

Massachusetts Institute of Technology

Department of Physics

Physics 8.033

Out: Friday 3 November 2006

Due: Friday 17 November 2006

Problem Set 8 (Cosmology)

Due: Friday 17 November 2006 at 4:00PM. Please deposit the problem set in the appropriate 8.033 bin, labeled with name and recitation section number and stapled as needed (3 points).

Readings: Taylor & Wheeler until page 2-18, also project G. Also the three items linked to under “handouts” on the course web site:

- (a). the popular article on cosmology
- (b). the spacetime review article
- (c). Ned Wrights cosmology Tutorial

The lecture notes state a number of properties of the expanding Universe (the FRW metric). To strengthen your intuition and get used to dealing with metrics, you will now derive some of them.

Problem 1: “Geodesics in the expanding universe:(6+3+3 points)” For the special case of flat space ($k = 0$), the FRW metric can be written in Cartesian coordinates as (using units where $c = 1$)

$$\begin{aligned}d\tau^2 &= dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2) \\ &= [1 - a(t)^2 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)] dt^2 \\ &= [1 - a(t)^2 |\dot{\mathbf{r}}|^2] dt^2,\end{aligned}$$

where dots denote d/dt and $\mathbf{r} \equiv (x, y, z)$ is called the *comoving position*.

- (a). Find the geodesic between the two spacetime events $(t, \mathbf{r}) = (t_0, \mathbf{r}_0)$ and $(t, \mathbf{r}) = (t_1, \mathbf{r}_0)$. Hint: Find the curve of maximal aging. The result is very simple and a straightforward mathematical argument suffices to show this — there is no need to use calculus of variations for this.
- (b). What is the total time that elapses on a clock moving on this geodesic curve between these two events?
- (c). An object is said to be *comoving* if $\dot{\mathbf{r}} = \mathbf{0}$. Consider a galaxy in the this FRW spacetime that is comoving early on. Based on your result in (a), describe in one sentence its future motion.

Problem 2: “Cosmic expansion”(6+3+3 points): Consider two comoving galaxies in a flat FRW universe (same metric as above), one at $\mathbf{r} = (x, y, z) = (0, 0, 0)$ and one at $\mathbf{r} = (x, y, z) = (x_0, 0, 0)$.

- (a). Calculate the distance σ between them at a fixed time $t = t_0$, *i.e.*, the length of the spacelike geodesics between the event A with $(x, y, z, t) = (0, 0, 0, t_0)$ and the event B with $(x, y, z, t) = (x_0, 0, 0, t_0)$.

Hint: This geodesic is simply the straight line $(x, y, z, t) = (x, 0, 0, t_0)$ where x goes from 0 to x_0 (you don’t need to prove this), so you merely need to integrate along this curve:

$$\sigma = \int d\sigma = \int_0^{x_0} \frac{d\sigma}{dx} dx,$$

where $d\sigma$ is the proper space interval defined by $d\sigma^2 = -d\tau^2$.

- (b). Give the ratio of this separation σ at two different times, $\frac{\sigma(t_2)}{\sigma(t_1)}$. You would have obtained this exact same result for *any* pair of comoving galaxies, regardless their positions, so as the Universe expands, all distances increase by the same factor.

- (c). Using your result from part (a), compute the Hubble parameter defined as recession velocity over distance, *i.e.*,

$$H \equiv \frac{\dot{\sigma}}{\sigma}.$$

Problem 3: “Deriving the Friedmann equation” : Although the rigorous way to do this is using Einstein’s theory of General Relativity, it turns out that you can obtain exactly the same result with classical mechanics as you will now show. This is helpful for intuitively understanding the Friedmann equation, arguably the most important equation in all of cosmology. N.B. Make sure to use *classical* mechanics throughout this problem, not special relativity.

I (3+3+3+3+3 points) Consider a large mass M at rest and a negligibly small mass m a distance a away, moving straight away from M with a velocity v .

