Welcome back to 8.033!

Christian Doppler

1803-1853, Austrian

Image Courtesy of Wikipedia.

Why opposite sense?

Summary of last lecture:

- Time dilation
- Length contraction
- Relativity of simultaneity
- Problem solving tips

MIT Course 8.033, Fall 2006, Lecture 6

Max Tegmark

Today: Relativistic Kinematics

- Space/time unification: η, imaginary rotations, etc.
- Proper time, rest length, timelike, spacelike, null
- More 4-vectors: **U**, **K**
- Velocity addition
- Doppler effect
- Aberration

Velocity addition

Transformation toolbox: velocity addition

SIMPLER WITH 2x2 MATRICES

- If the frame S' has velocity v_1 relative to S and the frame S'' has velocity v_2 relative to S' (both in the x-direction), then what is the speed v_3 of S'' relative to S?
- $\mathbf{x}' = \mathbf{\Lambda}(v_1)\mathbf{x}$ and $\mathbf{x}'' = \mathbf{\Lambda}(v_2)\mathbf{x}' = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)\mathbf{x}$, so
- $\mathbf{\Lambda}(\mathbf{v}_3) = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)$, *i.e.*

$$\begin{pmatrix} \gamma_3 & 0 & 0 & -\gamma_3\beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_3\beta_3 & 0 & 0 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 & 0 & -\gamma_2\beta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2\beta_2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{pmatrix}$$

$$= \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_2 & 0 & 0 & -[\beta_1+\beta_2] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -[\beta_1+\beta_2] & 0 & 0 & 1+\beta_1\beta_2 \end{pmatrix}$$

- Take ratio between $(1,4)$ and $(1,1)$ elements:

$$\beta_3 = -\frac{\mathbf{\Lambda}(v_3)_{41}}{\mathbf{\Lambda}(v_3)_{11}} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}.$$

- In other words,

$$v_3 = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

Transformation toolbox: perpendicular velocity addition

- Here's an alternative derivation of velocity addition that easily gives the non-parallel components too

- If the frame S' has velocity v in the x-direction relative to S and a particle has velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in S', then what is its velocity $\mathbf{u}$ in S?

- Applying the inverse Lorentz transformation

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + vx'/c^2) \end{aligned}$$

to two nearby points on the particle's world line and subtracting gives

$$\begin{aligned} dx &= \gamma(dx' + vdt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma(dt' + vdx'/c^2). \end{aligned}$$

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- Answer:

$$
\begin{aligned}
u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \\
u_y &= \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\gamma^{-1}\frac{dy'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \\
u_z &= \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\gamma^{-1}\frac{dz'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}}
\end{aligned}
$$

Unification of space & time

“Everything is relative” — or is it?

- All observers agree on rest length
- All observers agree on proper time
- All observers (as we’ll see later) agree on rest mass

Analogy: Australians and Bostonians agree on length of a 3-vector.

Transformation toolbox: boosts as generalized rotations

- A "boost" is a Lorentz transformation with no rotation
- A rotation around the z-axis by angle θ is given by the transformation
$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- We can think of a boost in the x-direction as a rotation by an imaginary angle in the (x, ct)-plane:
$$\mathbf{\Lambda}(-v) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},$$
where $\eta \equiv \tanh^{-1}\beta$ is called the *rapidity.*
- Proof: use hyperbolic trig identities on next page
- Implication: for multiple boosts in same direction, rapidities add and hence the order doesn't matter

Hyperbolic trig reminders

$$\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \\
\tanh^{-1} x &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)
\end{aligned}$$

$$\cosh \tanh^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\sinh \tanh^{-1} x = \frac{x}{\sqrt{1-x^2}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

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The Lorentz invariant

The only difference between space & time is a minus sign!

