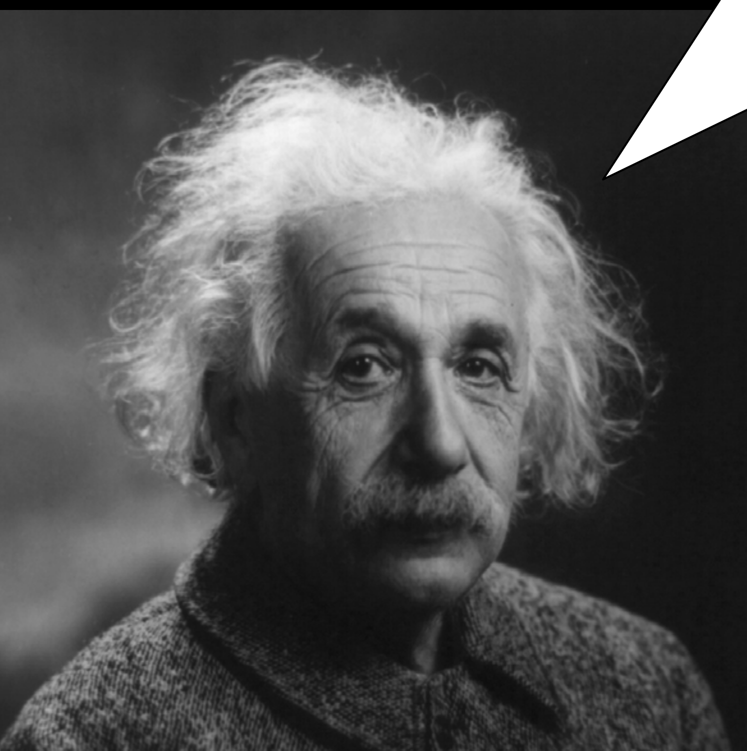
Welcome back to 8.033!



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MIT Course 8.033, Fall 2006, Lecture 11

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Today's topics:

- Dynamics problem-solving toolbox
- More on mass-energy equivalence, kinetic energy, rest energy
- Acceleration & force
- Particle accelerators
- Interstellar rocket travel

Momentum & energy toolbox:

- Relativistic mass:

$$m = \gamma m_0$$

- Mass-energy unification:

$$E = mc^2$$

- Momentum 4-vector (momentum-energy unification):

$$\mathbf{P} \equiv m_0 \mathbf{U} = m_0 \frac{d\mathbf{X}}{d\tau} = m_0 \gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = m \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix},$$

(Use upper case **X**, **U** and **P** for the 4-vectors to avoid confusion with the **x**, **u** and **p** 3-vectors.)

- Handy velocity formula follows straight from this:

$$\beta = \frac{cp}{E}$$

- Rest energy:

$$E_0 = m_0 c^2$$

is total energy of particle in the frame where it is at rest

- Kinetic enery:

$$K = E - E_0 = mc^2 - m_0 c^2 = m_0(\gamma - 1) = \frac{1}{2} m_0 u^2 + O\left(\frac{u}{c}^4\right)$$

- Rest mass invariant:

$$m_0 = \frac{1}{c}\sqrt{-\mathbf{P}^t\boldsymbol{\eta}\mathbf{P}} = \frac{1}{c^2}\sqrt{E^2 - c^2p^2},$$

giving the handy relations

$$E = \sqrt{(m_0c^2)^2 + (cp)^2},$$

$$p \equiv |\mathbf{p}| = \sqrt{\frac{E^2}{c^2} - (m_0c)^2}.$$

- Low-speed limit $|\beta| \ll 1$:

$$E \approx m_0c^2 + \frac{1}{2}m_0u^2,$$

$$p = m_0\gamma u \approx m_0u.$$

- High-speed limit $|\beta| \approx 1$ ($\gamma \gg 1$, $E \gg E_0$):

$$E \approx cp$$

This becomes exact ($E = cp$) for particles moving with speed of light, like photons and gravitons.

