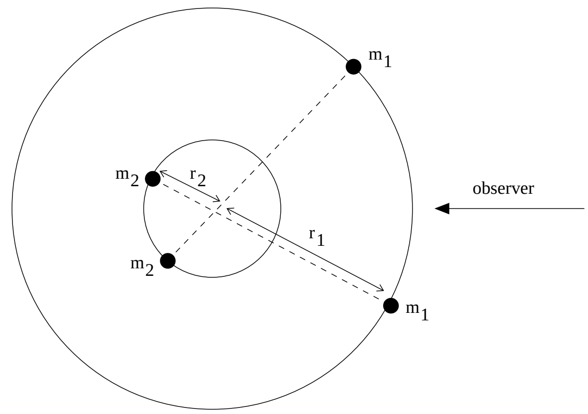
MIT 8.03 Fall 2005 – Solutions to Problem Set 8

Problem 8.1 — Doppler shifts of EM radiation ⇒ a black-hole X-ray binary

The following figure shows the binary system at two different times.



(a) Since the two orbits are circular, the speed of the two masses are $v_1 = 2\pi r_1/T$ and $v_2 = 2\pi r_2/T$. Applying Newton's second law to m_1,

$$\begin{aligned} F &= m_1 a \\ \frac{Gm_1m_2}{(r_1+r_2)^2} &= m_1\frac{v_1^2}{r_1} \\ &= m_1\frac{1}{r_1}\left(2\pi\frac{r_1}{T}\right)^2 \\ &= m_1\frac{4\pi^2 r_1}{T^2} \\ \Rightarrow T^2 &= \frac{4\pi^2(r_1+r_2)^3}{G(m_1+m_2)}. \end{aligned}$$

Note that we could have applied Newton's law to m_2 and gotten the same result.

(b) Since the absorption line is in the visible spectrum, it must be produced by the donor (m_1). The figure below shows the donor at the two positions where the observer measures the maximum radial velocity, v_{max}. Note that $v_1 = v_{\text{max}}$.

The minimum and maximum of the observed wavelengths correspond to position 1 (blue shift) and position 2 (red shift) respectively. The doppler shift is given by

$$\frac{\lambda'}{\lambda} = \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}},$$

where $\beta = v/c$, λ' and λ are the wavelengths in the reference frames of the observer and the star, respectively. For this problem, $\theta = 0$. Hence,

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1-\beta}{1+\beta}} \approx 1 - \beta$$
$$\Rightarrow \beta \approx 1 - \frac{\lambda'}{\lambda}.$$

Since $\lambda = (499.75 \text{ nm} + 500.25 \text{ nm})/2 = 500$ nm,

$$\beta \approx 1 - \frac{499.75}{500} = 5 \times 10^{-4}.$$

Thus, $v = 150$ km/s. Finally, the period of the spectrum shift of the donor equals its orbital period, i.e, $T = 5.6$ days.

(c) Using $v_1 = 2\pi r_1/T$, $r_1 \approx 1.16 \times 10^{10}$ m.

(d) Let $x = r_2/r_1$. Then, eliminating r_2 in the equation derived in part (a),

$$T^2 = \frac{4\pi^2 r_1^3 (x+1)^3}{G(m_1 + m_2)}$$

From the definition of center of mass, $m_1 r_1 = m_2 r_2$. Then, $m_2 = m_1/x$. Eliminating m_2,

$$T^2 = \frac{4\pi^2 r_1^3 (x+1)^3}{G m_1 (1 + 1/x)}.$$

Substituting for the known values of G, T, r_1 and m_1, we arrive at the following equation

$$15.32 = x^3 + 2x^2 + x.$$

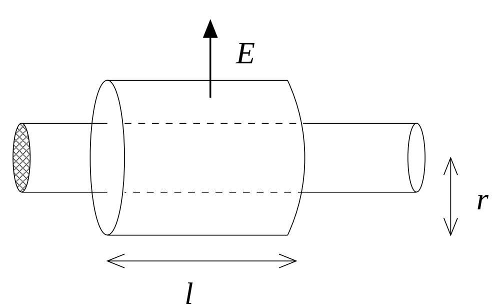
The only real solution to this equation is $x = 1.87$. Hence, $r_2 = x r_1 = 2.17 \times 10^{10}$ m.

(e) $m_2 = m_1/x \approx 16 M_{sun} \approx 3.19 \times 10^{31}$ kg.

Problem 8.2 (Bekefi & Barrett 5.3) — Transmission line

(a) (i) **Capacitance**

In order to find the capacitance of the system, we need to assume that one wire holds a linear charge density μ and the other one holds $-\mu$. Consider only the positively charged wire, then we can use Gauss' law to find the electric field outside the wire. We can imagine a cylindrical gaussian surface of radius r and length l concentric to the wire with flat end pieces.