- The Minkowski metric

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  is left invariant by all Lorentz matrices $\mathbf{\Lambda}$:

$$\mathbf{\Lambda}^t \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta}$$

  (indeed, this equation is often used to define the set of Lorentz matrices — for comparison, $\mathbf{\Lambda}^t \mathbf{I} \mathbf{\Lambda} = \mathbf{I}$ would define rotation matrices)

- Proof: Show that works for boost along x-axis. Show that works for rotation along y-axis or z-axis. General case is equivalent to applying such transformations in succession.

- All Lorentz transforms leave the quantity

$$\mathbf{x}^t \boldsymbol{\eta} \mathbf{x} = x^2 + y^2 + z^2 - (ct)^2$$

invariant

- Proof:

$$\mathbf{x}'^t \boldsymbol{\eta} \mathbf{x}' = (\boldsymbol{\Lambda}\mathbf{x})^t \boldsymbol{\eta} (\boldsymbol{\Lambda}\mathbf{x}) = \mathbf{x}^t (\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda}) \mathbf{x} = \mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$$

(Also easy to see directly from top equation)

- (More generally, the same calculation shows that $\mathbf{x}^t \boldsymbol{\eta} \mathbf{y}$ is invariant)

- So just as the usual Euclidean squared length $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = \mathbf{r}^t \mathbf{r} = \mathbf{r}^t \mathbf{I} \mathbf{r}$ of a 3-vector is rotaionally invariant, the generalized "length" $\mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$ of a 4-vector is Lorentz-invariant.

- It can be positive or negative

4-vectors are null, spacelike or timelike:

- For events $\mathbf{x}_1$ and $\mathbf{x}_2$, their Lorentz-invariant separation is defined as

$$\Delta\sigma^2 \equiv \Delta\mathbf{x}^t \boldsymbol{\eta} \Delta\mathbf{x} = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$

- A separation $\Delta\sigma^2 = 0$ is called *null*
- A separation $\Delta\sigma^2 > 0$ is called *spacelike*, and

$$\Delta\sigma \equiv \sqrt{\Delta\sigma^2}$$

is called the *proper distance* (the distance measured in a frame where the events are simultaneous)

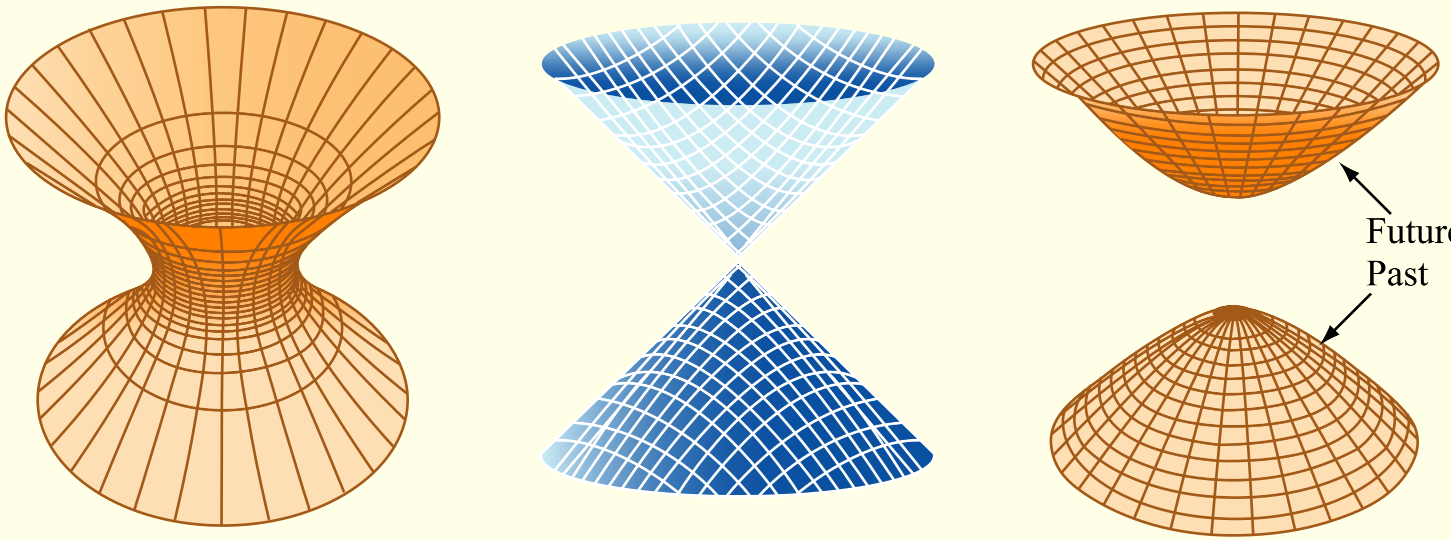
- A separation $\Delta\sigma^2 < 0$ is called *timelike*, and