- $-\mathbf{P}^t \boldsymbol{\eta} \mathbf{P} = (E/c)^2 - p^2$ is invariant also for *system* of particles, since

$$\mathbf{P}_{\mathrm{tot}}{}' \equiv \sum_i \mathbf{P}_i' = \sum_i \boldsymbol{\Lambda} \mathbf{P}_i = \boldsymbol{\Lambda} \left(\sum_i \mathbf{P}_i \right) = \boldsymbol{\Lambda} \mathbf{P}_{\mathrm{tot}} .$$

- We derived $\mathbf{p} = m_0 \gamma u$ only for 1-dimensional collision. But *any* collision is 1-dimensional in the frame where the total momentum is zero!

Acceleration & force (optional!)

- The acceleration 4-vector $\mathbf{A}$ and the Force 4-vector $\mathbb{F}$ are less useful than their 4-vector cousins $\mathbf{X}$, $\mathbf{U}$, $\mathbf{P}$ and $\mathbf{K}$. We'll use $\mathbb{F}$ mainly for deriving the force transformation law, which will in turn give us the transformation law for electromagnetic fields. We'll use upper case $\mathbf{A}$ for the acceleration 4-vector to avoid confusion with the the acceleration 3-vector $\mathbf{a}$, and the annoying symbol $\mathbb{F}$ for the force 4-vector to avoid confusion with the the force 3-vector $\mathbf{F}$.

- Acceleration 4-vector:

$$\mathbf{A} \equiv \frac{d\mathbf{U}}{d\tau} = \gamma_u \frac{d\mathbf{U}}{dt} = \gamma_u \frac{d}{dt}\gamma_u \begin{pmatrix} u_x \\ u_y \\ u_z \\ c \end{pmatrix} = \gamma_u \frac{d}{dt}\gamma_u \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix}$$

$$= \gamma_u^2 \begin{pmatrix} \dot{\mathbf{u}} \\ 0 \end{pmatrix} + \gamma_u \dot{\gamma}_u \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix} = \gamma_u \begin{pmatrix} \mathbf{a} + \dot{\gamma}_u \mathbf{u} \\ \dot{\gamma}_u c \end{pmatrix}$$

$$= \gamma_u^2 \begin{pmatrix} \mathbf{a} \\ 0 \end{pmatrix} + \gamma_u^4 \frac{\mathbf{u}\cdot\mathbf{a}}{c^2} \begin{pmatrix} \mathbf{u} \\ c \end{pmatrix},$$

where in the last step, we have used the fact that

$$\dot{\gamma}_u = \gamma_u^3 \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}.$$

- Force 4-vector:

$$\mathbb{F} \equiv \frac{d}{d\tau}\mathbf{P} = \gamma_u \frac{d}{dt}\mathbf{P} = \gamma_u \frac{d}{dt} m_0 \mathbf{U} = m_0 \frac{d}{d\tau}\mathbf{U},$$

so by definition, we have

$$\mathbb{F} = m_0 \mathbf{A}.$$

(Note that this does *not* apply the Newtonian result $\mathbf{F} = ma$!)

- Interpretation of Force 4-vector:

$$\mathbb{F} = \gamma_u \frac{d}{dt}\mathbf{P} = \gamma_u \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{E}/c \end{pmatrix} = \gamma_u \begin{pmatrix} \mathbf{F} \\ P/c \end{pmatrix},$$

where $\mathbf{F} = \dot{p}$ is the familiar force 3-vector and $P = \dot{E}$ is the power, the energy change per unit time (in Watts).

- Work-energy theorem:

$$dE = \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \mathbf{F} \cdot \mathbf{u} dt,$$

so the power satisfies

$$P = \dot{E} = \mathbf{u} \cdot \mathbf{F}.$$

- Force 3-vector explicitly: Dividing the above equation $\mathbb{F} = m_0 \mathbf{A}$ by γ_u gives

$$\frac{\mathbf{F}}{m_0 \gamma_u} = \mathbf{a} + \gamma_u^2 \frac{\mathbf{u} \cdot \mathbf{a}}{c^2} \mathbf{u}.$$

- Special case where $\mathbf{u}$ and $\mathbf{a}$ are parallel, *e.g.*, for linear motion:

$$\frac{\mathbf{F}}{m_0 \gamma_u} = \mathbf{a} + \gamma_u^2 \frac{u^2 \mathbf{a}}{c^2} = \left(1 + \gamma_u^2 \beta^2\right) \mathbf{a} = \gamma_u^2 \mathbf{a}.$$

- Special case where $\mathbf{u}$ and $\mathbf{a}$ are perpendicular, eg, for circular motion:

$$\frac{\mathbf{F}}{m_0 \gamma_u} = \mathbf{a}$$

- Note that in relativity, $\mathbf{F}$ and $\mathbf{a}$ are generally *not* parallel, but that they are parallel for these two special cases.

