

Massachusetts Institute of Technology

Department of Physics

Physics 8.033

Out: Friday 6 October 2006

Due: Friday 20 October 2006

Problem Set 5

Due: Friday 20 October 2006 at 4:00PM. Please deposit the problem set in the appropriate 8.033 bin, labeled with name and recitation section number and stapled as needed (3 points).

Readings: Relativistic Dynamics Chapters in Resnick and French.

Problem 1: “The Great Neutron Shootout” (3+3+3+3+3 points)

Sue and Jim are two experimenters at rest with respect to one another at different points in space. They “fire” neutrons at each other, each neutron leaving its “gun” with a relative speed of $0.60c$. Jim makes five observations about what is going on: (a) “My separation from Sue is 10 km.” (b) “The speed of ‘my’ neutron is $0.60c$.” (c) “Two of our neutrons have collided; relativistic momentum and kinetic energy are conserved.” (d) “After this collision, one of the neutrons was scattered through an angle of 30° .” (e) “I am firing neutrons at the rate of $10,000 \text{ s}^{-1}$.”

For each of these observations, state the corresponding observation that would be reported by a third person (Sam, say), who is in a frame S chosen so that Sue’s neutrons are at rest in it.

Problem 2: “A Ton of Sunlight” (3 points)

The Sun radiates energy at the rate of $4.0 \times 10^{26} \text{ W}$. How many “tons of sunlight” (mass equivalent) does the Earth intercept each day?

Problem 3: “An Enormous Source of Energy” (3 points)

Quasars are thought to be the nuclei of active galaxies in the early stages of their formation. A typical quasar radiates energy at the rate of 10^{41} W . At what rate is the mass of this quasar being reduced to supply this energy? Express your answer in solar mass units per year, where one solar mass unit, ($\text{smu} = 2 \times 10^{30} \text{ kg}$), is the mass of our sun.

Problem 4: (9 points) “Relativistic Transformation of Acceleration”: Starting from the definition of acceleration, $a = du/dt$, derive the x and y components of acceleration in an arbitrary inertial frame S' moving with speed v along the x -axis of S . The answers are:

$$a_x = \frac{a'_x}{\gamma^3(1 + vu'_x/c^2)^3}, \quad (4.1)$$

$$a_y = \frac{a'_y}{\gamma^2(1 + vu'_x/c^2)^2}, \quad (4.2)$$

where the expression for a_y is valid only for the case where either $u'_y = 0$ or $a'_x = 0$. In general, the transformation of a_y is rather messy.

Problem 5: “Constant Acceleration”: (6 points) French, problem #7-8, Chapter 7, page 226. Hint: You should end up solving $d(\gamma u)/dt = g$ to find $u(t)$; then integrate $u dt = dx$ to find $x(t)$.

Problem 6: “Relativistic Rocket Problem”: (3+3+6+6+3 points)

(a). Show that in the rest frame of the rocket, for non-relativistic dynamics:

$$\Delta M_0 V_{\text{ex}} = -M_0 \Delta u' \quad (6.1)$$

or equivalently

$$\frac{dM_0}{dt'} V_{\text{ex}} = -M_0 \frac{du}{dt'}, \quad (6.2)$$

where M_0 is the rest mass of the rocket, ΔM_0 is the differential mass loss in the exhaust, V_{ex} is the speed of the rocket exhaust (in the rocket's frame, S'), and $\Delta u'$ is the change in the rocket's speed after ejecting ΔM_0 . This is correct unless V_{ex} is comparable with the speed of light.

- (b). Consider first non-relativistic motion of the rocket in a fixed (stationary frame). Note that in the non-relativistic case $a = a'$ and $F = F'$. Thus, the rocket equation in S is:

$$\frac{dM_0}{dt} V_{\text{ex}} = -M_0 \frac{du}{dt}. \quad (6.3)$$

If the rate of mass loss in the exhaust is known as a function of time, then the above expression can be directly integrated to find $u(t)$ since $M_0(t)$ is known. Whether or not the rate of exhaust ejection is known, the dt 's can be canceled on both sides and the equation integrated to find u as a function of the initial and final masses of the rocket. Show that this leads to:

$$u = V_{\text{ex}} \ln(M_{\text{initial}}/M_{\text{final}}). \quad (6.4)$$

- (c). Suppose that the rocket in part (b) is firing its engine vertically in a gravitational field. Further assume that the atmospheric drag is negligible (not the case in the Earth's atmosphere, but okay on the Moon), and that the rocket motor is on while the rocket covers a distance that is relatively short compared with the radius of the gravitating body—such that the acceleration due to gravity, g , can be taken to be approximately constant. Show that if the rocket starts from rest, and ejects its exhaust uniformly in time (i.e., $dM_0/dt = \dot{M}_0 = \text{constant}$, and $M_0 = M_{0,\text{initial}} + \dot{M}_0 t$), the rocket equation becomes:

$$\frac{du}{dt} = -\frac{V_{\text{ex}} \dot{M}}{M_{\text{initial}} + \dot{M}t} - g. \quad (6.5)$$

Show that if the rocket is fired for a total time T , the above equation is easily integrated to yield $u(t)$:

$$u(T) = V_{\text{ex}} \ln \left[\frac{M_{\text{initial}}}{M_{\text{initial}} + \dot{M}T} \right] - gT = V_{\text{ex}} \ln \left[\frac{M_{\text{initial}}}{M_{\text{final}}} \right] - gT. \quad (6.6)$$

