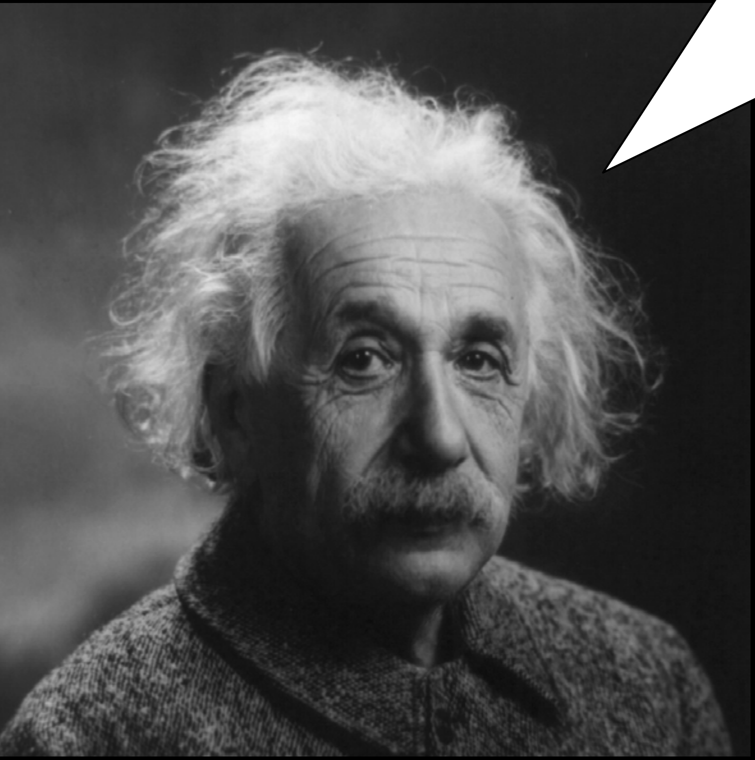


Welcome back to 8.033!

Image courtesy of Wikipedia.

Summary of electromagnetism:

Key formula summary

- Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{u} \times \mathbf{B})$$

- Lorentz transforming the electromagnetic field:

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - \beta B_z) \\ E'_z &= \gamma(E_z + \beta B_y) \\ B'_x &= B_x \\ B'_y &= \gamma(B_y + \beta E_z) \\ B'_z &= \gamma(B_z - \beta E_y). \end{aligned}$$

E^2-B^2 *is Lorentz-invariant*

E•B *is Lorentz-invariant*

Summary of electromagnetism:

- Current 4-vector:

$$\mathbb{J} \equiv \begin{pmatrix} J_x \\ J_y \\ J_z \\ \rho c \end{pmatrix} = \rho_0 \mathbf{U},$$

where the proper charge density ρ_0 is the local charge density in a frame where $\mathbf{J} = 0$.

- Electric field from stationary charge q (Coulomb's law):

$$\mathbf{E} = \frac{q}{r^2}\hat{\mathbf{r}} = \frac{q}{\mathbf{x}^2 + y^2 + z^2}\hat{\mathbf{r}}$$

- Electric field from charge q moving in x-direction:

$$\mathbf{E}' = \frac{\gamma q r'}{\left(\gamma^2 x'^2 + y'^2 + z'^2\right)^{3/2}}\hat{\mathbf{r}}'$$

Maxwell details and Greek index stuff won't be on tests

The Standard Model Lagrangian

Summary of last lecture:

