MIT Course 8.033, Fall 2005, Relativistic dynamics
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Topics

- Formula summary
- Momentum & energy
- Acceleration & force (optional)
- Transformation of force (optional)
- Transformation of acceleration (optional)

Dynamics toolbox: formula summary

- Mass-energy unification:

$$E = mc^2 = m_0 \gamma c^2$$

- Momentum 4-vector:

$$\mathbf{P} \equiv m_0 \mathbf{U} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

- Energy formula:

$$E = \sqrt{(m_0 c^2)^2 + (cp)^2}$$

- Velocity formula:

$$\beta = \frac{cp}{E}$$

- **Optional material:**
  - Acceleration 4-vector:

$$\mathbf{A} \equiv \frac{d\mathbf{U}}{d\tau} = \gamma_u^2 \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} + \gamma_u^4 \frac{\mathbf{u} \cdot \mathbf{a}}{c^2} \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix}$$

  - Force 4-vector:

$$\mathbb{F} \equiv \frac{d}{d\tau} \mathbf{P} = \gamma_u \begin{pmatrix} \mathbf{F} \\ P/c \end{pmatrix} = m_0 \mathbf{A}$$

  - Power:

$$P = \dot{E} = \mathbf{u} \cdot \mathbf{F} = m_0 \gamma_u^3 \, \mathbf{u} \cdot \mathbf{a}$$

  - Force 3-vector:

$$\frac{\mathbf{F}}{m_0 \gamma_u} = \mathbf{a} + \gamma_u^2 \frac{\mathbf{u} \cdot \mathbf{a}}{c^2} \mathbf{u} = \begin{cases} \gamma_u^2 \mathbf{a} & (\mathbf{u} \text{ \& } \mathbf{a} \text{ parallel}) \\ \mathbf{a} & (\mathbf{u} \text{ \& } \mathbf{a} \text{ perpendicular}) \end{cases}$$

  - Acceleration 3-vector:

$$m\mathbf{a} = \mathbf{F} - \frac{P\mathbf{u}}{c^2}$$

  - Force transformation:

$$\begin{aligned} F'_x &= \frac{F_x - \frac{v}{c^2} P}{1 - \frac{u_x v}{c^2}}, \\ F'_y &= \frac{F_y}{\gamma \left(1 - \frac{u_x v}{c^2}\right)}, \\ F'_z &= \frac{F_z}{\gamma \left(1 - \frac{u_x v}{c^2}\right)}, \\ P' &= \frac{P - vF_x}{1 - \frac{u_x v}{c^2}} \end{aligned}$$

Momentum & energy toolbox:

- Relativistic mass:

$$m = \gamma m_0$$

- Mass-energy unification:

$$E = mc^2$$

- Momentum 4-vector (momentum-energy unification):

$$\mathbf{P} \equiv m_0\mathbf{U} = m_0\frac{d\mathbf{X}}{d\tau} = m_0\gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = m \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix},$$

(Use upper case **X**, **U** and **P** for the 4-vectors to avoid confusion with the **x**, **u** and **p** 3-vectors.)

- Handy velocity formula follows straight from this:

$$\beta = \frac{cp}{E}$$

- Rest energy:

$$E_0 = m_0c^2$$

is total energy of particle in the frame where it is at rest

- Kinetic enery:

$$K = E - E_0 = mc^2 - m_0c^2 = m_0c^2(\gamma - 1) = \frac{1}{2}m_0u^2 + O\left(\frac{u}{c}^4\right)$$

- Rest mass invariant:

$$m_0 = \frac{1}{c}\sqrt{-\mathbf{P}^t\boldsymbol{\eta}\mathbf{P}} = \frac{1}{c^2}\sqrt{E^2 - c^2p^2},$$

giving the handy relations

$$E = \sqrt{(m_0c^2)^2 + (cp)^2},$$

$$p \equiv |\mathbf{p}| = \sqrt{\frac{E^2}{c^2} - (m_0c)^2}.$$

- Low-speed limit $|\beta| \ll 1$:

$$E \approx m_0c^2 + \frac{1}{2}m_0u^2,$$

$$p = m_0\gamma u \approx m_0u.$$

- High-speed limit $|\beta| \approx 1$ ($\gamma \gg 1$, $E \gg E_0$):

$$E \approx cp$$

This becomes exact ($E = cp$) for particles moving with speed of light, like photons and gravitons.