- Acceleration 3-vector explicitly:

$$\mathbf{a} = \frac{\mathbf{F}}{m_0\gamma_u} - \frac{\mathbf{u}\cdot\mathbf{F}}{m_0\gamma_u c^2}\mathbf{u} = \frac{\mathbf{F}}{m} - \frac{P}{mc^2}\mathbf{u}.$$

The last term (the departure from $\mathbf{F} = m\mathbf{a}$) is seen to have the form of a friction term proportional to the power put into the particle. Derivation: the three steps below.

$$\dot{\gamma}_u = \frac{d}{dt}\frac{m_0\gamma_u c^2}{m_0c^2} = \frac{d}{dt}\frac{E}{m_0c^2} = \frac{\dot{E}}{m_0c^2} = \frac{\mathbf{u}\cdot\mathbf{F}}{m_0c^2} = \frac{P}{m_0c^2}.$$

Combining this with the other expression for $\dot{\gamma}_u$ above gives

$$\mathbf{u}\cdot\mathbf{a} = \frac{\mathbf{u}\cdot\mathbf{F}}{\gamma_u^3 m_0}.$$

The above equation for $\mathbf{F}$ now becomes

$$\frac{\mathbf{F}}{m_0\gamma_u} = \mathbf{a} + \frac{P}{m_0\gamma_u c^2}\mathbf{u} = \mathbf{a} + \gamma_u^2\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u}.$$

Transformation of force

- Let's compute the transformation law for force by transforming to a frame S' moving with velocity v in the x-direction relative to S:

$$\begin{aligned}
\mathbb{F}' &= \gamma_{u'} \begin{pmatrix} F'_x \\ F'_y \\ F'_z \\ P'/c \end{pmatrix} = \mathbf{\Lambda}\mathbb{F} = \gamma_u \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \\ P/c \end{pmatrix} \\
&= \gamma_u \begin{pmatrix} \gamma[F_x - \beta P/c] \\ F_y \\ F_z \\ \gamma[P/c - \beta F_x] \end{pmatrix} = \frac{\gamma_{u'}}{\gamma(1-\frac{u_x v}{c^2})} \begin{pmatrix} \gamma[F_x - \beta P/c] \\ F_y \\ F_z \\ \gamma[P/c - \beta F_x] \end{pmatrix} .
\end{aligned}$$

In the last step, we used the relation $\gamma_{u'} = \gamma_u \gamma[1 - u_x v/c^2]$ which we proved earlier when transforming the velocity 4-vector U — it followed from the fact that its normalization is Lorentz invariant, *i.e.*, $\mathbf{U}'^t \boldsymbol{\eta} \mathbf{U}' = \mathbf{U}^t \boldsymbol{\eta} \mathbf{U}$.

- The 4 components now give our desired force transformation equations:

$$\begin{aligned}
F'_x &= \frac{F_x - \frac{v}{c^2}P}{1 - \frac{u_x v}{c^2}}, \\
F'_y &= \frac{F_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \\
F'_z &= \frac{F_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \\
P' &= \frac{P - vF_x}{1 - \frac{u_x v}{c^2}},
\end{aligned}$$

  where $P = \mathbf{u} \cdot \mathbf{F}$ as usual.

- If we take S to be the rest frame of the particle, then $\mathbf{u} = 0$, $P = \mathbf{u} \cdot \mathbf{F} = 0$ and this simplifies to $F'_x = F_x$, $F'_y = F_y/\gamma$, $F'_z = F_z/\gamma$, so in the frame S' where the particle is moving, the force is unaffected in the parallel direction and suppressed by γ in the transverse directions.