$$
\begin{aligned}
\mathcal{L} = \; & -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4} g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \\
& \tfrac{1}{2} i g_s^2 (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu + \bar{G}^a \partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
& M^2 W^+_\mu W^-_\mu - \tfrac12 \partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac12 \partial_\mu A_\nu \partial_\mu A_\nu - \tfrac12 \partial_\mu H \partial_\mu H - \\
& \tfrac12 m_h^2 H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - M^2\phi^+\phi^- - \tfrac12\partial_\mu\phi^0\partial_\mu\phi^0 - \tfrac{1}{2c_w^2} M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} + \\
& \tfrac{2M}{g}H + \tfrac12(H^2+\phi^0\phi^0+2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - \\
& W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
& W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - \\
& W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \tfrac12 g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \\
& \tfrac12 g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + \\
& g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - \\
& W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \\
& \tfrac18 g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] - \\
& gMW^+_\mu W^-_\mu H - \tfrac12 g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac12 ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - \\
& W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac12 g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \\
& \phi^+\partial_\mu H)] + \tfrac12 g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + \\
& igs_wMA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + \\
& igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \tfrac14 g^2W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \\
& \tfrac14 g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2-1)^2\phi^+\phi^-] - \tfrac12 g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + \\
& W^-_\mu\phi^+) - \tfrac12 ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac12 g^2 s_wA_\mu\phi^0(W^+_\mu\phi^- + \\
& W^-_\mu\phi^+) + \tfrac12 ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- - \\
& g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar\nu^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + \\
& m_d^\lambda)d^\lambda_j + igs_wA_\mu[-(\bar{e}^\lambda\gamma e^\lambda) + \tfrac23(\bar{u}^\lambda_j\gamma u^\lambda_j) - \tfrac13(\bar{d}^\lambda_j\gamma d^\lambda_j)] + \tfrac{ig}{4c_w}Z^0_\mu[(\bar\nu^\lambda\gamma^\mu(1+ \\
& \gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac43 s_w^2 - 1 - \gamma^5)u^\lambda_j) + \\
& (\bar{d}^\lambda_j\gamma^\mu(1 - \tfrac83 s_w^2 - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt2}W^+_\mu[(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+ \\
& \gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt2}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1+\gamma^5)u^\lambda_j)] + \\
& \tfrac{ig}{2\sqrt2}\tfrac{m_e^\lambda}{M}[-\phi^+(\bar\nu^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)] - \tfrac{g}{2}\tfrac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + \\
& i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt2}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1+ \\
& \gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt2}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j] - \\
& \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \\
& \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + \\
& igc_wW^+_\mu(\partial_\mu\bar{X}^0X^- - \partial_\mu\bar{X}^+X^0) + igs_wW^+_\mu(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+Y) + \\
& igc_wW^-_\mu(\partial_\mu\bar{X}^-X^0 - \partial_\mu\bar{X}^0X^+) + igs_wW^-_\mu(\partial_\mu\bar{X}^-Y - \partial_\mu\bar{Y}X^+) + \\
& igc_wZ^0_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) + igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \partial_\mu\bar{X}^-X^-) - \\
& \tfrac12 gM[\bar{X}^+X^+H + \bar{X}^-X^-H + \tfrac{1}{c_w^2}\bar{X}^0X^0H] + \tfrac{1-2c_w^2}{2c_w}igM[\bar{X}^+X^0\phi^+ - \\
& \bar{X}^-X^0\phi^-] + \tfrac{1}{2c_w}igM[\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-] + igMs_w[\bar{X}^0X^-\phi^+ - \\
& \bar{X}^0X^+\phi^-] + \tfrac12 igM[\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0]
\end{aligned}
$$

Note to self: do we need to permission the image of this equation?

(From T.D. Gutierrez)

MIT Course 8.033, Fall 2006, Lecture 16

Max Tegmark

Practical stuff:

- Taylor & Wheeler notation alerts: c=G=1, m_0=m, fluffy

Today's topic: General Relativity basics

- Principle of equivalence
- Light bending, gravitational redshift
- Metrics

Journalist: ***Professor Eddington, is it really true that only three people in the world understand Einstein's theory of general relativity?***