- $-\mathbf{P}^t \boldsymbol{\eta} \mathbf{P} = (E/c)^2 - p^2$ is invariant also for *system* of particles, since

$$\mathbf{P}_{\text{tot}}{}' \equiv \sum_i \mathbf{P}_i' = \sum_i \mathbf{\Lambda} \mathbf{P}_i = \mathbf{\Lambda} \left(\sum_i \mathbf{P}_i \right) = \mathbf{\Lambda} \mathbf{P}_{\text{tot}}.$$

- We derived $\mathbf{p} = m_0 \gamma u$ only for 1-dimensional collision. But *any* collision is 1-dimensional in the frame where the total momentum is zero!

Acceleration & force (optional!)

- The acceleration 4-vector $\mathbf{A}$ and the Force 4-vector $\mathbb{F}$ are less useful than their 4-vector cousins $\mathbf{X}$, $\mathbf{U}$, $\mathbf{P}$ and $\mathbf{K}$. We'll use $\mathbb{F}$ mainly for deriving the force transformation law, which will in turn give us the transformation law for electromagnetic fields. We'll use upper case $\mathbf{A}$ for the acceleration 4-vector to avoid confusion with the the acceleration 3-vector $\mathbf{a}$, and the annoying symbol $\mathbb{F}$ for the force 4-vector to avoid confusion with the the force 3-vector $\mathbf{F}$.

- Acceleration 4-vector:

$$\begin{aligned}
\mathbf{A} &\equiv \frac{d\mathbf{U}}{d\tau} = \gamma_u \frac{d\mathbf{U}}{dt} = \gamma_u \frac{d}{dt}\gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = \gamma_u \frac{d}{dt}\gamma_u \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix} \\
&= \gamma_u^2 \begin{pmatrix} \dot{\mathbf{u}} \\ 0 \end{pmatrix} + \gamma_u \dot{\gamma}_u \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix} = \gamma_u \begin{pmatrix} \mathbf{a} + \dot{\gamma}_u \mathbf{u} \\ \dot{\gamma}_u c \end{pmatrix} \\
&= \gamma_u^2 \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} + \gamma_u^4 \frac{\mathbf{u}\cdot\mathbf{a}}{c^2} \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix},
\end{aligned}$$

  where in the last step, we have used the fact that

$$\dot{\gamma}_u = \gamma_u^3 \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}.$$

- Force 4-vector:

$$\mathbb{F} \equiv \frac{d}{d\tau}\mathbf{P} = \gamma_u \frac{d}{dt}\mathbf{P} = \gamma_u \frac{d}{dt} m_0 \mathbf{U} = m_0 \frac{d}{d\tau}\mathbf{U},$$

  so by definition, we have

$$\mathbb{F} = m_0 \mathbf{A}.$$

  (Note that this does *not* apply the Newtonian result $\mathbf{F} = ma$!)

- Interpretation of Force 4-vector:

$$\mathbb{F} = \gamma_u \frac{d}{dt}\mathbf{P} = \gamma_u \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{E}/c \end{pmatrix} = \gamma_u \begin{pmatrix} \mathbf{F} \\ P/c \end{pmatrix},$$

  where $\mathbf{F} = \dot{p}$ is the familiar force 3-vector and $P = \dot{E}$ is the power, the energy change per unit time (in Watts).

- Work-energy theorem:

$$dE = \mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot \frac{d\mathbf{r}}{dt} dt = \mathbf{F}\cdot \mathbf{u} dt,$$

  so the power satisfies

$$P = \dot{E} = \mathbf{u}\cdot\mathbf{F}.$$

- Force 3-vector explicitly: Dividing the above equation $\mathbb{F} = m_0\mathbf{A}$ by γ_u gives

$$\frac{\mathbf{F}}{m_0\gamma_u} = \mathbf{a} + \gamma_u^2 \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u}.$$

- Special case where **u** and **a** are parallel, *e.g.*, for linear motion:

$$\frac{\mathbf{F}}{m_0\gamma_u} = \mathbf{a} + \gamma_u^2 \frac{u^2\mathbf{a}}{c^2} = \left(1 + \gamma_u^2\beta^2\right)\mathbf{a} = \gamma_u^2\mathbf{a}.$$

- Special case where **u** and **a** are perpendicular, eg, for circular motion:

$$\frac{\mathbf{F}}{m_0\gamma_u} = \mathbf{a}$$

- Note that in relativity, **F** and **a** are generally *not* parallel, but that they are parallel for these two special cases.

