
Welcome back to 8.033!

Orbits

- For a particle moving in the Schwarzschild metric, the energy E and and angular momentum L are conserved. It's convenient to divide these two by the rest mass of the particle and work with the energy per unit rest enery $\tilde{E} \equiv E/m$ (dimensionless, since $c = 1$) and the angular momentum per unit rest mass, $\tilde{L} \equiv L/m$ (units of length).

- In terms of these two constants, the equations of motion become

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \tilde{E}^2 - \tilde{V}(\tilde{L}, r)^2, \\ \frac{d\varphi}{d\tau} &= \frac{\tilde{L}}{r^2}, \end{aligned}$$

  where the *effective potential* per unit rest mass is

$$\tilde{V}(\tilde{L}, r)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

  and the proper time τ is related to the t-coordinate i
  (far-away time) by

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} = \gamma_r^2 \tilde{E}.$$

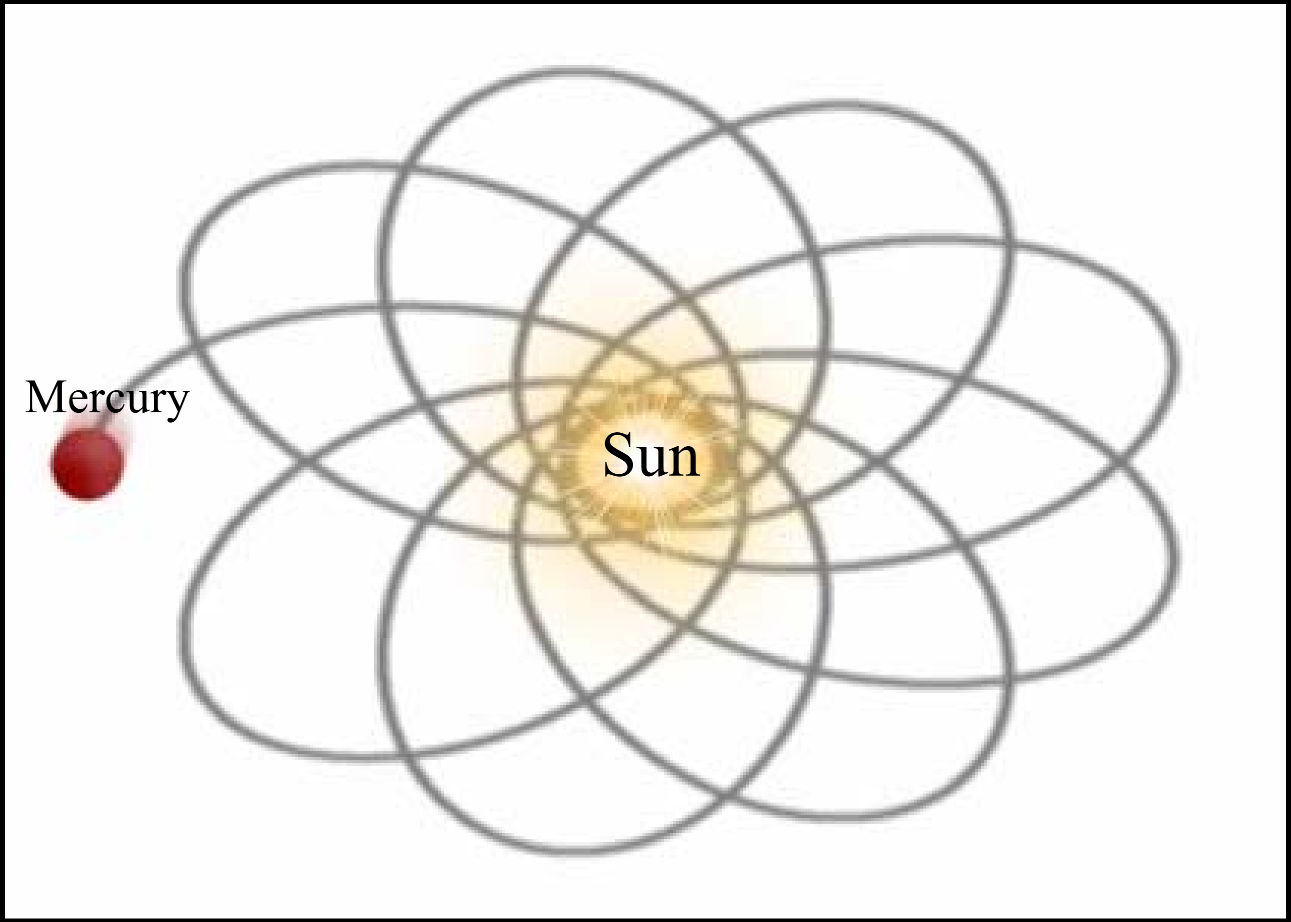
- $\tilde{E} \geq 1$ is a neccessary condition for being able to escape to $r = \infty$ (where $\tilde{V} = 0$).

- To build intuition for Schwarzschild orbits and the effective potential, I highly recommend the interactive simulator at `http://www.fourmilab.ch/gravitation/orbits/`. Note that it crashes and requires reloading if you accidentally fall in.

Interesting circular orbits:

	r	v_{shell}	$\tilde{E}$	$\tilde{L}$
Photon orbit	$3M$	1	∞	∞
Tourist orbit	$4M$	$\frac{1}{\sqrt{2}}$	1	$4M$
Last stable orbit	$6M$	$\frac{1}{2}$	$\sqrt{\frac{8}{9}}$	$\sqrt{12}M$
Classical limit	∞	0	1	∞

Perihelion advance: 43 arcseconds/century



MIT Course 8.033, Fall 2006, Lecture 24

Max Tegmark

TODAY'S TOPICS:

- Lensing
- Shapiro time delay
- Evidence for GR
- Time travel
- Mysteries for you to figure out
- o

Simulation by Andrew Hamilton.