By symmetry, the electric field is normal to the curved surface of the cylinder but it is parallel to the flat ends. The area of the curved portion of the gaussian surface is $2\pi rl$. Then, applying Gauss' Law,

$$\begin{aligned} \oint_S \vec{E}\cdot d\vec{A} &= \frac{Q}{\epsilon_0} \\ \underbrace{\int_{\text{sides}} \vec{E}\cdot d\vec{A}}_{=0} + \int_{\text{curved}} \vec{E}\cdot d\vec{A} &= \frac{Q}{\epsilon_0} \\ E2\pi rl &= \frac{l\mu}{\epsilon_0} \\ \Rightarrow E &= \frac{\mu}{2\pi r\epsilon_0}, \end{aligned}$$

where E is the electric field magnitude for $r \geq a$. The potential difference between the wire and a distance b away is

$$\begin{aligned} V' &= \int_a^b \vec{E}\cdot d\vec{r} \\ &= \int_a^b \frac{\mu}{2\pi\epsilon_0 r}\, dr \\ &= \frac{\mu}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right). \end{aligned}$$

Due to the symmetry of the two wires, the electric potential between the wires is twice the potential due to one wire. Hence, the capacitance per unit length is

$$C_0 = \frac{\mu}{V} = \frac{\mu}{2V'} = \frac{\pi\epsilon_0}{\ln(b/a)}.$$

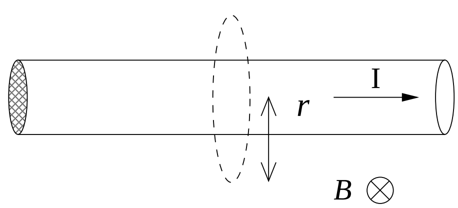
Notice that C_0 and ϵ_0 have the same units, i.e. F/m.

(ii) **Inductance**

Let's assume that each wire carries a current I in opposite directions. Recall

$$\oint_L \vec{B}\cdot d\vec{r} = \mu_0 I_0 + \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$

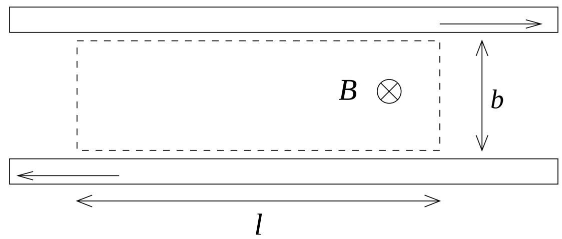
The integral is over a closed path L. If we attach any surface with boundary L then I_0 is the current penetrating that surface and Φ_E is the electric flux through that surface. Consider the magnetic field due to one wire only and a circular Amperian loop of radius r concentric to the wire. Since the current is constant, the E-field is constant so $\partial\Phi_E/\partial t = 0$.



By symmetry, the magnetic field $\vec{B}$ is constant around the loop. Hence,

$$\begin{aligned} \oint_L \vec{B}\cdot d\vec{r} &= 2\pi rB \\ \Rightarrow B &= \frac{\mu_0 I}{2\pi r} \qquad \text{for } a \leq r \leq b. \end{aligned}$$

Consider a rectangular surface of length l and width b, as shown in the figure.



Then, the magnetic flux through that surface due to the magnetic field from only one wire is

$$\begin{aligned}\phi_B &= \int_S \vec{B}\cdot d\vec{A} \\ &= l\int_a^b \frac{\mu_0 I}{2\pi r}\,dr \\ &= \frac{l\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) I\end{aligned}$$

If we now add the second wire into the picture, the magnetic flux doubles. Hence,

$$\Phi_B = \frac{l\mu_0 I}{\pi}\ln\left(\frac{b}{a}\right).$$

Inductance is defined as $L = \Phi_B/I$. Hence,

$$L_0 = \frac{\Phi_B}{I}\frac{1}{l} = \frac{\mu_0}{\pi}\ln\left(\frac{b}{a}\right).$$

Notice that L_0 and μ_0 have the same units, i.e. H/m.

(b)

$$v = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

(c)

$$Z_0 = \sqrt{\frac{L_0}{C_0}} = \frac{\ln(b/a)}{\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}$$

(d) $C_0 \approx 11$ pF/m.
$L_0 \approx 1$ μH/m.
$Z_0 \approx 302$ Ω.

