbf{U} = \frac{q}{c}\mathbf{M}'\mathbf{U}',$$
where the transformed matrix is
$$\mathbf{M}' \equiv \mathbf{\Lambda}\mathbf{M}\mathbf{\Lambda}^{-1}. \tag{2}$$

- Plugging in our $\mathbf{M}$-matrix above, this gives
$$\begin{pmatrix} 0 & -\beta\gamma E_y & -\beta\gamma E_z & E_x \\ \beta\gamma E_y & 0 & 0 & \gamma E_y \\ \beta\gamma E_z & 0 & 0 & \gamma E_z \\ E_x & \gamma E_y & \gamma E_z & 0 \end{pmatrix}.$$

- This shows two things. First we see that, hardly surprisingly by now, the $\mathbf{E}$-field is picks up some $\gamma$-factors — specifically, $E'_x = E_x$ whereas $E'_y = \gamma E_y$ and $E'_z = \gamma E_z$. Second, we see that new terms appear in the matrix that don't correspond to an $E$-field! The component $M_{23}$ would also become non-zero if we transformed to a frame moving in a different direction. So to be able to describe the general case, we need to introduce more field components in $\mathbf{M}$. It's easy to show that the upper left $3\times 3$ block of the matrix is antisymmetric regardless of how we Lorentz transform, *i.e.*, $M_{11} = M_{22} = M_{33} = 0$ and $M_{32} = -M_{23}$, $M_{13} = -M_{31}$, $M_{21} = -M_{12}$, so we simply need to keep track of the three quantities $M_{23}$, $M_{31}$ and $M_{12}$. We could denote these three numbers by whatever symbols we want — let's call them $B_x$, $B_y$ and $B_z$. This means that, by definition, the **M**-matrix takes the form

$$\mathbf{M} \equiv \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}.$$

- Plugging this back into equation (1) now gives

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}), \qquad (3)$$

  for the first three components, *i.e.*, the famous Lorentz force law from 8.02! The 4th component gives $P = q\mathbf{E} \cdot \mathbf{u} = \mathbf{F} \cdot \mathbf{u}$ as should be.

- For simplicity, we've used c.g.s. units here, where $B$ has the same units as $E$. To switch to m.k.s. units, replace $\mathbf{B}$ by $c\mathbf{B}$.

- In conclusion, starting with a pure electric field in $S$, we found that in $S'$, the force on our particle will also depend on its velocity according to equation (3), *i.e.*, there is a magnetic field!

## Transforming the electromagnetic field

- Having figured out that these three new components correspond to a **B**-field, let us now use equation (2) to derive the transformation properties of an arbitrary electromagnetic field:

$$\begin{pmatrix} 0 & B'_z & -B'_y & E'_x \\ -B'_z & 0 & B'_x & E'_y \\ B'_y & -B'_x & 0 & E'_z \\ E'_x & E'_y & E'_z & 0 \end{pmatrix} =$$

$$= \mathbf{\Lambda} \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} \mathbf{\Lambda}^{-1} =$$

$$= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ E_x & E_y & E_z & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \gamma(B_z-\beta E_y) & -\gamma(B_y+\beta E_z) & E_x \\ -\gamma(B_z-\beta E_y) & 0 & B_x & \gamma(E_y-\beta B_z) \\ \gamma(B_y+\beta E_z) & -B_x & 0 & \gamma(E_z+\beta B_y) \\ E_x & \gamma(E_y-\beta B_z) & \gamma(E_z+\beta B_y) & 0 \end{pmatrix},$$

  so

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y-\beta B_z) \\ E'_z &= \gamma(E_z+\beta B_y) \\ B'_x &= B_x \\ B'_y &= \gamma(B_y+\beta E_z) \\ B'_z &= \gamma(B_z-\beta E_y). \end{aligned}$$

- In case you're familiar with tensor notation, the Einstein summation convention and raising/lowering indices, the matrix denoted **M** above is the electromagnetic field tensor $F_\mu^{\ \nu}$ which, unlike $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$, is not antisymmetric.

## Transforming charge and current densities

- The theory of electromagnetism consists of two parts: how matter affects fields and how fields affect matter. Above we studied the latter — let us now study the former.

- Analogy: the theory of gravity consists of two parts: how matter affects fields (the gravitational field) and how fields affect matter. In general relativity, the role of the gravitational field is played by the metric, and we will find that both parts of the theory get a geometric interpretation: the former that matter moves along geodesics through spacetime and the latter that matter curves spacetime.

- The source of electromagnetic fields is matter carrying electric charge, characterized at each spacetime eventy by a *charge density* $\rho(\mathbf{r}, t)$ and a *current density* $\mathbf{J}(\mathbf{r}, t)$.