Gravitational lensing

lensing.mov

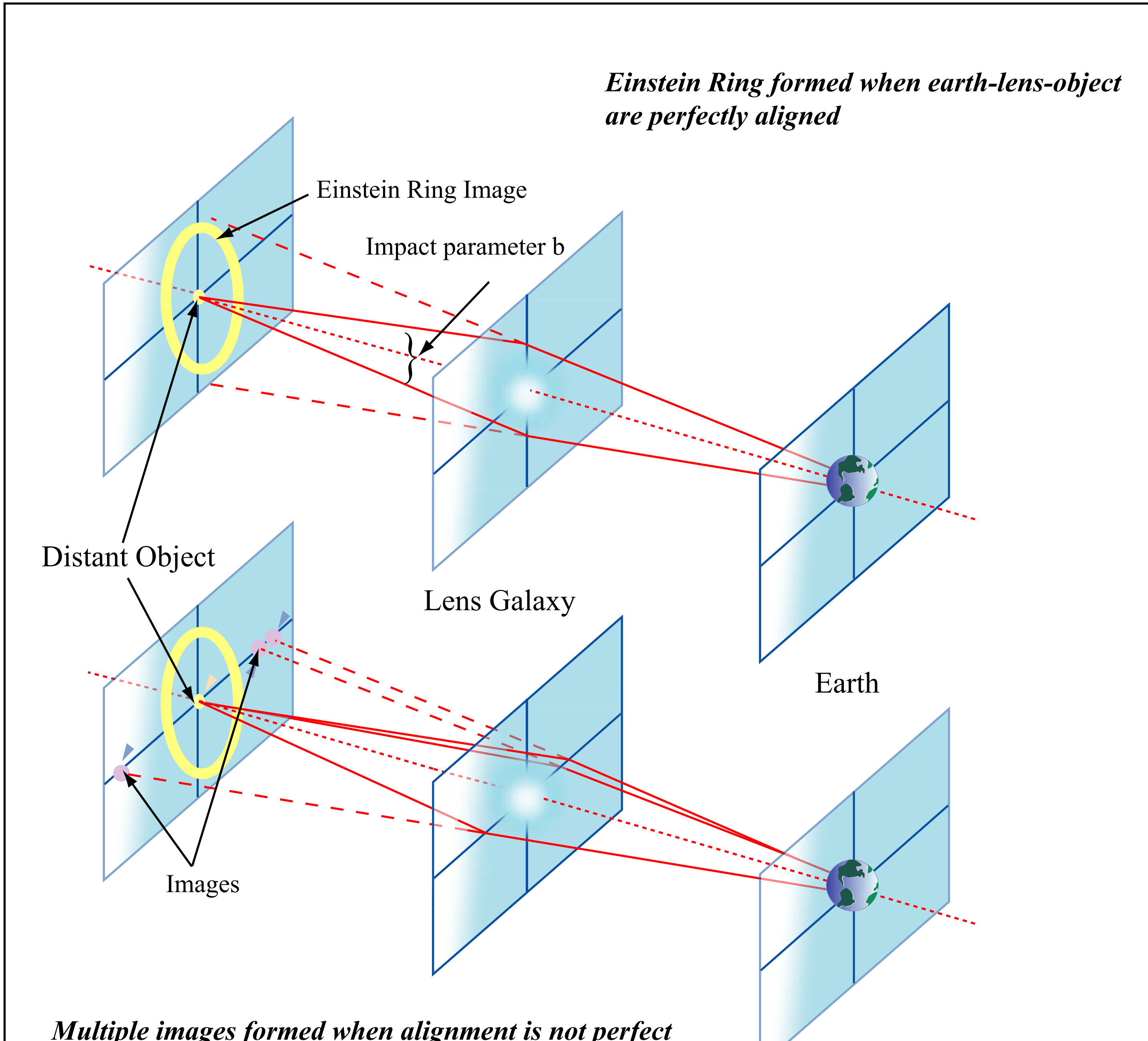
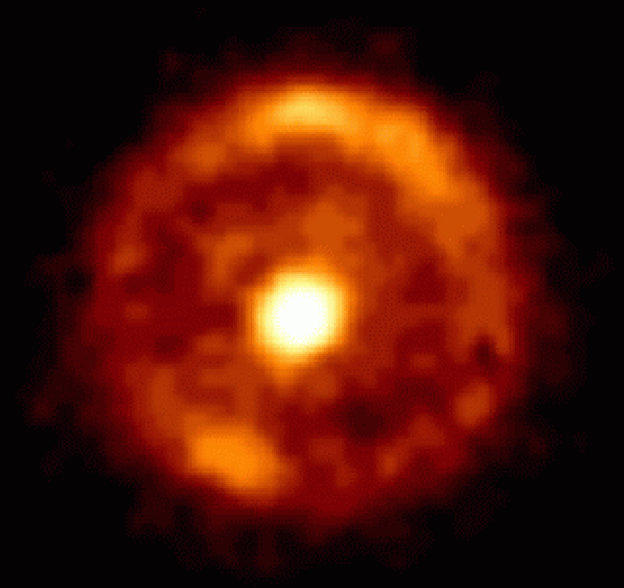
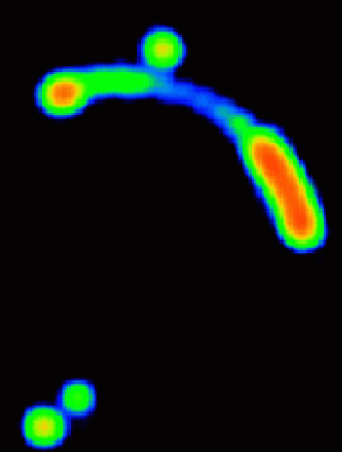


Figure by MIT OCW.





Images courtesy of NASA.

(Upper) The Hubble Space Telescope picture of the distant galaxy 1938+666 which has been imaged into an Einstein ring by an intervening galaxy. The intervening galaxy shows up as the bright spot in the centre of the ring. The picture was taken in the infra-red region of the spectrum and the computer-generated colour of the image has been chosen simply for ease of viewing.