Problem 8.3 — Coaxial cable

(a) By convention, the current of the reflected wave is negative. Also, remember that the reflected wave travels in the opposite direction to the transmitted wave, so the wavenumber k flips its sign. Using $V = IZ$,

$$I(z,t) = \frac{V_i}{Z_0}e^{j(\omega t - kz)} - \frac{V_r}{Z_0}e^{j(\omega t + kz)}$$

(b) Since the resistor and the capacitor are in series, the total impedance is the sum of their impedances:

$$Z_L = R - \frac{j}{\omega C} = R + \frac{1}{j\omega C}.$$

(c)

$$\begin{aligned} V(0,t) &= (V_i + V_r)e^{j\omega t} \\ I(0,t) &= \frac{1}{Z_0}(V_i - V_r)e^{j\omega t} \end{aligned}$$

The boundary conditions are

$$V(0,t) = V_L \qquad I(0,t) = I_L$$

Then, using the result derived in part (b),

$$\begin{aligned} Z_L &= \frac{V_L}{I_L} = \frac{V(0,t)}{I(0,t)} \\ &= Z_0 \frac{V_i + V_r}{V_i - V_r}. \end{aligned}$$

Solving for V_r gives

$$V_r = V_i \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(R - Z_0) - j/\omega C}{(R + Z_0) - j/\omega C}$$

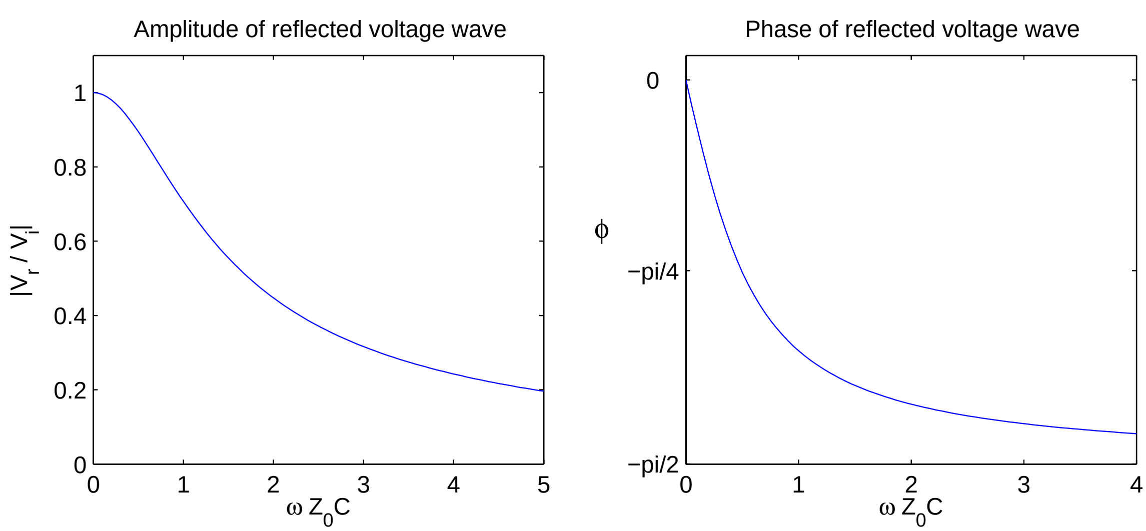
(d) Yes, our expression for V_r/V_i is consistent with the general expression.

(e) If $R = Z_0$,

$$V_r = V_i \frac{-j/\omega C}{2Z_0 - j/\omega C}.$$

Then, the magnitude and phase of V_r are

$$\begin{aligned} |V_r| &= \frac{|V_i|}{\sqrt{1 + (2Z_0\omega C)^2}} \\ \tan\phi(V_r) &= -2Z_0\omega C. \end{aligned}$$



Problem 8.4 (Bekefi & Barrett 5.4) — Rectangular waveguide

We are given

$$E_y = E_{0y}\sin(k_x x)\cos(k_y y)\cos(\omega t - k_z z).$$

However, we are not told anything about E_x or E_z. We cannot assume that $E_x = E_z = 0$. Instead, we use Maxwell's equations to find the components of $\vec{E}$.

Since the wave propagates in the z-direction, $E_z = 0$. Recall

$$\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= 0 \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} &= 0 \\
\Rightarrow \frac{\partial E_x}{\partial x} &= E_{0_y} k_y \sin(k_x x)\sin(k_y y)\cos(\omega t - k_z z)
\end{aligned}$$

So,

$$E_x = -E_{0_y}\frac{k_y}{k_x}\cos(k_x x)\sin(k_y y)\cos(\omega t - k_z z)$$

We then have

$$\vec{E} = E_x\hat{x} + E_y\hat{y}$$

(a) The y-component of the wave equation gives

$$\begin{aligned}
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \\
k_x^2 + k_y^2 + k_z^2 &= \frac{1}{c^2}\omega^2 \\
\Rightarrow \omega &= c\sqrt{k_x^2 + k_y^2 + k_z^2}
\end{aligned}$$

We encourage you to check that the x-component of the wave equation gives the same result.