$$\Delta\tau \equiv \sqrt{-\Delta\sigma^2}$$

is called the *proper time interval* (the time interval measured in a frame where the events are at the same place)

The Three Types of 4-Vectors:

SPACELIKE	NULL	TIMELIKE



$\Delta X^t \eta \, \Delta x > 0$	$\Delta X^t \eta \, \Delta x = 0$	$\Delta X^t \eta \, \Delta x > 0$
$\lvert \Delta x \rvert > c\Delta t$	$\lvert \Delta x \rvert = c\Delta t$	$\lvert \Delta x \rvert > c\Delta t$

Figure by MIT OCW.

Timelike,
spacelike
or null?

Transformation toolbox: perpendicular velocity addition

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\end{aligned}
$$

Application:

$d\tau$ is invariant

Transformation toolbox: velocity as a 4-vector

- For a particle moving along its world-line, define its velocity 4-vector

$$\mathbf{U} \equiv \frac{d\mathbf{X}}{d\tau} = \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix},$$

where

$$\gamma_u \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

- This is the derivative of its 4-vector $\mathbf{x}$ w.r.t. its proper time τ, since $d\tau = dt/\gamma_u$

- $\mathbf{U}' = \mathbf{\Lambda U}$:

$$\mathbf{U}' = \frac{d\mathbf{X}'}{d\tau'} = \frac{d\mathbf{\Lambda X}}{d\tau} = \mathbf{\Lambda}\frac{d\mathbf{X}}{d\tau} = \mathbf{\Lambda U},$$

since the proper time interval $d\tau$ is Lorentz-invariant

- This means that all velocity 4-vectors are normalized so that

$$\mathbf{U}^t \boldsymbol{\eta} \mathbf{U} = -c^2.$$

Which type?

- This immediately gives the velocity addition formulas:

$$\mathbf{U}' = \gamma_{u'} \begin{pmatrix} u'_x \\ u'_y \\ u'_z \\ c \end{pmatrix} = \mathbf{\Lambda}(-\mathbf{v})\mathbf{U} = \gamma_u \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_u \gamma [u_x + v] \\ \gamma_u u_y \\ \gamma_u y_z \\ \gamma_u \gamma [1 + \frac{u_x v}{c^2}] c \end{pmatrix} = \gamma_{u'} \begin{pmatrix} \frac{u_x + v}{1 + u_x v/c^2} \\ \frac{u_y/\gamma}{1 + u_x v/c^2} \\ \frac{u_z/\gamma}{1 + u_x v/c^2} \\ c \end{pmatrix},$$

where $\gamma_{u'} = \gamma_u \gamma \left[1 + \frac{u_x v}{c^2}\right]$ — this last equation follows from the fact that the 4-vector normalization in Lorentz invariant, *i.e.*,$\mathbf{u}'^t \boldsymbol{\eta} \mathbf{u}' = \mathbf{u}^t \boldsymbol{\eta} \mathbf{u} = -1$.

- The 1st 3 components give the velocity addition equations we derived previously.

This is how it *is* in the frame S’.

But how does it *look*?