- These can be combined into the current 4-vector (or "4-current")

$$\mathbb{J} \equiv \begin{pmatrix} J_x \\ J_y \\ J_z \\ \rho c \end{pmatrix}.$$

  For a blob of charge of uniform density $\rho_0$ in its rest frame that moves with velocity 4-vector $\mathbf{U}$, the 4-current is simply

$$\mathbb{J} \equiv \rho_0 \mathbf{U},$$

  and the total 4-current from many sources (say electrons and ions moving in opposite directions) is simply the sum of all the individual 4-currents.

- $\rho_0$ is called the *proper charge density*.

- The first part of the theory (how fields affect matter) is given by the Lorentz force law that we derived,

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B}). \tag{4}$$

- The second part of the theory (how matter determines the fields) is given by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{5}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}, \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = \mathbf{0}. \tag{8}$$

- We derived magnetism from electricity by assuming that the first part of the theory was Lorentz invariant. Let's now show that the second part is Lorentz invariant too, so that everything is consistent. We've already shown that the wave equation (which gives solutions to Maxwell's equations in vacuum, *i.e.*, with $\mathbb{J} = \mathbf{0}$) is Lorentz invariant, but we need to show more: that the full Maxwell equations are Lorentz invariant in general, even in the presence of charges and currents.

- You won't be responsible for the material below in this course — I'm just presenting it here so that you can admire the full elegance of electromagnetism, which only becomes manifest in relativistic 4-vector notation.

- A standard vector calculus result is that a vector field with no curl can be written as a gradient of some scalar field, say $\phi$, and a vector field with no divergence can be written as a curl of some vector field, $\mathbf{A}$. Maxwell's last two equations above therefore imply that we can write

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}}, \tag{9}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{10}$$

  where $\phi$ and $\mathbf{A}$ are referred to as the scalar potential and the vector potential, respectively. Proof: $\nabla \cdot \mathbf{B} = 0$ gives $\mathbf{B} = \nabla \times \mathbf{A}$, after which the 4th Maxwell equation shows that $\nabla \times \left(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}\right) = \mathbf{0}$ so that we can write $\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}} = -\nabla\phi$.

- These are conveniently combined into a 4-vector

$$\mathbb{A} \equiv \begin{pmatrix} A_x \\ A_y \\ A_z \\ \phi \end{pmatrix}.$$

- The differential operator

$$\Box \equiv \left(\frac{\partial}{c\partial t}\right)^2 - \nabla^2$$

is called the d'Alembertian, and is a spacetime generalization of the Laplace operator $\nabla^2$. It is easy to show that it is Lorentz invariant.

- In terms of this operator, the wave equation for some scalar field $\psi$ can be written in the extremely compact form

$$\Box\psi = 0.$$

- It turns out that there is some slop (known as *gauge freedom*) involved in the choices of $\phi$ and $\mathbf{A}$: for any scalar field $\psi$ satisfying the wave equation, you can replace $\mathbf{A}$ by $\mathbf{A}+\nabla\psi$ and $\phi$ by $\phi+\frac{\partial\psi}{c\partial t}$ without changing the fields $\mathbf{E}$ and $\mathbf{B}$. Without loss of generality, we can use this freedom to make $\mathbb{A}$ satisfy the so-called *Lorentz gauge condition*

$$\nabla\cdot\mathbf{A} - \frac{1}{c}\dot{\phi} = 0. \tag{11}$$

- Plugging equations (9), (10) and (11) into Maxwell's first two equations and doing some vector algebra now gives the beautiful result

$$\Box\mathbb{A} = -\frac{4\pi}{c}\mathbb{J}.$$

  We have solved Maxwell's equations. This equation shows how matter determines the fields. For the special case $\mathbb{J}=\mathbf{0}$, we see that it simply reduces to the wave equation $\Box\mathbb{A}=\mathbf{0}$, *i.e.*, each of the four components of $\mathbb{A}$ must separately satisfy the wave equation.

- We set out to prove that the second half of the theory was Lorentz invariant. The last equation show this explicitly, since $\Box$ is Lorentz invariant and $\mathbb{A}$ and $\mathbb{J}$ are both 4-vectors.

- Here's some doubly optional material in case you're interested. If you're familiar with the tensor notation, raising and lowering of indices and the Einstein summation convention (certainly not neccesary for this course!), here's an electromagnetism synopsis:

$$\begin{aligned}
A^{\mu}{}_{,\mu} &= 0 \quad \text{(Lorentz gauge condition)},\\
J^{\mu}{}_{,\mu} &= 0 \quad \text{(charge conservation)},\\
F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} \quad \text{(definition of } F\text{)},\\
F_{\mu} &= F_{\mu}^{\nu} U_{\nu} \quad \text{(how fields affect matter)},\\
\Box A_{\mu} &= -\frac{4\pi}{c} J_{\mu} \quad \text{(how matter affects fields)},\\
F^{\mu\nu}{}_{,\mu} &= 4\pi J_{\nu} \quad \text{(Maxwell's 1st 2 equations)},\\
F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} &= 0 \quad \text{(Maxwell's 2nd 2 equations)}.
\end{aligned}$$