Transformation of acceleration

- We could derive expressions using an approach like for force, but the results are so messy that it's not particularly useful — it's better to deal with explicit problems as needed.

- Here's a useful special case that you get to derive on a problem set (probably PS7): For an arbitrary acceleration $\mathbf{a}$ in S, the acceleration $\mathbf{a}'$ in S' is related to $\mathbf{a}$ via

$$
\begin{aligned}
a_x &= \frac{a'_x}{\gamma^3(1 + vu'_x/c^2)^3} \\
a_y &= \frac{a'_y}{\gamma^2(1 + vu'_x/c^2)^2},
\end{aligned}
$$

  with the **important caveat** that the expression for a_y is only valid for the case where either $u'_y = 0$ or $a'_x = 0$.

OK, now wake up!

Dynamics toolbox: formula summary

- Mass-energy unification:

$$E = mc^2 = m_0\gamma c^2$$

- Momentum 4-vector:

$$\mathbf{P} \equiv m_0\mathbf{U} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

- Energy formula:

$$E = \sqrt{(m_0c^2)^2 + (cp)^2}$$

- Velocity formula:

$$\beta = \frac{cp}{E}$$

Who needs relativity?

- Particle physicists
- Astrophysicists
- Anyone using electromagnetism

How accelerate to near the speed of light?