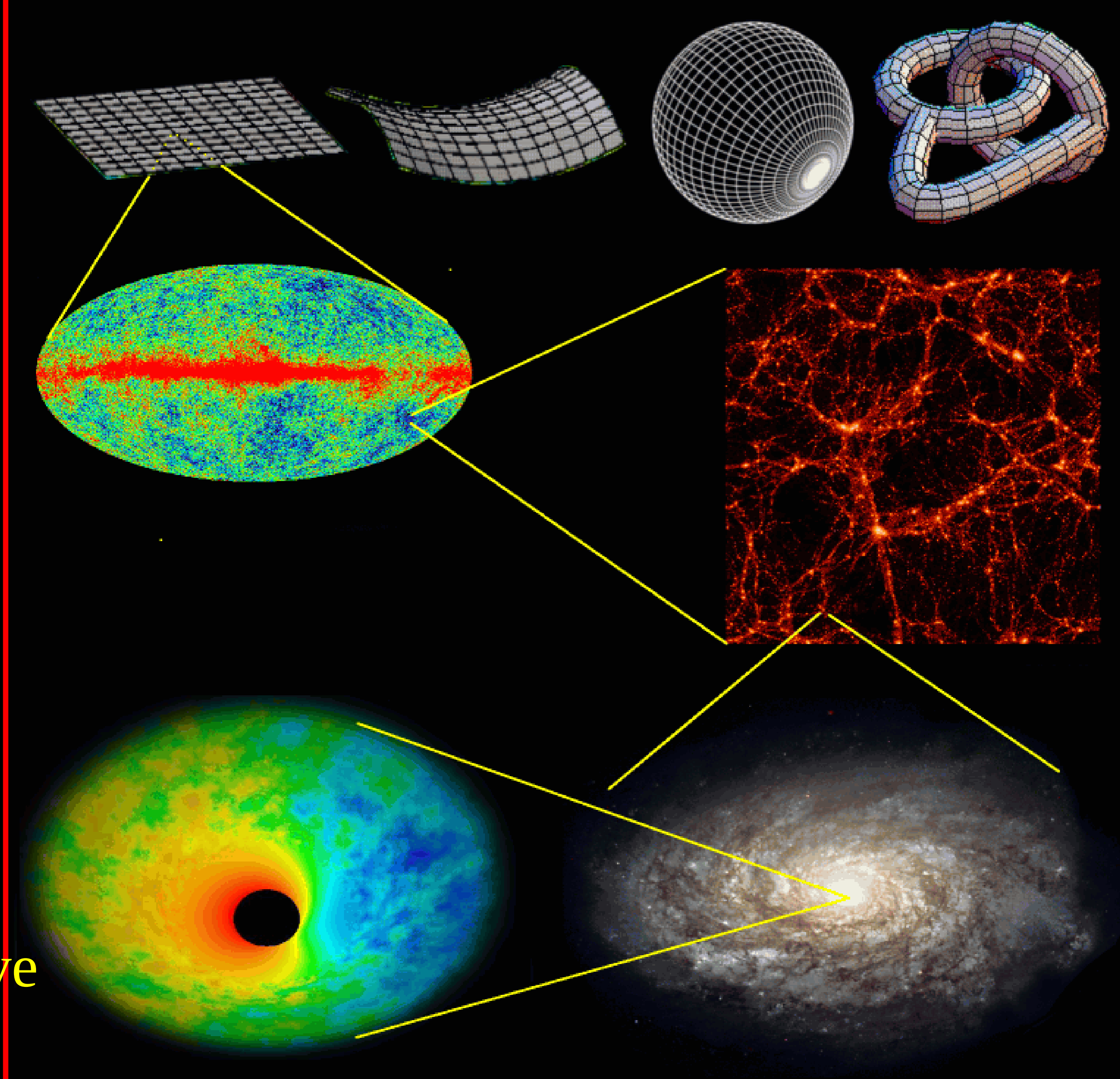
Eddington: ***Who is the third?***

Q: What medium is a gravitational wave a vibration of?

A: Space!

Tegmark 2002, Science, 296, 1427-1433

Note to self: we need to get IP permission from Science for this image?



Space can vibrate, stretch, curve – perhaps even “melt”!

The laws of physics are the same in all inertial frames

The laws of physics are the same in all ~~inertial~~ frames

Special relativity concept summary

- Space and time unified into 4D spacetime.
- Analogous unification for other 4-vectors (momentum+energy, *etc.*).
- Lorentz transform relates 4-vectors in different inertial frames. Example: fast moving clocks are slower, shorter and heavier.
- $E = mc^2$. Example: nuclear power.

General relativity concept summary

- Spacetime is not static but dynamic, globally expanding and locally curving and contracting to form black holes *etc.*
- Matter curves spacetime so that things moving “straight” (along geodesics) through curved spacetime appear deflected/accelerated (gravity).

Newtonian gravity

The "gravitational field" $\mathbf{g}$ is minus the gradient $\nabla\phi$ of the Newtonian gravitational potential ϕ. Units: ϕ/c^2 is dimensionless.

- How matter affects the gravitational field:

  $$\nabla^2\phi = 4\pi G\rho$$

  Implication: the gravitational potential from a single point mass M at the origin is

  $$\phi = -\frac{GM}{r},$$

  and fields from different masses simply add.

