Summary of last lecture:

(Assumptions leading to this?)

- We're done! The Lorentz transformation is

$$\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix},$$

$i.e.,$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} \gamma(x - \beta ct) \\ y \\ z \\ \gamma(ct - \beta x) \end{pmatrix}.$$

- Compare to Einstein's 1905 paper

MIT Course 8.033, Fall 2006, Lecture 5

Max Tegmark

Today: Relativistic Kinematics

- Time dilation
- Length contraction
- Relativity of simultaneity
- Proper time, rest length
- **Key people:** Einstein

IS IT RIGHT?

Implications: time dilation

- In the frame S, a clock is at rest at the origin ticking at time intervals that are Δt =1 seconds long, so the two consecutive ticks at $t = 0$ and $t = \Delta t$ have coordinates

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix}.$$

- In the frame S', the coordinates are

$$\mathbf{x}'_1 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{x}'_2 = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c\Delta t \end{pmatrix} = \begin{pmatrix} -\gamma v\Delta t \\ 0 \\ 0 \\ \gamma c\Delta t \end{pmatrix}$$

- So in S', the clock appears to tick at intervals $\Delta t' = \gamma\Delta t > \Delta t$, *i.e.*, slower! (Draw Minkowski diagram.)

Time dilation, cont'd

- The light clock movie says it all:
  http://www.anu.edu.au/Physics/qt/

Time dilation, cont'd

- The light clock movie says it all: *http://www.anu.edu.au/Physics/qt/*
- Cosmic ray muon puzzle
  - Created about 10km above ground
  - Half life 1.56×10^{-6} second
  - In this time, light travels 0.47 km
  - So how can they reach the ground?
  - $v \approx 0.99c$ gives $\gamma \approx 7$
  - $v \approx 0.9999c$ gives $\gamma \approx 71$
- Leads to twin paradox

Consider two frames in relative motion. For $t = 0$, the Lorentz transformation gives $x' = \gamma x$, where $\gamma > 1$.

Question: How long does a yard stick at rest in the unprimed frame look in the primed frame?

1. Longer than one yard
2. Shorter than one yard
3. One yard

Let’s measure the length of our moving eraser!

Implications: relativity of simultaneity

- Consider two events simultaneous in frame S:

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- In the frame S', they are

$$\begin{aligned} \mathbf{x}'_1 &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \mathbf{x}'_2 &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L \\ 0 \\ 0 \\ -\gamma\beta \text{Ł} \end{pmatrix} \end{aligned}$$

- So in S', the second event happened first!

- So S-clocks appear unsynchronized in S' - those with larger x run further ahead

Transformation toolbox: the inverse Lorentz transform

- Since $\mathbf{x}' = \mathbf{\Lambda}(v)\mathbf{x}$ and $\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}'$, we get the consistency requirement

$$\mathbf{x} = \mathbf{\Lambda}(-v)\mathbf{x}' = \mathbf{\Lambda}(-v)\mathbf{\Lambda}(v)\mathbf{x}$$

  for any event $\mathbf{x}$, so we must have $\mathbf{\Lambda}(-v) = \mathbf{\Lambda}(v)^{-1}$, the matrix inverse of $\mathbf{\Lambda}(v)$.

- Is it?

$$\mathbf{\Lambda}(-\mathbf{v})\mathbf{\Lambda}(\mathbf{v}) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

  i.e., yes!