- (d). Next, consider the case where the rocket motion is at least somewhat relativistic, but the exhaust velocity is not. Then, the equation in part (a) tells us that in S' the rocket acceleration is:

$$a' = \frac{du'}{dt'} = -\frac{V_{\text{ex}}}{M_0} \frac{dM_0}{dt'}. \quad (6.7)$$

Now, transform the acceleration to the S frame to find:

$$a = \frac{du}{dt} = -\frac{1}{\gamma^3} \frac{V_{\text{ex}}}{M_0} \frac{dM_0}{dt'}. \quad (6.8)$$

From time dilation show that:

$$\frac{du}{dt} = -\frac{1}{\gamma^2} \frac{V_{\text{ex}}}{M_0} \frac{dM_0}{dt}. \quad (6.9)$$

Finally, cancel the dt 's and integrate to find:

$$\frac{u}{c} = \frac{\left(\frac{M_i}{M_f}\right)^{2V/c} - 1}{\left(\frac{M_i}{M_f}\right)^{2V/c} + 1}, \quad (6.10)$$

where V , M_i , and M_f are shorthand for V_{ex} , M_{initial} , and M_{final} , respectively.

- (e). Relating the answers from parts (b) and (d): Show that for exhaust speeds much lower than c , the answer from part (d) matches the non-relativistic expression in part (b). Show that unless V_{ex} is a substantial fraction of the speed of light, the rocket will never attain truly relativistic speeds.

Optional Problem 7: “Human Power Output Per Unit Mass vs. That of the Sun”: Suppose that a “typical” person of mass ~ 60 kg, puts out 60 Watts of power while at rest. By contrast, the Sun has a luminosity (power output) of 4×10^{26} Watts, and mass of 2×10^{30} kg. Compare the power generated per unit mass of the Sun and a person. How do you make sense of this result?

Optional Problem 8: “Powering the Sun by Gravitational Potential Energy”: Suppose that the luminosity of the Sun were powered by the release of gravitational potential energy only. Its power output is 4×10^{26} Watts. If the Sun contracted from its current radius of $R_{\odot} = 7 \times 10^8$ m, to half that value, how long would the contraction have to last in order to supply the observed power output? Assume that the gravitational potential energy of the Sun is $\sim GM_{\odot}^2/R$, where $M_{\odot} = 2 \times 10^{30}$ kg, and R is the radius at any given time.

Optional Problem 9: “Snell’s Law Derived From Fermat’s Principle”: Fermat’s Principle states that the path of a light ray from one point to another is that which requires the least time. Suppose that light is emitted by a source at $(x = 0, y = y_0)$ in air, heads in the direction of a semi-infinite slab of glass located at $y = 0$, and enters the glass such as to reach the point $(x = x_0, y = -y_0)$. The geometry is shown in the diagram. Take the speed of light to be c in air, and c/n in glass, where n is called the “index of refraction”, and is typically a number in the range of 1.4-1.6.

Assume, from Fermat’s Principle, the obvious result that *within* any given medium (i.e., of fixed n) light travels in a straight line. Then use Fermat’s Principle to prove Snell’s law: $\sin \theta_i = n \sin \theta_t$. You need not bother with the calculus of variations for this problem.

Hint: take the distances traveled to be $y_0/\cos \theta_i$ and $y_0/\cos \theta_t$ in the air and glass, respectively. Then note that $x_0 = y_0(\tan \theta_i + \tan \theta_t)$, which defines the constraint between θ_i and θ_t .

Optional Problem 10: “Lagrange’s Equations of Motion”: Marion & Thornton’s book on “Classical Dynamics” introduces Lagrangian mechanics with the following two instructive statements:

1. “Minimal principles in physics have a long and interesting history. The search for such principles is predicated on the notion that nature always minimizes certain important quantities when a physical process takes place.”
2. In two papers in 1834 and 1835, Hamilton announced the dynamical principle on which it is possible to base all of mechanics and, indeed, most of classical physics. Hamilton’s Principle may be stated as follows: “Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.”

Let’s use this principle and the Euler equations for extremizing path integrals to derive Lagrange’s equations of motion and solve a couple of simple mechanics problems. First, identify the “Lagrangian” as the difference between the kinetic and potential energies:

$$\mathcal{L} \equiv K - V, \tag{10.1}$$

where K and V are the kinetic and potential energies, respectively. Show that if K and V depend on coordinates q_k , then the equations that satisfy Hamilton’s principle are:

$$\frac{\partial \mathcal{L}}{\partial q_k} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = 0. \tag{10.2}$$

- (a). As a first illustrative mechanics problem, consider a mass M suspended (in gravity) on a spring with a spring constant k . Call the vertical direction z . Take the position of the mass to be $z = 0$ when the spring is neither stretched nor compressed. What is the potential energy as a function of z . Write down the kinetic energy in terms of \dot{z} . Use Lagrange’s equation to find the equation of motion of the mass. How does the equation compare with that found by starting with $F = ma$?
- (b). As a second illustrative problem, consider a pendulum consisting of a mass, M , suspended on a string of length l . Displace the pendulum by an angle ϕ with respect to the vertical and release the mass. Use Lagrange’s equation to describe the subsequent motion of the mass. The problem is effectively one-dimensional in that a complete description of the motion is given in terms of a single dependent variable (ϕ) and its derivative ($\dot{\phi}$). Simple analytic solutions to this equation are found by making a small-angle approximation, i.e., $\sin \phi \approx \phi$. You also made the same approximation in the 8.01 version of this problem.

Feedback: Roughly how much time did you spend on this problem set?