- Illustrate this with a picture. Write down the total energy E of the system, including both the kinetic energy and the gravitational potential energy.
- Calculate the escape velocity v , defined as the smallest velocity that would let m escape to infinity eventually. (Hint: This corresponds to the case $E = 0$.)
- For the special case $E = 0$, solve for the position a as a function of time t . Make the assumption that $m \ll M$, so that the large mass remains at rest throughout. Define $t = 0$ as the time when $a = 0$ (this fixes your integration constant).
Hint: If you have no experience with differential equations, try plugging in a solution of the form $a(t) = At^B$ and solve for the constants A and B .
- Using your solution from part (c), how long ago were the two masses at the same place if the current separation is a_0 ?
- In words, describe what would happen eventually for the two cases $E < 0$ and $E > 0$.

II (6+3+3+3+3 points) Perhaps without knowing it, you have just described the expanding Universe. All that remains to be done is to reinterpret your equations in terms of a different problem.

- In problem 2, you showed that the a comoving object moves radially away from the origin with a distance $\sigma(t) \propto a(t)$ and speed $v = H\sigma$. This means that no matter overtakes any other matter, *i.e.*, that the total mass contained within an expanding sphere of radius $a(t)$ remains constant. Assuming that the Universe is full of uniformly distributed comoving matter of density $\rho(t)$, this constant enclosed mass is

$$M = \frac{4}{3}\pi a^3 \rho.$$

You will determine the function $a(t)$ by studying the motion of a tiny blob of stuff of mass m at the edge of this sphere. Illustrate this with a picture.

Since the mass distribution is spherically symmetric about the origin, the blob will feel a gravitational force equivalent to that of single mass $M = \frac{4}{3}\pi a^3 \rho$ at the origin. (This is a famous theorem by Newton. The result that we can ignore all the matter outside of the sphere holds in general relativity as well, and is known as Birkoff’s theorem.) Use this to eliminate M from your answer in part I (a) and write down the combined kinetic and gravitational energy E for the blob.

- Plug in the relation $v = Ha$ to eliminate v from this result, and solve for H^2 . This is the famous Friedmann equation. Tidy it up by replacing the integration constant E by the dimensionless curvature constant k defined as

$$k \equiv -\frac{2E}{mc^2}.$$

- Using your answer in part I (c) (for the $k = 0$ case), write down the solution to the Friedmann equation as $a(t) \propto t^B$ for some constant B .
- How much time has elapsed since the Big Bang, *i.e.*, since the time when $\rho = \infty$? Express your answer in terms of H . Hint: You simply need to relate H to t . A simple way to do this is to write $a = At^B$ as in part I (c) and compute $H = \dot{a}/a$.
- In words, describe what would happen eventually for the two cases $k < 0$ and $k > 0$.

Problem 4: “Solving the Friedmann equation:” (3+3+3 points) In the last problem, you derived the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

and solved it for the special case of flat space ($k = 0$) and ordinary matter ($\rho \propto a^{-3}$). Now you will solve it for more interesting cases. Different types of matter dilute differently as space expands:

- $\rho_\gamma \propto a^{-4}$ (photons)
 - $\rho_m \propto a^{-3}$ (ordinary matter, dark matter)
 - $\rho_k \propto a^{-2}$ (spatial curvature, *i.e.*, the k -term)
 - $\rho_\Lambda \propto a^0$ (vacuum energy, *i.e.*, cosmological constant)
- (a). Compute $a(t)$ up to a proportionality constant for a Universe containing only photons ($H^2 \propto a^{-4}$).
Hint: For the matter-dominated case $H^2 \propto a^{-3}$ that you solved above, the key steps would be $(a^{-1}da/dt)^2 \propto a^{-3}$, $a^{1/2}da \propto dt$, $a^{3/2} \propto t$, $a \propto t^{2/3}$.
- (b). Compute $a(t)$ up to a proportionality constant for an empty Universe with curvature only ($H^2 \propto a^{-2}$).
- (c). Compute $a(t)$ up to a proportionality constant for a Universe with only vacuum energy ($H = H_0 = \text{constant}$). (Express your answer in terms of H_0 .)
 Note: In the theory of cosmological inflation, this solution applies approximately in the very early Universe, creating a vast volume in a very short time. This solution may also apply to our distant future.