Implications: length contraction

- Trickier than time dilation, opposite result (interval appears shorter, not longer)
- In the frame S, a yardstick of length L is at rest along the x-axis with its endpoints tracing out world lines with coordinates

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix}.$$

- In the frame S', these world lines are

$$\mathbf{x}'_1 = \begin{pmatrix} x'_1 \\ y'_1 \\ z'_1 \\ ct'_1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} -\gamma\beta ct \\ 0 \\ 0 \\ \gamma ct \end{pmatrix}$$

$$\mathbf{x}'_2 = \begin{pmatrix} x'_2 \\ y'_2 \\ z'_2 \\ ct'_2 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \\ 0 \\ ct \end{pmatrix} = \begin{pmatrix} \gamma L - \gamma\beta ct \\ 0 \\ 0 \\ \gamma ct - \gamma\beta L \end{pmatrix}$$

- Let's work out the new world lines of the yard stick endpoints
- $\mathbf{x}'_1 + \beta c t'_1 = 0$, so left endpoint world line is

$$x'_1 = -vt'_1$$

- $\mathbf{x}'_2 - \gamma L + \beta(ct'_2 + \gamma\beta L) = 0$, so right endpoint world line is

$$x'_2 = \gamma L - \beta(ct'_2 + \gamma\beta L) = \frac{L}{\gamma} - vt'_2$$

- Length in S' is

$$x'_2 - x'_1 = \frac{L}{\gamma} + v(t'_1 - t'_2) = \frac{L}{\gamma}$$

since both endpoints measured at same time ($t'_1 = t'_2$)

- Draw Minkowski diagram of this

- An observer in S' measures length as $x'_2 - x'_1$ at the same time t', - *not* at the same time t.
- Let's measure at $t' = 0$.
- $t'_1 = 0$ when $t = 0$ — at this time, $x'_1 = 0$
- $t'_2 = 0$ when $ct = \beta L$ - at this time, $\mathbf{x}'_2 = \gamma L - \gamma\beta^2 L = L/\gamma$
- So in S'-frame, measured length is $L' = L/\gamma$, *i.e., shorter*

Transformation toolbox: velocity addition

SIMPLER WITH 2x2 MATRICES

- If the frame S' has velocity v_1 relative to S and the frame S'' has velocity v_2 relative to S' (both in the x-direction), then what is the speed v_3 of S'' relative to S?
- $\mathbf{x}' = \mathbf{\Lambda}(v_1)\mathbf{x}$ and $\mathbf{x}'' = \mathbf{\Lambda}(v_2)\mathbf{x}' = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)\mathbf{x}$, so
- $\mathbf{\Lambda}(\mathbf{v}_3) = \mathbf{\Lambda}(v_2)\mathbf{\Lambda}(v_1)$, *i.e.*

$$\begin{pmatrix} \gamma_3 & 0 & 0 & -\gamma_3\beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_3\beta_3 & 0 & 0 & \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 & 0 & -\gamma_2\beta_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2\beta_2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{pmatrix}$$

$$= \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_2 & 0 & 0 & -[\beta_1+\beta_2] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -[\beta_1+\beta_2] & 0 & 0 & 1+\beta_1\beta_2 \end{pmatrix}$$

- Take ratio between $(1,4)$ and $(1,1)$ elements:

$$\beta_3 = -\frac{\mathbf{\Lambda}(v_3)_{41}}{\mathbf{\Lambda}(v_3)_{11}} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}.$$

- In other words,

$$v_3 = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

Transformation toolbox: perpendicular velocity addition

- Here's an alternative derivation of velocity addition that easily gives the non-parallel components too

- If the frame S' has velocity v in the x-direction relative to S and a particle has velocity $\mathbf{u}' = (u'_x, u'_y, u'_z)$ in S', then what is its velocity $\mathbf{u}$ in S?

- Applying the inverse Lorentz transformation

$$
\begin{aligned}
x &= \gamma(x' + vt') \\
y &= y' \\
z &= z' \\
t &= \gamma(t' + vx'/c^2)
\end{aligned}
$$

  to two nearby points on the particle's world line and subtracting gives

$$
\begin{aligned}
dx &= \gamma(dx' + vdt') \\
dy &= dy' \\
dz &= dz' \\
dt &= \gamma(dt' + vdx'/c^2).
\end{aligned}
$$

$$
\begin{aligned}
dx &= \gamma(dx' + v dt') \\
dy &= dy' \\
dz &= dz' \\
dt &= \gamma(dt' + v dx'/c^2).
\end{aligned}
$$