## Retarded positions

- As above, Maxwell's equations determine the field from arbitrary colllections of moving charges from their 4-current density. To boost our intuition, let us look at the special case of a single charge.
- The electric and magnetic fields from a stationary charge $q$ at $\mathbf{r} = 0$ are (in c.g.s. units)

$$\begin{aligned}\mathbf{E} &= q\frac{\hat{\mathbf{r}}}{r^2} = q\frac{\mathbf{r}}{r^3},\\ \mathbf{B} &= \mathbf{0}.\end{aligned}$$

- What are the fields created by a charge moving with velocity $v$ in the $x$-direction? Since the last two equations give the answer in the rest frame $S$ of the charge, all we need to do is Lorentz transform them into the frame $S'$ where the charge is moving. Doing this gives the new electric field

$$\begin{aligned}E'_x &= E_x = \frac{q}{r^3}x,\\ E'_y &= \gamma E_y = \frac{q}{r^3}\gamma y,\\ E'_z &= \gamma E_z = \frac{q}{r^3}\gamma z.\end{aligned}$$

- All that remains is to reexpress this result in terms of the new coordinates $(x', y', z')$. In $S'$, the charge is moving, so $\mathbf{E}'$ will depend on the new time $t'$. Let us calculate the field at the time $t' = 0$ in $S'$ (this is when the charge is at the origin of the frame $S'$). At this instant, $x = \gamma x'$, since more generally $x = \gamma x' + \gamma v t'$. Since $y' = y$ and $z' = z$ at all time, this gives

$$\begin{aligned}E'_x &= \frac{q}{r^3}\gamma x',\\ E'_y &= \frac{q}{r^3}\gamma y',\\ E'_z &= \frac{q}{r^3}\gamma z',\end{aligned}$$

*i.e.*

$$\mathbf{E}' = \gamma\frac{q}{r^3}\mathbf{r}' = \frac{\gamma q}{\left(\gamma^2 x'^2 + y'^2 + z'^2\right)^{3/2}}\mathbf{r}' = \gamma q\frac{\left(x'^2 + y'^2 + z'^2\right)^{1/2}}{\left(\gamma^2 x'^2 + y'^2 + z'^2\right)^{3/2}}\hat{\mathbf{r}}'$$

- **Conclusion 1:** The *magnitude* $E = |\mathbf{E}|$ of the field becomes anisotropic: decreased by a factor $\gamma^2$ in front of and behind the moving charge and increased by a factor $\gamma$ in the perpendicular direction.
- **Conclusion 2:** The *direction* of the field ($\hat{\mathbf{r}}'$) still points straight away from the *instantaneous* position of the charge. This is remarkable, since the electromagnetic field can only propagate at the speed of light, so the charge must have caused this field at a time when it was in a different position (the so-called *retarded position*). Sure enough, this remarkable property no longer holds if the charge accelerates, leading to the Abraham-Lorentz force fiasco.

## A fly in the ointment

- Finally, you should know that as beautiful as it is, this whole theory has a lethal flaw (resolved by quantum mechanics). There is an instability whereby the electric field created by an accelerating electron acts back on the the electron causing it to undergo runaway linear acceleration! Aside from the fact that $E$ is formally infinite at the location of a point charge (a problem avoided if the electron somehow were to have a finite size), one might think that there should be no net force by symmetry. This is clearly the case for a stationary charge and, as we saw on the previous page, it is even true for a charge in uniform motion since all field lines point towards the current rather than retarded position. However, it breaks down for accelerated motion: in c.g.s. units and for speeds $v \ll c$, this so-called Abraham-Lorentz force on a particle of charge $q$ is

  $$\mathbf{F} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{a}},$$

  *i.e.*, it depends on the time-derivative of the acceleration, the *third* derivative of the position with respect to time. For $v \ll c$ we have $\mathbf{F} = m\mathbf{a}$ and hence

  $$\dot{\mathbf{a}} = \omega\mathbf{a},$$

  where the frequency

  $$\omega = \frac{3}{2}\frac{c^3 m}{q^2} \approx \left(6.266 \times 10^{-24} s\right)^{-1}$$

  for the case of an electron. The solution to this equation is

  $$\mathbf{a} = \mathbf{a}_0 e^{\omega t}, \quad \mathbf{u} = \mathbf{u}_0 + \frac{1}{\omega}\mathbf{a}_0 e^{\omega t},$$

  *i.e.*, the electron will all on its own increase its velocity exponentially, doubling on a timescale around $10^{-23}$ seconds, in stark contrast to what we actually observe!

- Note: This problem is alleviated in quantum field theory, but remains important — it is intimately linked with the so-called self-energy of the electron and the issue known as renormalization.