If you want something harder, one of the optional problems below is to show that the case with both matter and curvature makes $a(t)$ a cycloid.

Problem 5: “Age of the Universe” (9×3 points): The Friedmann equation implies that we can write

$$H(a) = H_0 [\Omega_\gamma a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda]^{1/2},$$

where a is normalized so that $a = 1$ at the present time and where the parameters H_0 , Ω_γ , Ω_m , Ω_k and Ω_Λ are all constants. $H = a^{-1}da/dt$ implies $dt = da/aH$, so the age of the Universe is

$$t_0 = \int_0^1 \frac{da}{aH} = \int \frac{H_0^{-1} da}{a [\Omega_\gamma a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda]^{1/2}}.$$

- (a). Compute the dimensionless age of the Universe $H_0 t_0$ for the matter-dominated case ($\Omega_m = 1$, $\Omega_\gamma = \Omega_k = \Omega_\Lambda = 0$).
- (b). Compute the dimensionless age of the Universe $H_0 t_0$ for the photon-dominated case ($\Omega_\gamma = 1$, $\Omega_m = \Omega_k = \Omega_\Lambda = 0$).
- (c). Compute the dimensionless age of the Universe $H_0 t_0$ for the empty Universe case ($\Omega_k = 1$, $\Omega_\gamma = \Omega_m = \Omega_\Lambda = 0$).
- (d). Compute the dimensionless age of the Universe $H_0 t_0$ for the currently favored case ($\Omega_k = 0$, $\Omega_\gamma \approx 0$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$). **Hint:** This integral can be done analytically, so feel free to use the result below.

$$\begin{aligned} H_0 t_0 &= \int_0^1 \frac{da}{(\Omega_m/a + \Omega_\Lambda a^2)^{1/2}} = \frac{2}{3}(1 - \Omega_m)^{-1/2} \sinh^{-1} \left[\left(\frac{1 - \Omega_m}{\Omega_m} \right)^{1/2} \right] \\ &= \frac{2}{3}(1 - \Omega_m)^{-1/2} \ln \left[\Omega_m^{-1/2} + \left(\frac{1 - \Omega_m}{\Omega_m} \right)^{1/2} \right], \end{aligned}$$

where the second of the expressions on the right-hand-side is the more useful form. (Optionally: prove this formula — it is helpful to make a change of variables, where $x = a^{3/2}$.)

- (e). For the above cases, give the age of the Universe t_0 in Gigayears, using the fact that $H_0^{-1} \approx h^{-1} \times 9.7846$ Gyr. Assume a dimensionless Hubble parameter $h = 0.7$ ($H_0 = h \times 100\text{km/s/Mpc}$.)
- (f). Check your answers against Ned Wright's Javascript calculator at <http://www.astro.ucla.edu/~wright/CosmoCalc.html> (make sure to enter $H_0 = 70\text{km/s/Mpc}$ and $z = 0$) Does it agree with your results? (Note that it can't handle the photon-dominated case). Use it to calculate the age of a closed Universe with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.8$.
- (g). Using either analytic arguments or this calculator, answer the following qualitative questions:
- Increasing H_0 makes the Universe YOUNGER / OLDER (**circle one**).
 - Increasing Ω_m makes the Universe YOUNGER / OLDER (**circle one**).
 - Increasing Ω_Λ makes the Universe YOUNGER / OLDER (**circle one**).
- (h). The age of the Universe at redshift z (at the time of emission of photons that are just now arriving at the Earth with a redshift of z) is given by changing variables from $a = (1+z)^{-1}$ to z in the integral above:

$$t(z) = \int_z^\infty \frac{dz'}{(1+z')H} = \int_z^\infty \frac{H_0^{-1} dz'}{(1+z') [\Omega_\gamma (1+z')^4 + \Omega_m (1+z')^3 + \Omega_k (1+z')^2 + \Omega_\Lambda]^{1/2}}.$$

Compute $H_0 t(z)$ by doing this integral for the matter-dominated case ($\Omega_m = 1$, $\Omega_\gamma = \Omega_k = \Omega_\Lambda = 0$).