(Lower) The MERLIN radio picture of the radio source 1938+666 embedded in the distant galaxy. The incomplete ring (or arc) shows that the radio source is not perfectly aligned with the lens galaxy and the Earth. The lens galaxy does not contain a radio source and hence does not show up in this picture. The colours are computer-generated and represent different levels of radio brightness.

Orbits

How deal with photons?

- For a particle moving in the Schwarzschild metric, the energy E and and angular momentum L are conserved. It's convenient to divide these two by the rest mass of the particle and work with the energy per unit rest enery $\tilde{E} \equiv E/m$ (dimensionless, since $c = 1$) and the angular momentum per unit rest mass, $\tilde{L} \equiv L/m$ (units of length).

- In terms of these two constants, the equations of motion become

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}(\tilde{L}, r)^2,$$

$$\frac{d\varphi}{d\tau} = \frac{\tilde{L}}{r^2},$$

where the *effective potential* per unit rest mass is

$$\tilde{V}(\tilde{L}, r)^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

and the proper time τ is related to the t-coordinate (far-away time) by

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} = \gamma_r^2 \tilde{E}.$$

- $\tilde{E} \geq 1$ is a neccessary condition for being able to escape to $r = \infty$ (where $\tilde{V} = 0$).

- To build intuition for Schwarzschild orbits and the effective potential, I highly recommend the interactive simulator at `http://www.fourmilab.ch/gravitation/orbits/`. Note that it crashes and requires reloading if you accidentally fall in.

Photon orbits and the deflection of light

- The orbit equations above aren't useful for photons since they involve $d\tau$, which is zero for photons.

- Taylor & Wheeler show that eliminating $d\tau$ between the equations and taking the limit where the rest mass $m \to 0$ gives

$$
\begin{aligned}
\left(\frac{dr}{dt}\right)^2 &= \gamma_r^{-4}\left[1-\left(\frac{b}{\gamma_r r}\right)^2\right] \\
r\frac{d\varphi}{dt} &= \pm\frac{b}{r\gamma_r^2},
\end{aligned}
\tag{4}
$$

where the *impact parameter*

$$b \equiv \frac{L}{E}$$

is the only constant of motion that we need to keep track of.

- The impact parameter is related to $r_{\rm min}$, the distance of closest approach, by

$$b = \gamma_{r_{\rm min}} r_{\rm min}.$$

  Since this shows that $b = r_{\rm min}$ if $M = 0$, we can interpret the impact parameter b as the smallest r-coordinate that the photon would ever get if the black hole were not there and the photon simply moved in a straight line.

- For a photon, the only closed orbit is a circular one with $r = 3M$.

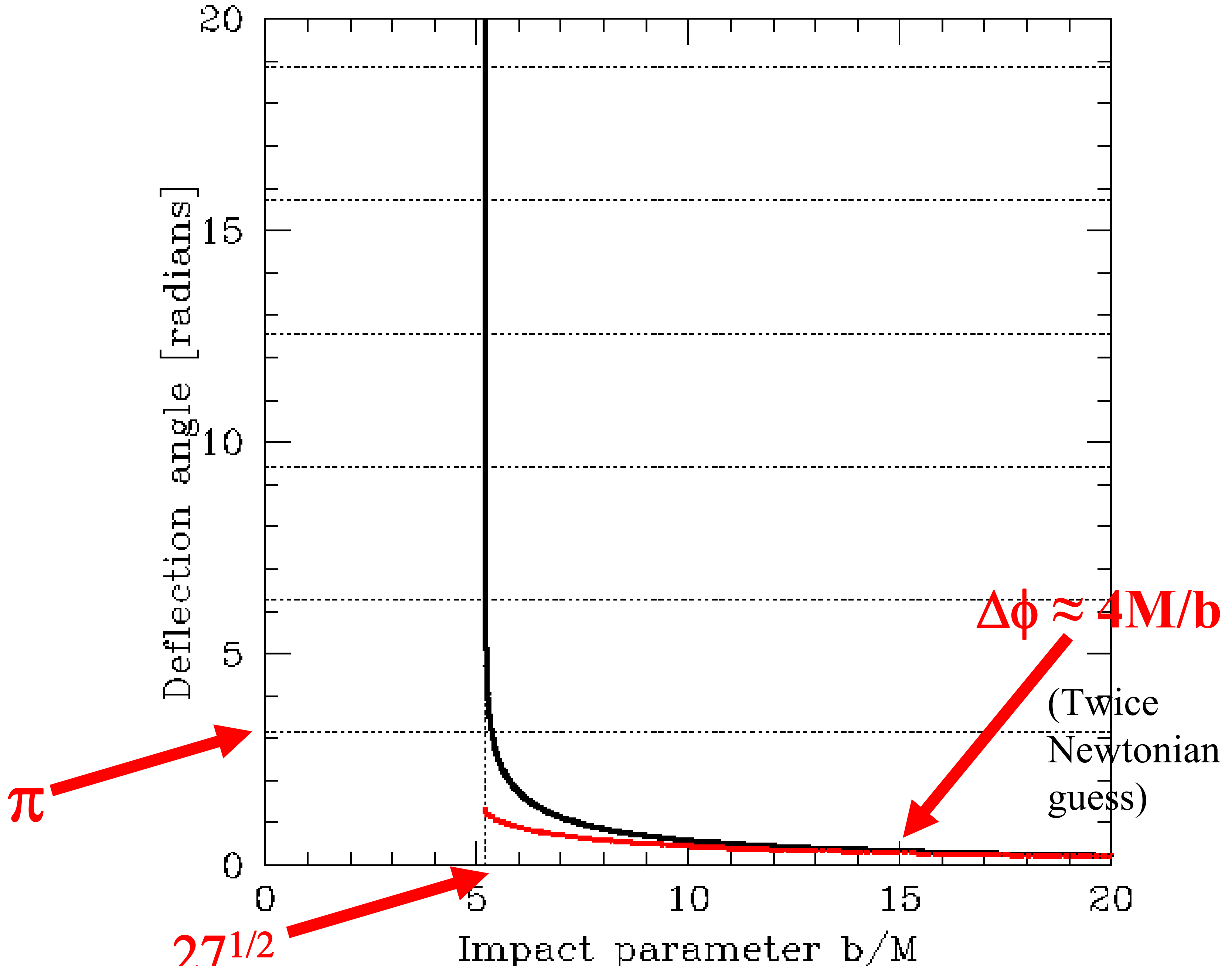
- **Deflection angle**: Consider a photon arriving from far away, getting deflected by the gravity of a star or a Schwarzschild black hole and flying off to infinity again. As shown in Taylor & Wheeler project D, the total deflection angle is