- How matter affects the gravitational field:

  $$F = m\mathbf{g} = -m\nabla\phi.$$

Equivalence principle (1911)

- General relativity (GR) consists of two parts: how matter (particles, electromagnetic fields, *etc.*) affects spacetime and how spacetime affect matter. The second part is specified by the strong equivalence principle.

- **Weak equivalence principle:** No local experiment can distinguish between a uniform gravitational field $\mathbf{g}$ and a frame accelerated with $\mathbf{a} = \mathbf{g}$.

- **Strong equivalence principle:** The laws of physics take on their special-relativistic form in any locally inertial frame frame.

- A freely falling elevator is a locally inertial frame (if the elevator is small enough and our experiment short enough), so the strong version says that special relativity applies in all such elevators anywhere and anytime in the universe, *i.e.*, independently of the spacetime position and velocity of the elevator.

Who’s gone bungee jumping?

ELEVATOR MOVIES, BUZZ MOVIE

- Where did this idea come from? Combining

$$F = ma$$

with

$$F = \frac{GmM}{r^2}$$

shows that the gravitational acceleration

$$a = \frac{GM}{r^2}$$

is mass-independent as long as

"inertial mass" = "gravitational mass".

Is it?

- Galileo's Pisa experiment showed it with low precision.

- Eötvös (1890) and later others showed with high precision that a independent of both mass and composition (density, atomic element, matter/antimatter, etc). Coincidence? Einstein thought that no, it was telling us something.

- In other words, if you know the direction of the worldline of an object freely floating through a spacetime event (*i.e.*, the direction of the velocity 4-vector), then the continuation of the worldline under the influence of gravity is the same regardless of the mass and composition of the object. This suggested to Einstein that gravity was a purely geometric effect.

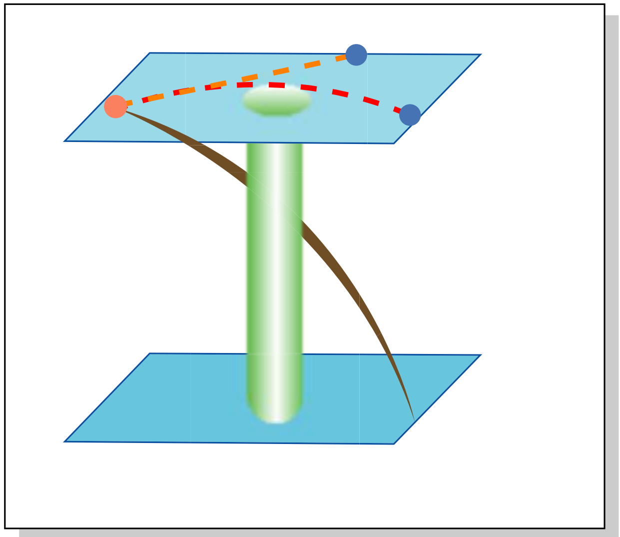
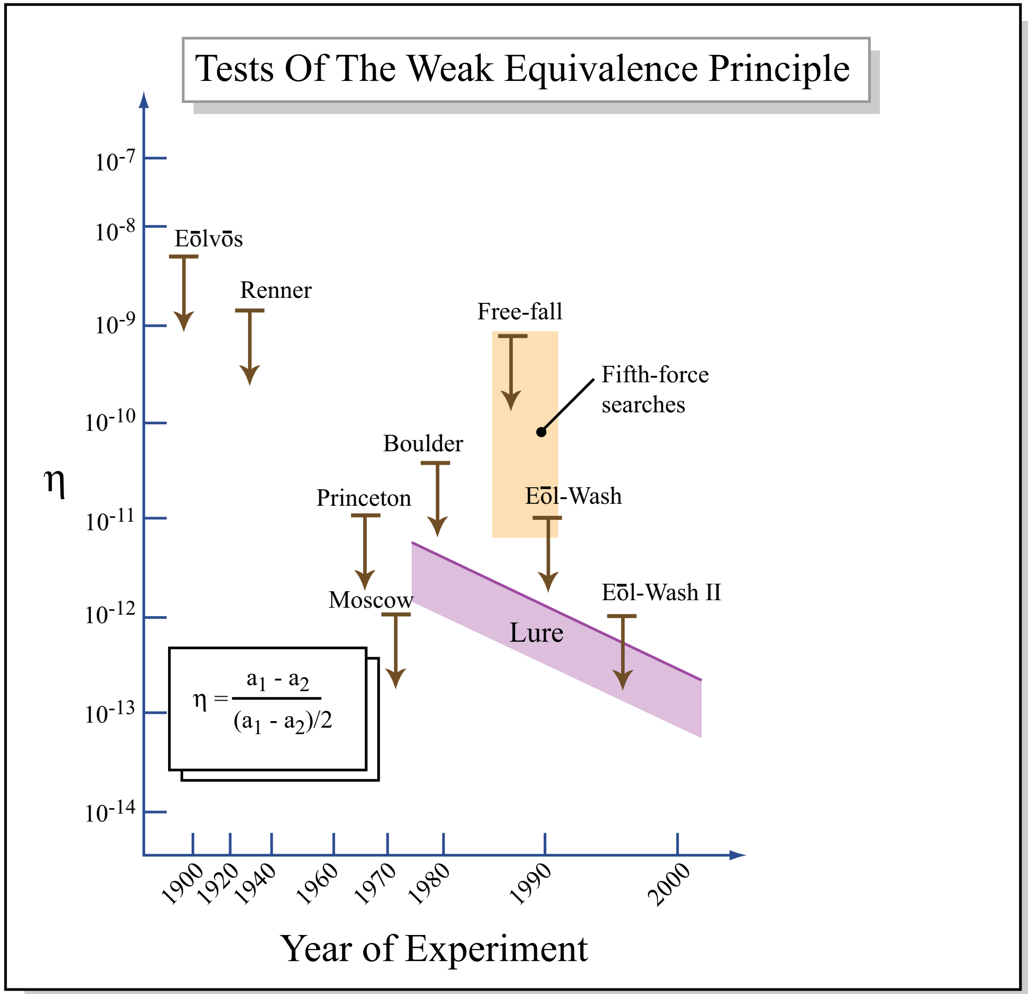