Transforming a wave vector

- A plane wave

$$E(\mathbf{x}) = \sin(k_x x + k_y y + k_z z - \omega t) \tag{1}$$

is defined by the four numbers

$$\mathbf{k} \equiv \begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega/c \end{pmatrix} .$$

- If the wave propagates with the speed of light c (like for an electromagnetic or gravitational wave), then the frequency is determined by the 3D wave vector (k_x, k_y, k_z) through the relation $\omega/c = k$, where $k \equiv \sqrt{k_x^2 + k_y^2 + k_z^2}$

- How does the 4-vector $\mathbf{k}$ transform under Lorentz transformations? Let's see.

Using the Minkowski matrix, we can rewrite equation (1) as

$$E(\mathbf{x}) = \sin(\mathbf{k}^t \boldsymbol{\eta} \mathbf{x}).$$

Let's Lorentz transform this: $\mathbf{x} \to \mathbf{x}'$, $\mathbf{k} \to \mathbf{k}'$. Using that $\mathbf{x}' = \boldsymbol{\Lambda}\mathbf{x}$, let's determine $\mathbf{k}'$.

$$E' = \sin(\mathbf{k}'^t \boldsymbol{\eta} \mathbf{x}') = \sin(\mathbf{k}'^t \boldsymbol{\eta} \boldsymbol{\Lambda} \mathbf{x}) = \sin[(\boldsymbol{\Lambda}^{-1}\mathbf{k}')^t (\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda})\mathbf{x}] = \sin[(\boldsymbol{\Lambda}^{-1}\mathbf{k}')^t \boldsymbol{\eta} \mathbf{x}]$$

This equals E if $\boldsymbol{\Lambda}^{-1}\mathbf{k}' = \mathbf{k}$, *i.e.*, if the wave 4-vector transforms just as a normal 4-vector:

$$\mathbf{k}' = \boldsymbol{\Lambda}\mathbf{k}$$

This argument assumed that $E' = E$. Later we'll see that the electric and magnetic fields *do* in fact change under Lorentz transforms, but not in a way that spoils the above derivation (in short, the phase of the wave, $\mathbf{k}^t \boldsymbol{\eta} \mathbf{x}$, must be Lorentz invariant)

So a plane wave $\mathbf{k}$ in S is also a plane wave in S', and the wave 4-vector transforms in exactly the same way as $\mathbf{x}$ does.

Aberration and Doppler effects

- Consider a plane wave propagating with speed c in the frame S:

$$\mathbf{k} = k \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \\ 1 \end{pmatrix},$$

  where ck is the wave frequency and the angles θ and ϕ give the propagation direction in polar coordinates.

- Let's Lorentz transform this into a frame S' moving with speed v relative to S in the z-direction: $\mathbf{k}' = \mathbf{\Lambda k}$, *i.e.*,

$$\begin{aligned} \mathbf{k}' &= k' \begin{pmatrix} \sin\theta' \cos\phi' \\ \sin\theta' \sin\phi' \\ \cos\theta' \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\gamma\beta \\ 0 & 0 & -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \\ 1 \end{pmatrix} \\ &= k \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \gamma(\cos\theta - \beta) \\ \gamma(1 - \beta\cos\theta) \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} \phi' &= \phi \\ \cos\theta' &= \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \\ k' &= k\gamma(1 - \beta\cos\theta) \end{aligned}$$

- This matches equations (1)-(4) in the Weiskopf et al ray tracing handout

- The change in the angle θ is known as *aberration*

- The change in frequency ck is known as the Doppler shift — note that since $k = 2\pi/\lambda$, we have $\lambda'/\lambda = k/k'$.

- If we instead take the ratio $\sqrt{{k'}_x^2 + {k'}_y^2}/k'_z$ above, we obtain the mathematically equivalent form of the aberration formula given by Resnick (2-27b):
$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

- Examine classical limits

- Transverse Doppler effect: $\cos\theta = 0$ gives $\omega' = \omega\gamma$, *i.e.*, simple time dilation