$$\Delta\phi = -\pi + 2\int_0^1 \left[1 - u^2 - \frac{2M}{r_{\rm min}}(1-u^3)\right]^{-1/2} du,$$

where $r_{\rm min}$ is the distance of closest approach, related to the impact parameter b is given by

$$b = \gamma_{r_{\rm min}} r_{\rm min} = \left(1 - \frac{2M}{r_{\rm min}}\right)^{-1/2} r_{\rm min}$$

In the plot below, I've done this integral numerically.



Interesting special cases:

- $\Delta\phi \approx \frac{4M}{b}$ for $b \gg M$ (dotted red curve in the plot). Einstein applied this approximation to the deflection of starlight near the Sun as confirmed by Eddington in 1919 and later to exquisite 0.1% precision for radio waves from the quasar 3C273 passing near the Sun. A heuristic Newtonian estimate gives only $\Delta\phi \approx \frac{2M}{b}$.
- $\Delta\phi \to \infty$ as $b \to \sqrt{27}M \approx 5.196M$, corresponding to the photon getting captured and making infinitely many orbits as it spirals in toward the circular $r = 3M$ orbit.
- For $b < \sqrt{27}$, the photon disappears into the black hole. This also means that no photons can come from the black hole direction towards you with larger impact parameters, so that when far from a black hole, you will see it as a black disc of radius $\sqrt{27}M$, *not* $2M$. In other words, the black disk appears with $27/4 \approx 7$ larger area than you might naively expect.
- The plot shows that for $b \approx 5.357M$, $\Delta\phi = \pi$ (dotted horizontal lines show π, 2π, 3π, *etc.*), which means that you can see your own image in a circle around the black hole of thie radius.
- The plot also shows that there will be an infinite number of smaler circular images of you, piling up towards the innermost radius $r = \sqrt{27}M$, so pointing your telescope at the perimeter of that scary-looking black disk reveals quite an interesting "halo".

Chema lensing movie

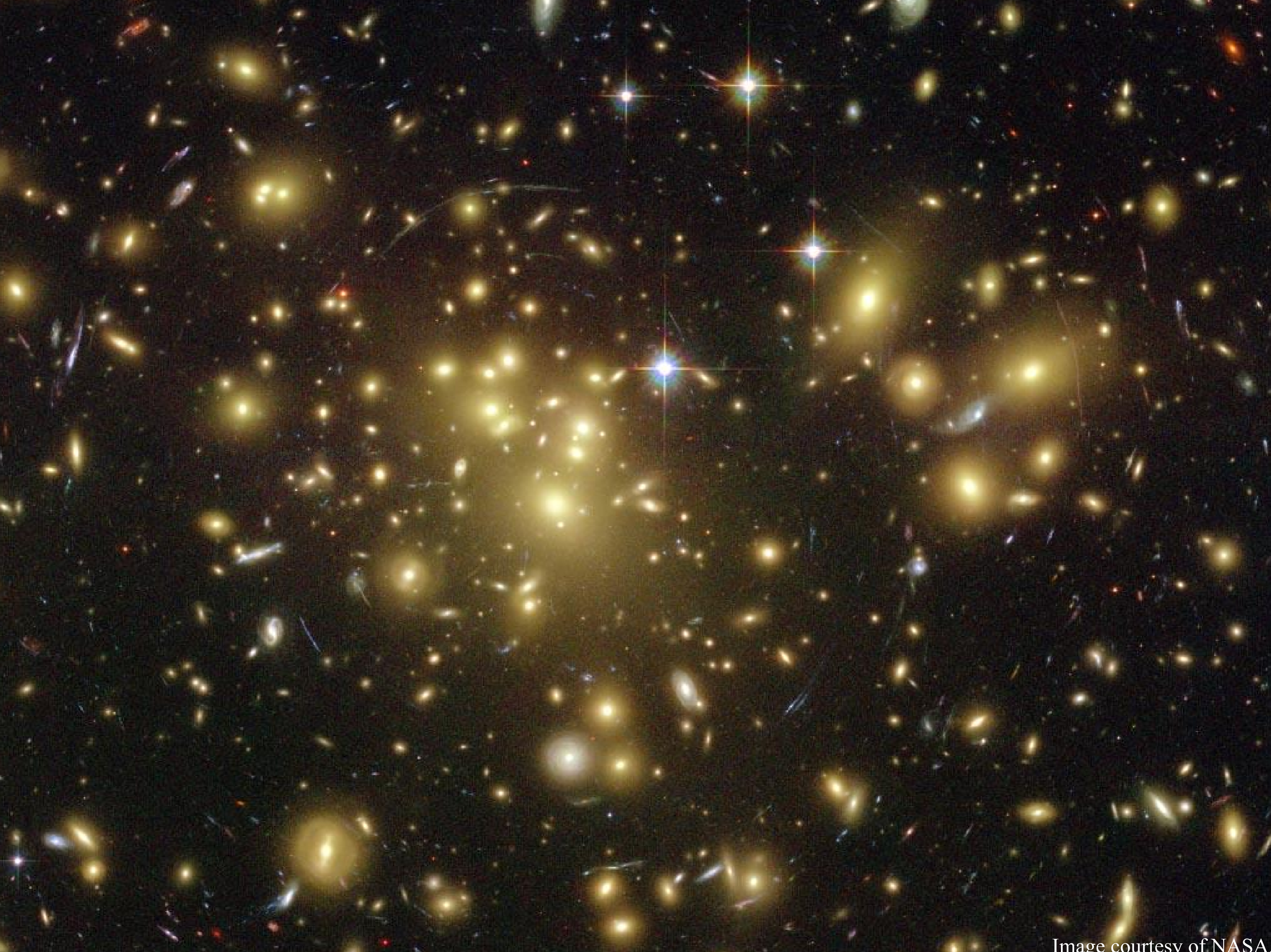


Image courtesy of NASA


What black holes look like

Galactic center movie

What’s wrong with this picture?