Figure by MIT OCW.



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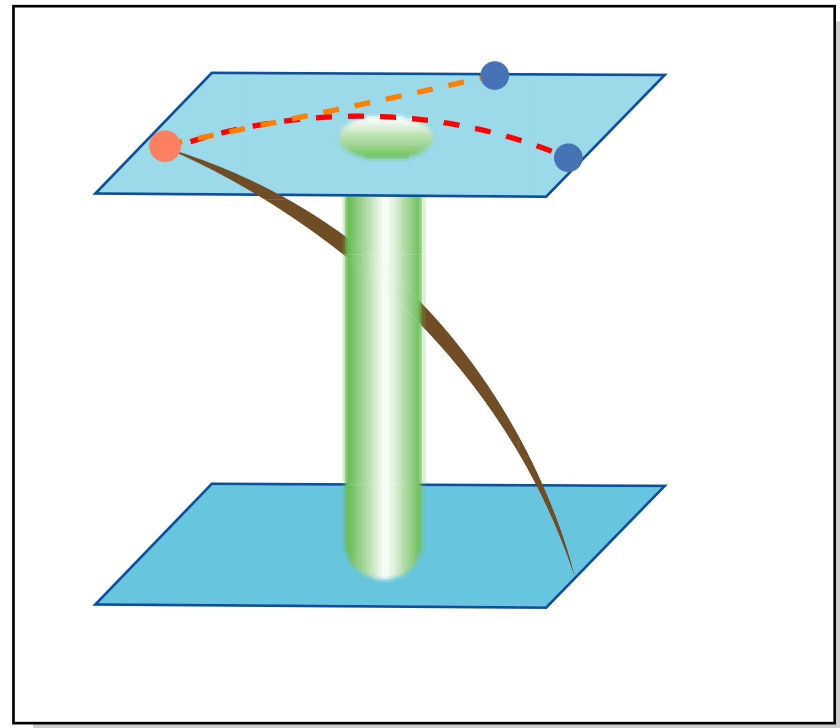
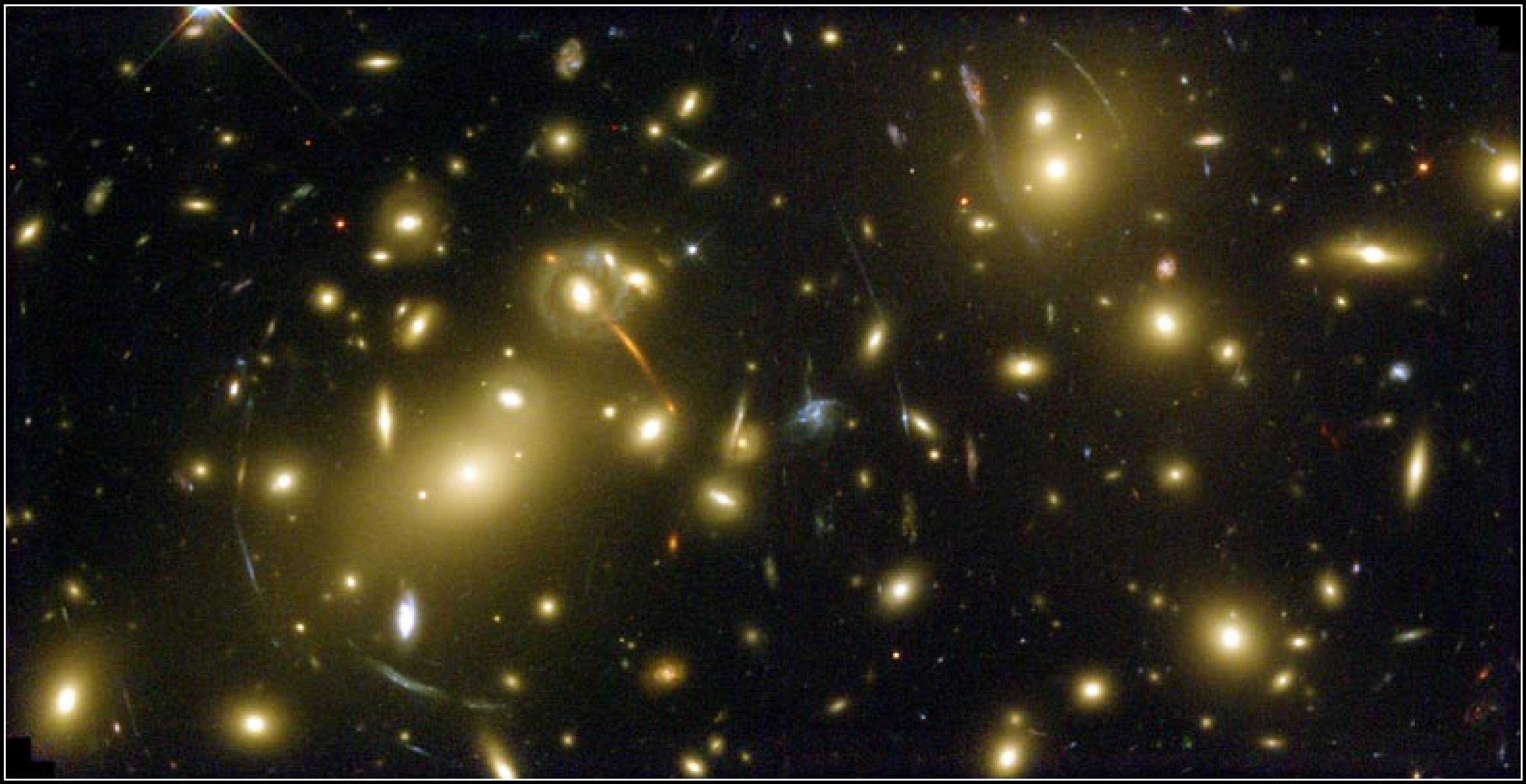


Figure by MIT OCW.

EP implication 1: Gravity bends light

Gravitational lensing

Lensing



Galaxy Cluster Abell 2218 **HST • WFPC2**

NASA, A. Fruchter and the ERO Team (STScI, ST-ECF) • STScI-PRC00-08

Image courtesy of NASA.