- (i). Using the Javascript calculator, compute the age $t(z)$ for for the currently favored cosmology ($\Omega_k = 0$, $\Omega_\gamma \approx 0$, $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$) when the cosmic microwave background radiation was released ($z = 10^3$) and corresponding to one of the most distant quasars ever observed ($z = 6$).

Problem 6: Key concepts (3+5+3+4):

- (a). Order the following epochs chronologically and give the approximate age of the Universe corresponding to each one: emission of the Cosmic Microwave Background, formation of typical galaxies, primordial nucleosynthesis, Planck time, today, death of Sun, inflation, formation of first stars. (Hint: see the Time Magazine handout.)
- (b). Give each of the following quantities to the nearest power of 10 (don't show calculations, being off by one power of 10 is OK):
- Number of stars in our Galaxy
 - Light travel time to closest star (Sun!) in minutes
 - Light travel time to Pluto in hours
 - Light travel time to 2nd closest star in years
 - Distance to Andromeda galaxy (M31) in lightyears
- (c). List (no explanations needed) three pieces of evidence supporting the Big Bang model.
- (d). Give rough current estimates of the Hubble parameter h , the dark energy density parameter Ω_Λ , the baryon density parameter Ω_b and the dark matter density parameter Ω_{dm} .

Optional Problem 7: "Accelerating Universe": Consider a currently fashionable Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. Ignoring the photon density, the squared expansion velocity is

$$\dot{a}^2 = a^2 H^2 \propto \frac{\Omega_m}{a} + \Omega_\Lambda a^3.$$

Graph the right hand side as a function of the scale factor a for this cosmology. Find the scale factor at the time of minimum expansion velocity (minimum \dot{a}). Is the Universe currently accelerating or not, *i.e.*, is \dot{a} currently increasing or decreasing?

Optional Problem 8: "Evolution of the Scale Factor for a Universe with $\Omega_\Lambda = 0$, and $\Omega_m > 1$ ": Show that the evolution of the scale factor of a closed Universe (only matter and curvature) can

be worked out analytically to yield the following parametric expressions:

$$a = \frac{1}{2} \left(\frac{\Omega_m}{\Omega_m - 1} \right) (1 - \cos \alpha), \quad (8.1)$$

$$t = \frac{1}{2H_0} \frac{\Omega_m}{(\Omega_m - 1)^{3/2}} (\alpha - \sin \alpha). \quad (8.2)$$

The curve $a(t)$ is a *cycloid*, *i.e.*, the same curve that solved the brachistochrone problem. Hint: To carry out the integral over a , make a substitution of variables where

$$a = \frac{\Omega_m}{\Omega_m - 1} \sin^2 \alpha/2. \quad (8.3)$$

Alternatively, simply plug in the above expressions and show that they satisfy the Friedmann equation, using the identity

$$\frac{da}{dt} = \frac{da/d\alpha}{dt/d\alpha}.$$

Optional Problem 9: For the above closed universe cycloid solution, show that a photon leaving the origin at the Big Bang arrives back at the same place at the Big Crunch, *i.e.*, just barely has time to go circumnavigate the Universe once. **Hint:** The photon trajectory is defined by $d\tau = 0$ which gives an expression for dr/dt that you can integrate.