The Einstein field equations

(how matter curves spacetime)

The field equations (optional)

The *Einstein field equations* are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where

$$\begin{aligned}
G_{\mu\nu} &\equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \\
R &\equiv g^{\mu\nu} R_{\mu\nu}, \\
R_{\mu\nu} &\equiv R^{\alpha}_{\mu\alpha\nu}, \\
R^{\alpha}_{\mu\nu\beta} &= \Gamma^{\alpha}_{\nu\beta,\mu} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\gamma}_{\mu\beta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\gamma}_{\nu\beta}\Gamma^{\alpha}_{\mu\gamma}, \\
\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right),
\end{aligned}$$

and $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$, *i.e.*

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}.$$

Here repeated indices are to be summed over from 0 to 3, commas denote derivatives, and G is Newton's gravitational constant. Throughout this section, we will use units where the speed of light $c = 1$. In the Einstein field equations, the dependent variables are the two tensors $g_{\mu\nu}$ and $T_{\mu\nu}$. They are both symmetric, and thus contain ten independent components each. g is the *metric tensor*, and describes the structure of spacetime at each spacetime point x^{μ}. $T_{\mu\nu}$ is called the *stress-energy tensor*, and describes the state of the matter (what is *in* space) at each point. The quantities $G_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$, $R^{\alpha}_{\mu\nu\beta}$ and $R_{\mu\nu}$ are named after Einstein, Christoffel, Riemann and Ricci, respectively.

The Schwarzschild metric is obtained by setting $T_{\mu\nu} = 0$ except at the origin and solving for the most general spherically symmetric time-independent metric. Requiring this requiring only azimuthal symmetric (ϕ-independence) gives the Kerr metric corresponding to rotating black holes. For the FRW metric, $T_{\mu\nu}$ is not zero but takes a simple diagonal form independent of position.

Evidence for GR

Evidence for GR:

- Gravitational redshift (in GPS, say)
- Factor of 2 in small light deflection formula (lensing angle)
- Mercury perihelion shift
- Shapiro time delay (movie)
- Apparent existence of black holes
- Cosmological redshift in expanding Universe
- Hulse-Taylor pulsar

Future tests:

- Gravity Probe B
- Gravitational waves (LIGO, LISA, …)

Russell Hulse & Joe Taylor 1993

Hulse-Taylor Pulsar (PSR 1913+16)

A binary pulsar found in 1974. It consists of a pulsar (a neutron star) with a pulsation period of 59 milliseconds) and a companion that move around each other in an elongated orbit (period 7.75 hours, periastron 1.1 Rsun, apastron 4.8 Rsun). The orbit is gradually shrinking, by about 3.1 mm per orbit, because of gravitational waves as predicted by the general theory of relativity. This will cause the two stars to merge - about 300 million years from now. The extreme density and small orbital radius of this system results in a huge orbital precession of 4.2° per year (Mercury does 1° every 8400 years).

Orbit shrinks by 3.1mm/year:

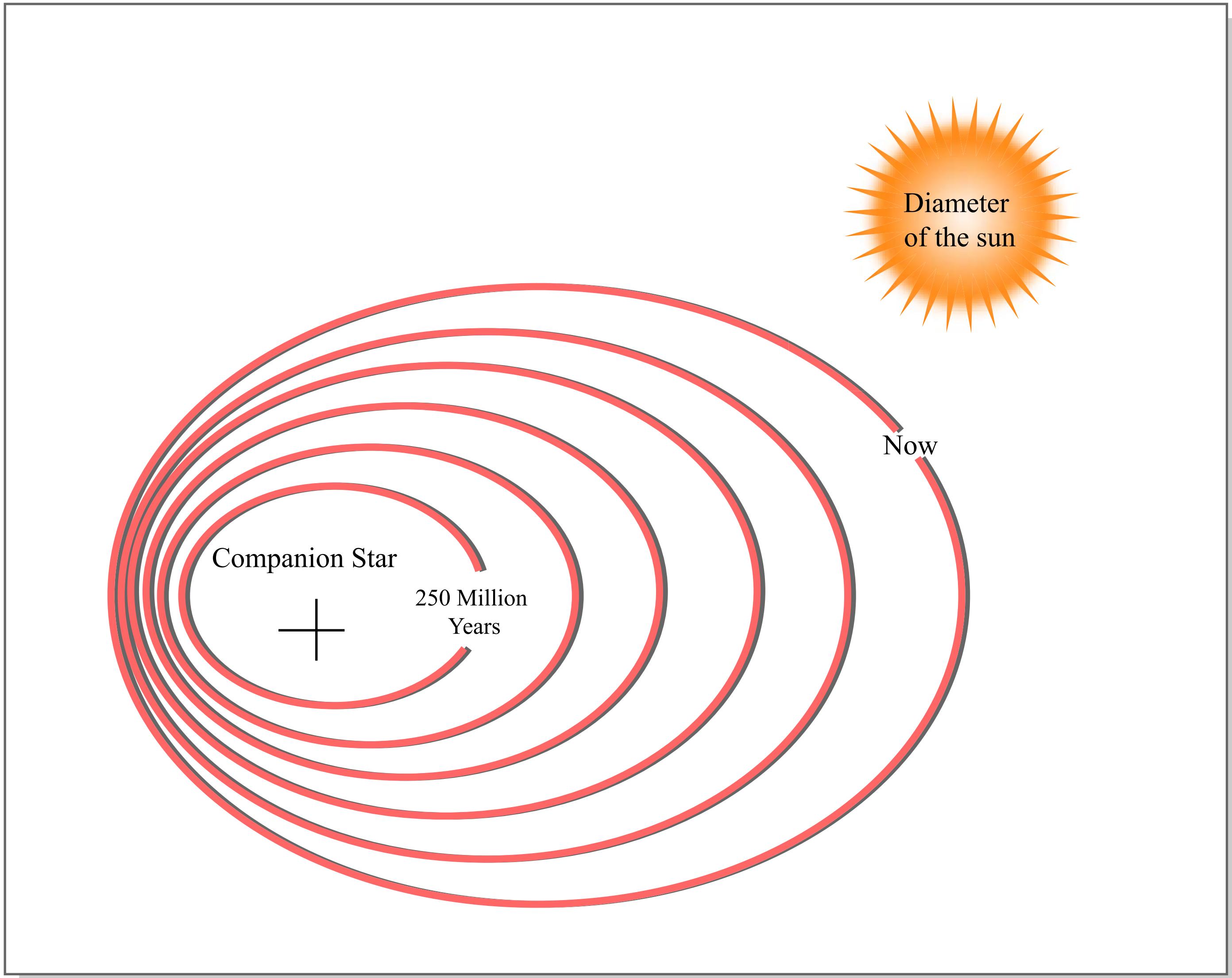
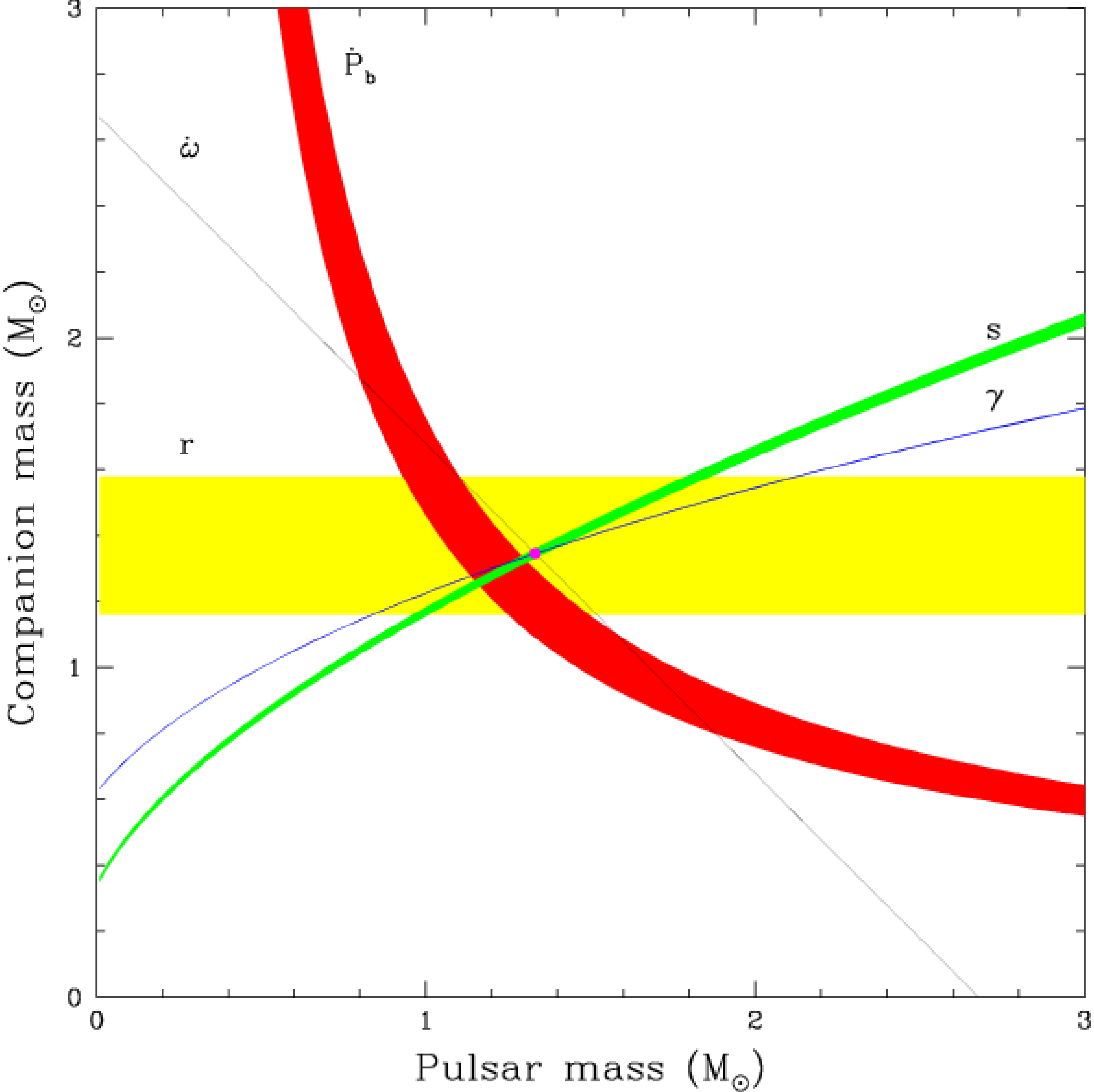


Figure by MIT OCW.



$\dot{P}_b$
$\dot{\omega}$
s
γ
r
Companion mass ($M_\odot$)
Pulsar mass ($M_\odot$)
0
1
2
3
0
1
2
3

What is the answer to the question of life, the Universe and everything?

42

42

But what, more specifically, is the question?