- Acceleration 3-vector explicitly:

$$\mathbf{a} = \frac{\mathbf{F}}{m_0\gamma_u} - \frac{\mathbf{u}\cdot\mathbf{F}}{m_0\gamma_u c^2}\mathbf{u} = \frac{\mathbf{F}}{m} - \frac{P}{mc^2}\mathbf{u}.$$

  The last term (the departure from $\mathbf{F} = m\mathbf{a}$) is seen to have the form of a friction term proportional to the power put into the particle. Derivation: the three steps below.

$$\dot{\gamma}_u = \frac{d}{dt}\frac{m_0\gamma_u c^2}{m_0c^2} = \frac{d}{dt}\frac{E}{m_0c^2} = \frac{\dot{E}}{m_0c^2} = \frac{\mathbf{u}\cdot\mathbf{F}}{m_0c^2} = \frac{P}{m_0c^2}.$$

  Combining this with the other expression for $\dot{\gamma}_u$ above gives

$$\mathbf{u}\cdot\mathbf{a} = \frac{\mathbf{u}\cdot\mathbf{F}}{\gamma_u^3 m_0}.$$

  The above equation for **F** now becomes

$$\frac{\mathbf{F}}{m_0\gamma_u} = \mathbf{a} + \frac{P}{m_0\gamma_u c^2}\mathbf{u} = \mathbf{a} + \gamma_u^2\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u}.$$

Transformation of force

- Let's compute the transformation law for force by transforming to a frame S' moving with velocity v in the x-direction relative to S:

$$
\begin{aligned}
\mathbb{F}' &= \gamma_{u'}\begin{pmatrix} F'_x \\ F'_y \\ F'_z \\ P'/c \end{pmatrix} = \mathbf{\Lambda}\mathbb{F} = \gamma_u \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} F_x \\ F_y \\ F_z \\ P/c \end{pmatrix} \\
&= \gamma_u \begin{pmatrix} \gamma[F_x - \beta P/c] \\ F_y \\ F_z \\ \gamma[P/c - \beta F_x] \end{pmatrix} = \frac{\gamma_{u'}}{\gamma(1-\frac{u_x v}{c^2})}\begin{pmatrix} \gamma[F_x - \beta P/c] \\ F_y \\ F_z \\ \gamma[P/c - \beta F_x] \end{pmatrix}.
\end{aligned}
$$

  In the last step, we used the relation $\gamma_{u'} = \gamma_u \gamma[1 - u_x v/c^2]$ which we proved earlier when transforming the velocity 4-vector U — it followed from the fact that its normalization is Lorentz invariant, *i.e.*, $\mathbf{U}'^t \boldsymbol{\eta} \mathbf{U}' = \mathbf{U}^t \boldsymbol{\eta} \mathbf{U}$.

- The 4 components now give our desired force transformation equations:

$$
\begin{aligned}
F'_x &= \frac{F_x - \frac{v}{c^2}P}{1 - \frac{u_x v}{c^2}}, \\
F'_y &= \frac{F_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \\
F'_z &= \frac{F_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \\
P' &= \frac{P - vF_x}{1 - \frac{u_x v}{c^2}},
\end{aligned}
$$

  where $P = \mathbf{u}\cdot\mathbf{F}$ as usual.

- If we take S to be the rest frame of the particle, then $\mathbf{u} = 0$, $P = \mathbf{u}\cdot\mathbf{F} = 0$ and this simplifies to $F'_x = F_x$, $F'_y = F_y/\gamma$, $F'_z = F_z/\gamma$, so in the frame S' where the particle is moving, the force is unaffected in the parallel direction and suppressed by γ in the transverse directions.

Transformation of acceleration

- We could derive expressions using an approach like for force, but the results are so messy that it's not particularly useful — it's better to deal with explicit problems as needed.

- Here's a useful special case that you get to derive on a problem set (probably PS7): For an arbitrary acceleration $\mathbf{a}$ in S, the acceleration $\mathbf{a}'$ in S' is related to $\mathbf{a}$ via

$$\begin{aligned} a_x &= \frac{a'_x}{\gamma^3(1 + vu'_x/c^2)^3} \\ a_y &= \frac{a'_y}{\gamma^2(1 + vu'_x/c^2)^2}, \end{aligned}$$

  with the **important caveat** that the expression for a_y is only valid for the case where either $u'_y = 0$ or $a'_x = 0$.