CERN particle accelerator

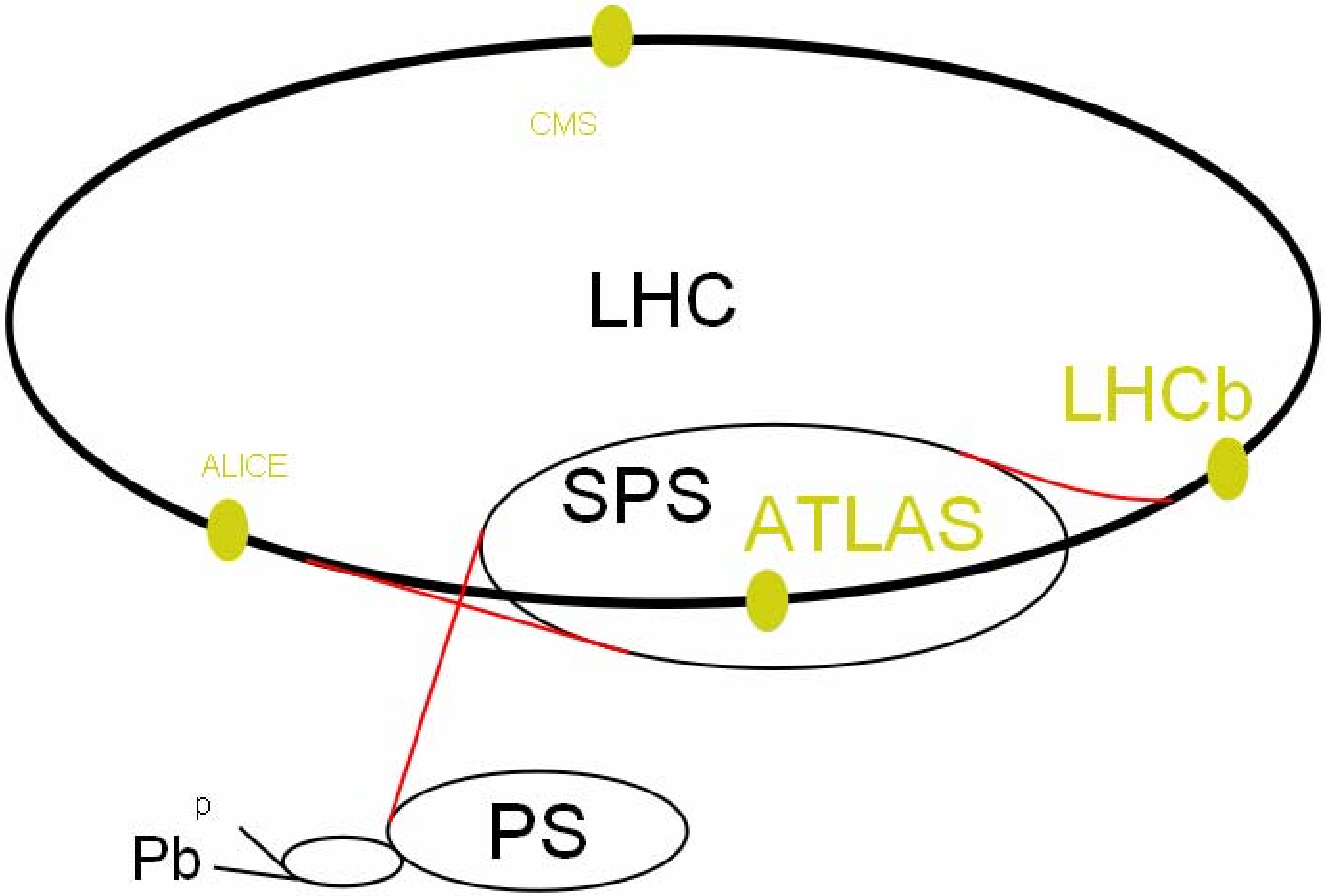

Image courtesy of Wikipedia.

Linear particle accelerator (Fermilab)

Image courtesy of Wikipedia.

How derive curvature radius?

INSTERSTELLAR SPACE TRAVEL

Where might we want to go?

M100 Galaxy

Why is it so hard?

Current Status of Space Travel

- The Galileo spacecraft swung by Jupiter and achieved a speed of 100,000 miles per hour.
- Most other space craft such as the Pioneers and Voyagers travel at about 25 - 35,000 miles/hour.

Interstellar Travel

- Time to get to the nearest star
  - Proxima Centauri
  - Galileo (100,000 mph) ~ 44,500 yrs
  - Voyager (30,000 mph) ~ 74,100 yrs

Some ideas:

* Antimatter-powered rockets
* Laser light sails
* Warp drive?
* Wormholes
* “Beam me up”,
  universal constructors

Rocket Limitation:

Propellant mass required to send one canister past Centauri Cluster within 900 years.

1. Chemical (500 sec): ~10^{137} kg. Not enough mass in the known universe to pull this off.
2. Fission (5,000 sec): ~10^{17} kg. A billion tankers worth of mass.
3. Fusion (10,000 sec): ~10^{11} kg. A thousand tankers worth of mass.
4. Ion/Antimatter (50,000 sec): ~10^{5} kg. Ten tanks worth of mass.

Conclusion: we need a propulsion breathrough; *no propellant!*

Anti-Matter as a Source of Fuel

- Matter-antimatter annihilation energy release is about ten billion times more powerful than that of chemical energy such as hydrogen and oxygen combustion.

- A thousand times more powerful than fission energy, which is used by nuclear power plants; and 300 times more powerful even than nuclear fusion energy.

- Antimatter would be the perfect rocket fuel.

2 Major Problems to Overcome

1- How do we get enough anti-matter?

2- How do we store anti-matter?

Anti-Matter Production

- All the antiprotons produced at CERN during one year would supply enough energy to light a 100 watt electric bulb for three seconds!

- In terms of the energy put in to produce high energy proton beams and store them, the efficiency of the antimatter energy production process would be 0.00000001%. The steam engine is millions of times more efficient!

It’s Expensive!!!

- Right now, antimatter is the most expensive substance on Earth.

- It costs about $62.5 trillion a gram ($1.75 quadrillion an ounce) to make.

- Can only create one billionth of a gram per year!

Sample Space Craft

Image courtesy of NASA.

Starship Propulsion

nuclear (H-bomb) powered

solar sail

interstellar ramjet

Some ideas:

* Antimatter-powered rockets
* Laser light sails
* Warp drive?
* Wormholes
* "Beam me up",
  universal constructors

Candidate Technologies for High-Speed Space Travel

1. Fission
   - Fission fragment
2. Fusion
   - Inertial Confinement Fusion (ICF)
3. Antimatter
   - Bean-core antimatter rocket
4. Beamed energy/momentum
   - Laser lightsail
   - Relativistic particle beam
5. Combinations
   - Antiproton-catalyzed micro-fission/fusion
   - Beamed-laser/ICF
   - Bussard interstellar ramjet