What is the outer radius of a rainbow, in degrees?

Q: What is the maximum accretion efficiency of a black hole, in percent?

A: $1-3^{-1/2} \approx 42\%$

Time travel

A Brief History of Time Travel:

1895 British author H.G. Wells publishes "The Time Machine"

1905 Einstein's special relativity: forward time travel possible

1916 Einstein's general relativity

1937 Kurt Gödel shows that the universe itself could be a time machine

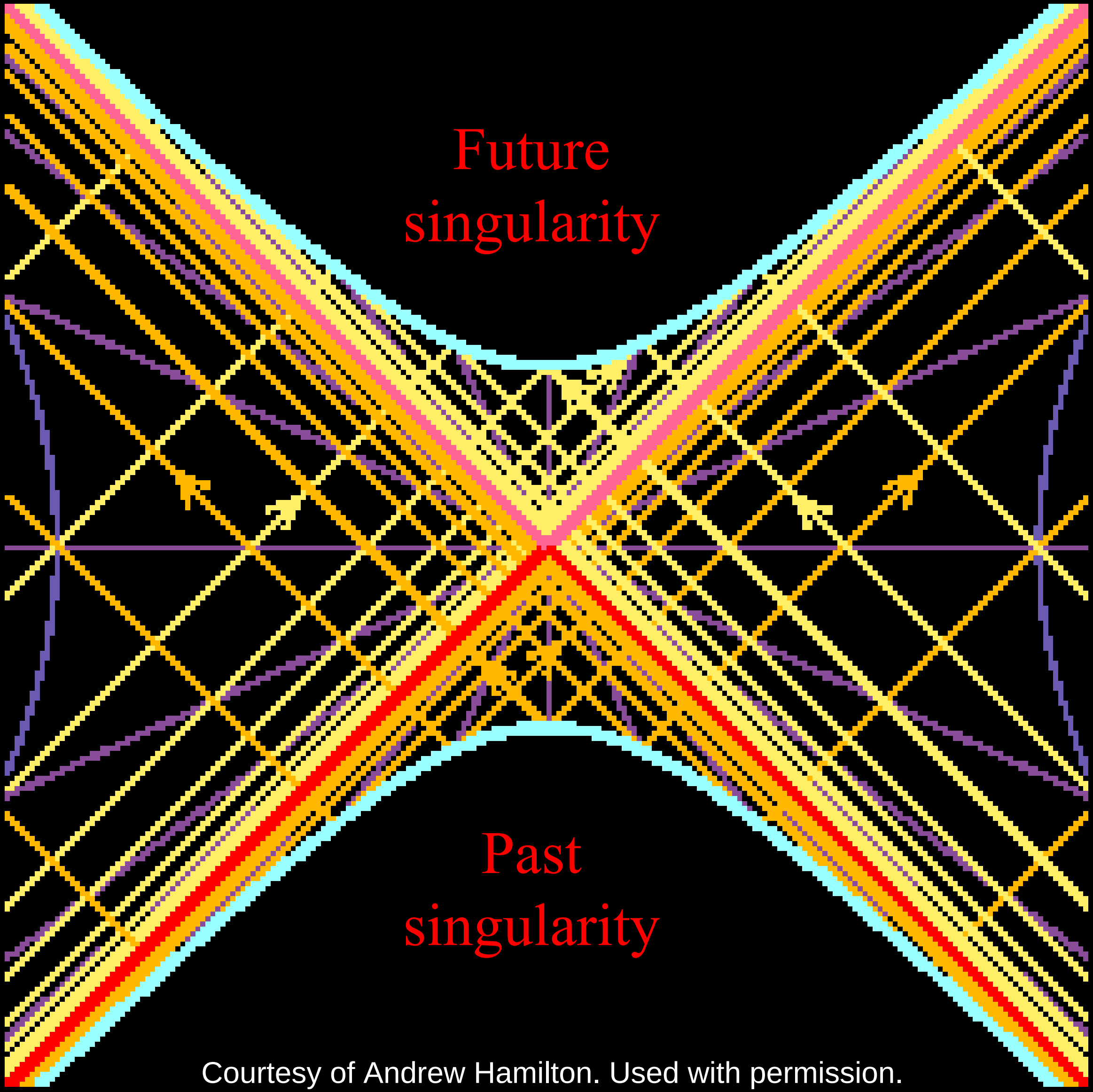
1974 Frank Tipler finds that vast, spinning cylinder permits time travel

1988 Kip Thorne suggests using wormholes as a means of time travel

1991 Richard Gott finds that cosmic strings permit time travel

Kruskal:
the singularity isn't simply a point!

Kruskal: black holes have another side!