EP implication 2: Gravitational redshift

Harvard Tower Experiment (Pound & Rebka 1960)

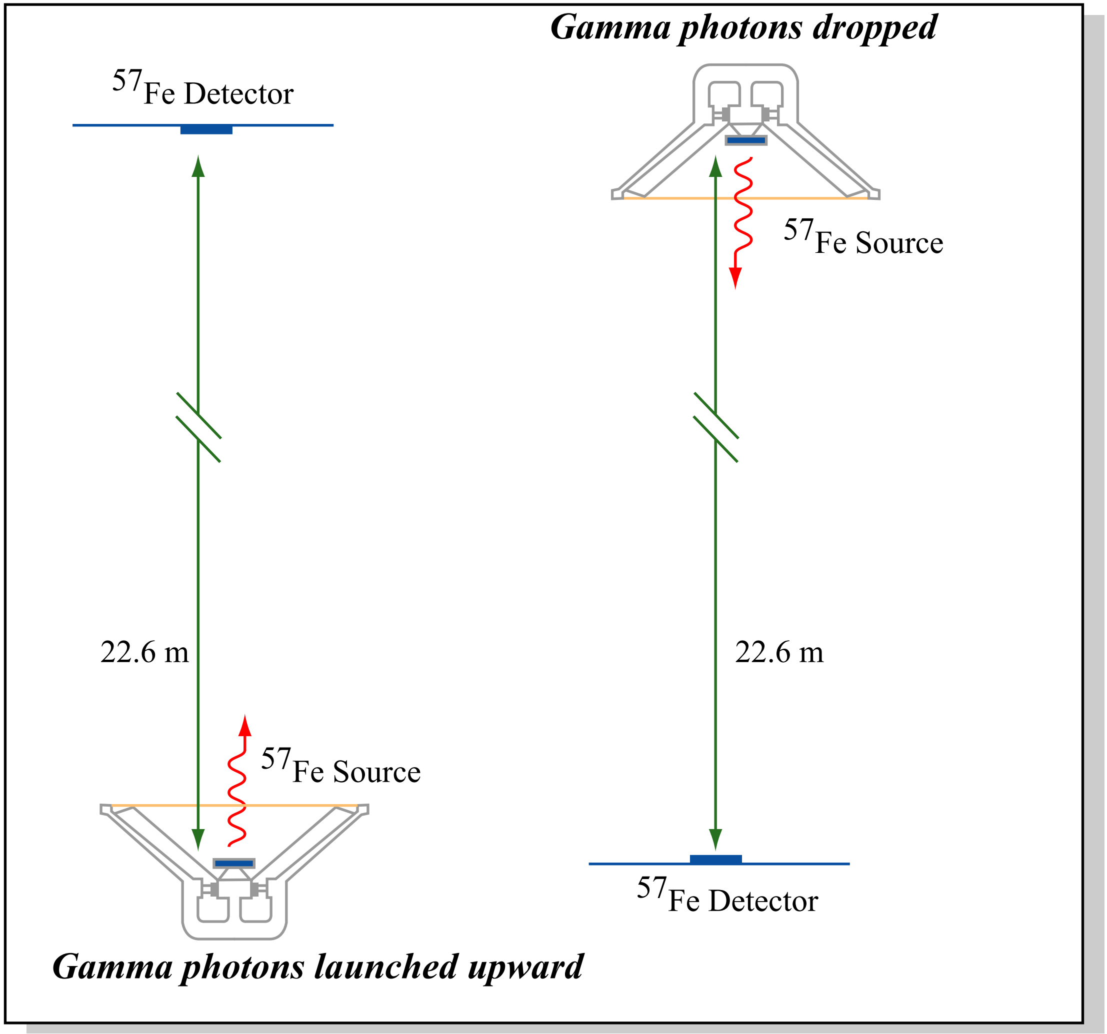


Figure by MIT OCW.

Over 22.6 meters, the gravitational redshift is only 5×10^{-15}, but the Mössbauer effect with the 14.4 keV gγ–ray from iron-57 has a high enough resolution to detect that difference.

EP implication 3: It’s all geometry (learn how to work with metrics!)

METRICS

Metrics and geodesics

- In an n-dimensional space, the *metric* is a (usually position-dependent) $n \times n$ symmetric matrix $\mathbf{g}$ that defines the way distances are measured. The length of a curve is $\int ds$, where

$$ds^2 = d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r},$$

and $\mathbf{r}$ are whatever coordinates you're using in the space. If you change coordinates, the metric is transformed so that ds stays the same (ds is invariant under all coordinate transformations).