- Answer:

$$
\begin{aligned}
u_x &= \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \\
u_y &= \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\gamma^{-1}\frac{dy'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_y\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}} \\
u_z &= \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + \frac{v dx'}{c^2})} = \frac{\gamma^{-1}\frac{dz'}{dt'}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_z\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}}
\end{aligned}
$$

Transformation toolbox: boosts as generalized rotations

- A "boost" is a Lorentz transformation with no rotation
- A rotation around the z-axis by angle θ is given by the transformation

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- We can think of a boost in the x-direction as a rotation by an imaginary angle in the (x, ct)-plane:

$$\mathbf{\Lambda}(-v) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix},$$

  where $\eta \equiv \tanh^{-1}\beta$ is called the *rapidity*.

- Proof: use hyperbolic trig identities on next page
- Implication: for multiple boosts in same direction, rapidities add and hence the order doesn't matter

Hyperbolic trig reminders

$$
\begin{aligned}
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}) \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}) \\
\tanh^{-1} x &= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)
\end{aligned}
$$

$$\cosh \tanh^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\sinh \tanh^{-1} x = \frac{x}{\sqrt{1 - x^2}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

The Lorentz invariant

- The Minkowski metric

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

  is left invariant by all Lorentz matrices $\boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta}$$

  (indeed, this equation is often used to define the set of Lorentz matrices — for comparison, $\boldsymbol{\Lambda}^t \mathbf{I} \boldsymbol{\Lambda} = \mathbf{I}$ would define rotation matrices)

- Proof: Show that works for boost along x-axis. Show that works for rotation along y-axis or z-axis. General case is equivalent to applying such transformations in succession.

- All Lorentz transforms leave the quantity

$$\mathbf{x}^t \boldsymbol{\eta} \mathbf{x} = x^2 + y^2 + z^2 - (ct)^2$$

invariant

- Proof:

$$\mathbf{x}'^t \boldsymbol{\eta} \mathbf{x}' = (\boldsymbol{\Lambda}\mathbf{x})^t \boldsymbol{\eta} (\boldsymbol{\Lambda}\mathbf{x}) = \mathbf{x}^t (\boldsymbol{\Lambda}^t \boldsymbol{\eta} \boldsymbol{\Lambda}) \mathbf{x} = \mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$$

- (More generally, the same calculation shows that $\mathbf{x}^t \boldsymbol{\eta} \mathbf{y}$ is invariant)
- So just as the usual Euclidean squared length $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = \mathbf{r}^t \mathbf{r} = \mathbf{r}^t \mathbf{I} \mathbf{r}$ of a 3-vector is rotaionally invariant, the generalized "length" $\mathbf{x}^t \boldsymbol{\eta} \mathbf{x}$ of a 4-vector is Lorentz-invariant.
- It can be positive or negative

- For events $\mathbf{x}_1$ and $\mathbf{x}_2$, their Lorentz-invariant separation is defined as

$$\Delta s^2 \equiv \Delta\mathbf{x}^t \boldsymbol{\eta} \Delta\mathbf{x} = \Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2$$

- A separation $\Delta s^2 = 0$ is called *null*

- A separation $\Delta s^2 > 0$ is called *spacelike*, and

$$\Delta\sigma \equiv \sqrt{\Delta s^2}$$

  is called the *proper distance* (the distance measured in a frame where the events are simultaneous)

- A separation $\Delta s^2 < 0$ is called *timelike*, and

$$\Delta\tau \equiv \sqrt{-\Delta s^2}$$

  is called the *proper time interval* (the time interval measured in a frame where the events are at the same place)

“Everything is relative” — or is it?

- All observers agree on rest length
- All observers agree on proper time
- All observers (as we’ll see later) agree on rest mass

Summary lecture:

- Time dilation
- Length contraction
- Relativity of simultaneity
- Problem solving tips