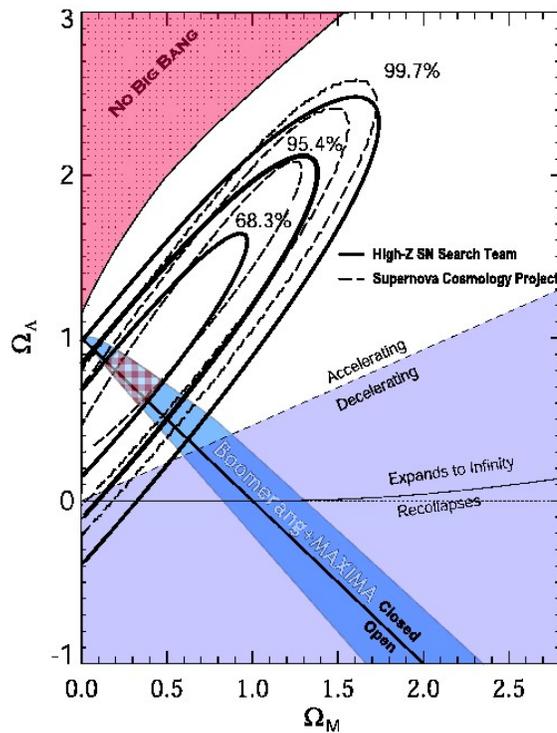
Optional Problem 10: “Minkowski space in disguise” (hard!): Show by a clever choice of coordinates that the FRW metric with $\Omega_\Lambda = \Omega_m = \Omega_\gamma = 0$, $\Omega_k = 1$ (this is the special case with $a(t) = t$, $k = -1$, corresponding to an empty and maximally open universe) is simply the Minkowski metric in disguise.

Optional Problem 11: “Ant Universe”: The following problem is taken from Shu’s book, “The Physical Universe”, Problem 15.5, page 371.

The premise: Imagine a world of ants on the surface of an expanding balloon of instantaneous radius $R(t)$. These ants have no concept of the existence of a third dimension perpendicular to the surface of the balloon. They are doomed to crawl forever only in the two dimensions parallel to the surface of the balloon. Because their world is a big place compared to any ant city, these ants have always imagined that they live on a flat surface. Moreover, being two-dimensional creatures, they, unlike us, cannot visualize their geometrically round world. (This is a slight rephrasing of Shu’s setup for the problem that follows.)

The problem: “Consider the great circle which corresponds to a line of sight from any given ant city. Let the radius of curvature of this great circle be $R(t)$, the instantaneous radius of the world relative to a center displaced in the third, unobservable dimension. Suppose that there are N ant cities at any time, distributed more or less evenly along this great circle. Show the average distances $s(t)$ between two ant cities along this (or any other) line of ant sight is given by $s(t) = \theta R(t)$, where $\theta = 2\pi/N$ radians. Argue that, in general, the distance between two ant cities which are initially separated by an angle θ (relative to the unobservable center of the world) is given at any later time t by the formula $S(t) = \theta R(t)$. The recessional velocity of one city with respect to the other is given by $v = \dot{s}(t)$ where the dot denotes differentiation with respect to time. Show that this recessional velocity satisfies Hubble’s law: $v = H(t)s$, where $H(t) = \dot{R}(t)/R(t)$ depends only on the history of the radius of curvature R . Notice, in particular, that all reference to the unobservable angle θ drops out.”

Optional Problem 12: “ Ω_Λ vs. Ω_m ”: Consider this plot showing various quantities graphed in the parameter space Ω_Λ vs. Ω_m :



There are three curves delineating regions (1) of expansion to infinity vs. recollapsing”, (2) “acceleration vs. deceleration”, and (3) “no big bang” vs. “big bang”. Try to either explain these curves qualitatively, or write a simple program to compute them quantitatively. The curve separating “acceleration vs. deceleration” is a simple analytic curve (i.e., a straight line of slope 1/2).

Feedback: Roughly how much time did you spend on this problem set?