- **Example:** 2D Euclidean space in Cartesian coordinates.

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$ds^2 = d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r} = \begin{pmatrix} dx & dy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = dx^2 + dy^2,$$

$$\int ds = \int \sqrt{d\mathbf{r}^t \, \mathbf{g} \, d\mathbf{r}} = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2} dx.$$

Applying the Euler-Lagrange equation to this shows that the shortest path between any two points is a straight line.

- **Example:** 4D Minkowski space in Cartesian coordinates ($c = 1$ for simplicity)

$$\mathbf{g} = \boldsymbol{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} d\tau^2 &= ds^2 = d\mathbf{x}^t\,\mathbf{g}\,d\mathbf{x} = \\ &= \begin{pmatrix} dx & dy & dz & dt \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \\ dt \end{pmatrix} \\ &= dt^2 - dx^2 - xy^2 - dz^2, \end{aligned}$$

$$\begin{aligned} \Delta\tau &= \int d\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}dt \\ &= \int \sqrt{1 - u^2}dt = \int \frac{dt}{\gamma}. \end{aligned}$$

Applying the Euler-Lagrange equation to this shows that the extremal interval between any two events is a straight line though spacetime.

Spherical coordinates

- Spherical coordinates (r, θ, φ) are defined by

$$\begin{aligned} x &= r \sin\theta \cos\varphi, \\ y &= r \sin\theta \sin\varphi, \\ z &= r \cos\theta. \end{aligned}$$

- This implies

$$\begin{aligned} dx &= \sin\theta \cos\varphi dr + r \cos\theta \cos\varphi d\theta - r \sin\theta \sin\varphi d\varphi, \\ dy &= \sin\theta \sin\varphi dr + r \cos\theta \sin\varphi d\theta + r \sin\theta \cos\varphi d\varphi, \\ dz &= \cos\theta dr - r \sin\theta d\theta. \end{aligned}$$

- This let's us reexpress the Minkowski metric in spherical coordinates:

$$\begin{aligned} d\tau^2 &= dt^2 - dx^2 - dy^2 - dz^2 \\ &= dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2. \end{aligned}$$

(To get the second line, we simply plugged in the expressions for dx, dy and dz and simplified the result.)

General covariance

- The analogous procedure is used to transform *any* metric into *any* coordinate system.

- **Key concept:** this means that we can do our calculations with metrics and geodesics in *any* system of space and time coordinates we like. In Minkowski space, inertial frames are just a special class of coordinate systems (the standard spacetime coordinates (x, y, z, ct) and Lorentz transforms thereof), so we're *not* limited to working in inertial frames in GR.

- Einstein insisted that not only the metric but indeed all laws of physics should be expressible using *any* coordinate system. This requirement is called *general covariance*.

- This is why GR is called *General* relativity, special relativity being merely the special case where you were allowed to start with an inertial frame and make a Lorentz transformation (a particular linear coordinate transformation).

- If you think of Lorentz transformations as coordinate transformations, they are simply the ones that have the property

$$d\tau^2 = d(ct)^2 - dx^2 - dy^2 - dz^2 = d(ct')^2 - dx'^2 - dy'^2 - dz'^2,$$

  since we previously proved that $d\tau$ is Lorentz invariant.

- Note that it's not at all obvious just from staring at a metric that someone writes down whether it's really just Minkowski space in disguise, expressed in some funny coordinates.

- In GR, it's convenient to use units where $c = G = 1$, simplifying these metrics:
- Minkowski metric:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

- Newtonian metric:

$$d\tau^2 = (1 + 2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

- Minkowski metric in polar coordinates:

$$d\tau^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

- Friedman-Robertson-Walker (FRW) metric:

$$d\tau^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right)$$

- Schwartzschild metric ($r_s = 2M$):

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2,$$

- In GR, it's convenient to use units where $c = G = 1$, simplifying these metrics:
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$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

- Newtonian metric:

$$d\tau^2 = (1 - 2\phi)dt^2 - dx^2 - dy^2 - dz^2$$

- Minkowski metric in polar coordinates:

- Schwartzschild metric ($r_s = 2M$):

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) d(ct)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2,$$

- Derive Newtonian metric from gravitational redshift
- Test that Newtonian metric reproduces Newton

Next lecture:

cosmology