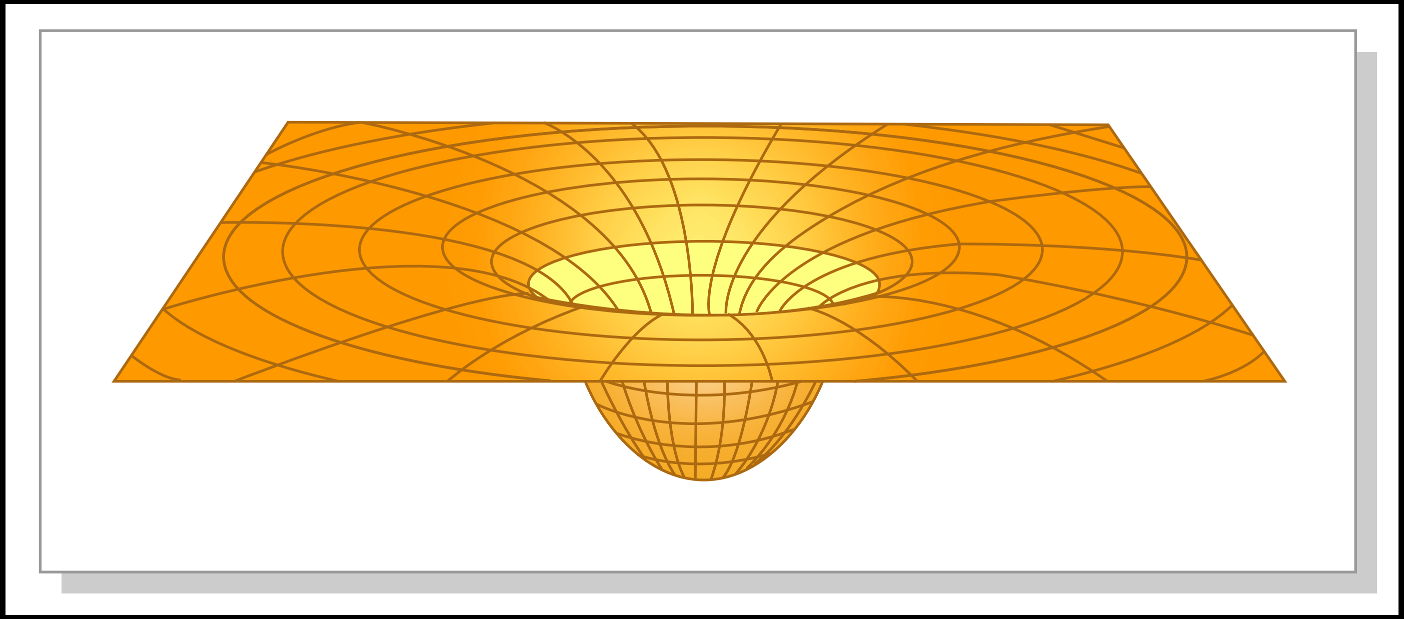


Figure by MIT OCW.

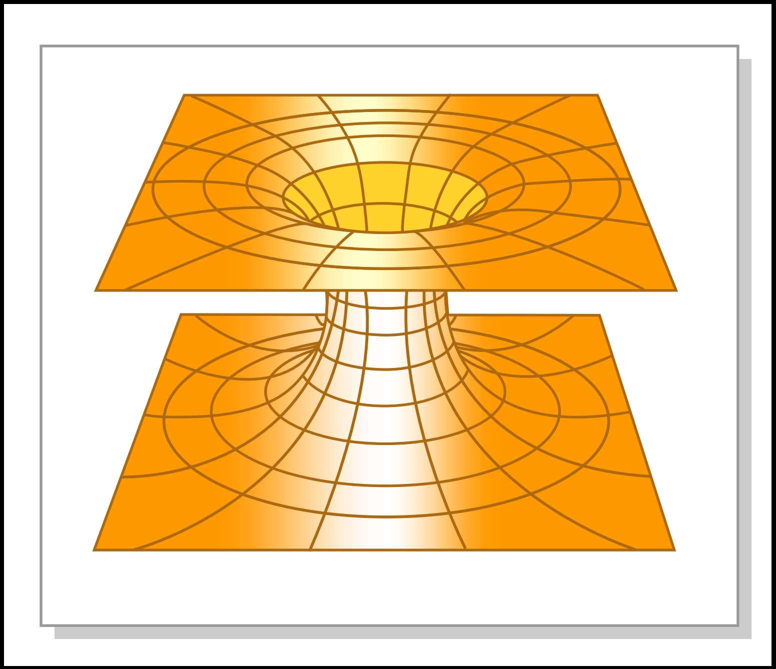


Figure by MIT OCW.

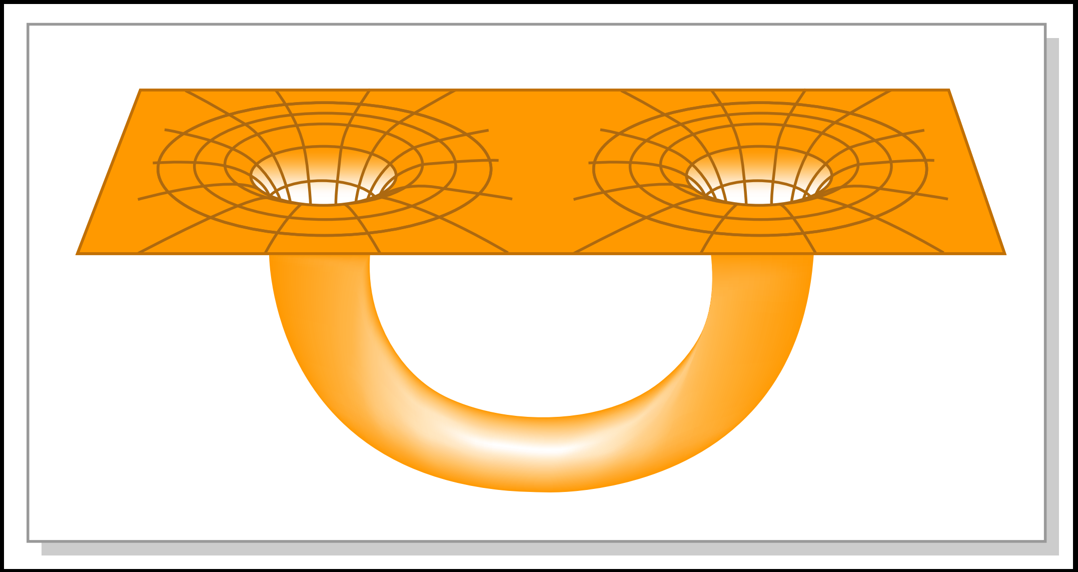


Figure by MIT OCW.

But can the wormhole be made stable?

Mysteries

UNSOLVED PROBLEMS:

- Where do the “constants” come from? Why 3+1 dimensions?
- Is there a quantum gravity TOE? (M-theory? Black hole evaporation?)
- Proton decay?
- SUSY?
- Higgs?
- Neutrino properties?
- Dark energy?
- Dark matter?
- Inflation?

Is GR correct? (applies to: Dark energy, Dark matter, Inflation)

Many questions for you to answer!

Some final words…

Think for yourself!