(b) The parallel component of the electric field must vanish at the walls, i.e. $E_{\|}(x=0) = E_{\|}(x=a) = 0$. Hence,

$$\begin{aligned}
E_y(x=a) &= E_{0_y}\sin(k_x a)\cos(k_y y)\cos(\omega t - k_z z) = 0 \\
\sin(k_x a) &= 0 \\
\Rightarrow k_x &= \frac{m\pi}{a} \qquad \text{for integers } m \geq 0
\end{aligned}$$

(c)

$$\begin{aligned}
E_x(y=b) &= -E_{0_y}\frac{k_y}{k_x}\cos(k_x x)\sin(k_y b)\cos(\omega t - k_z z) = 0 \\
\sin(k_y b) &= 0 \\
\Rightarrow k_y &= \frac{n\pi}{b} \qquad \text{for integers } n \geq 0
\end{aligned}$$

There is an additional constraint on the integers m and n. If $m = n = 0$ then we have the trivial solution $\vec{E} = 0$. Hence, we restrict m and n such that either $m = 0$ or $n = 0$ but not both at the same time.

(d) Combining the results from the previous parts,

$$\omega = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2} \tag{1}$$

$$k_z = \frac{\omega}{c}\sqrt{1 - \left(\frac{m\pi c}{\omega a}\right)^2 - \left(\frac{n\pi c}{\omega b}\right)^2}$$

Let

$$\omega_{m,n} = \pi c\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}.$$

Then,

$$k_z c = \sqrt{\omega^2 - \omega_c^2}$$

Since $a > b$, the restriction $\omega^2 - \omega_c^2 \geq 0$ implies that the lowest frequency ω for which propagation is possible is $\omega = \omega_{1,0}$.

Note that there is no set of discrete values of ω that can propagate through the waveguide. The problem is very different from resonance frequencies. Here, **all** values $\omega \geq \omega_{1,0}$ can propagate.

The phase and group velocities are

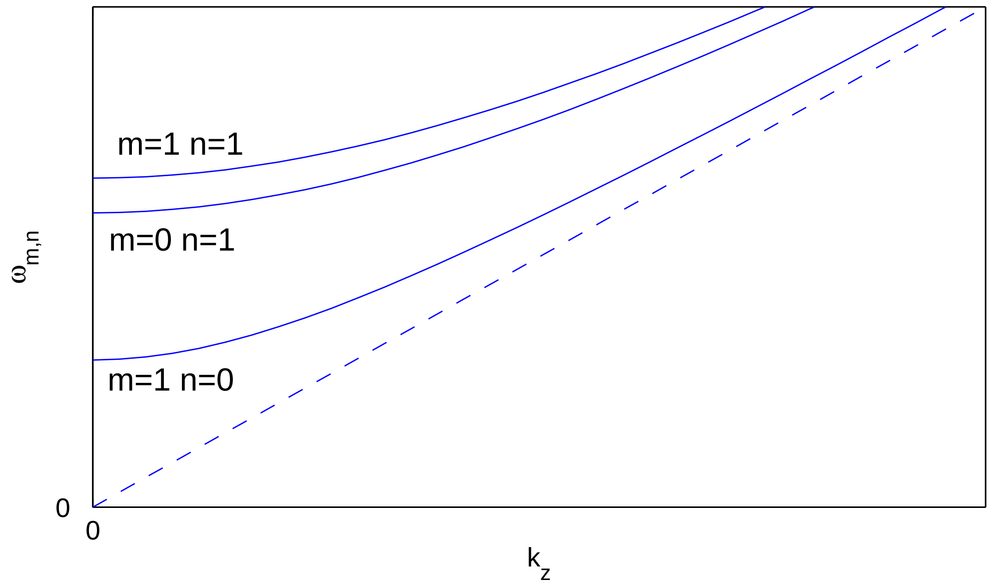
$$\begin{aligned} v_{p_z} &= \frac{\omega_z}{k_z} = \frac{c}{\sqrt{1 - (\omega_c/\omega)^2}} \\ v_{g_z} &= \frac{d\omega}{dk_z} = \frac{c^2 2k_z}{2c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2}} = \frac{c^2 k_z}{\omega} = \frac{c^2}{v_{p_z}} \end{aligned}$$

The limiting values are

$$\begin{array}{lllll} \omega \to \infty & \Rightarrow & k_z \to \infty & v_{p_y} \to c & v_{g_y} \to c \\ \omega \to \omega_c & \Rightarrow & k_z \to 0 & v_{p_y} \to \infty & v_{g_y} \to 0. \end{array}$$

Notice that $v_{p_z} \geq c$ and $v_{g_z} \leq c$. The group velocity—and not the phase velocity—must be less or equal than the speed of light. Notice also that here $v_{p_z} v_{g_z} = c^2$.

We can graphically display equation **??** for various values of m and n. Recall that there is no solution for $m = n = 0$. Since $a > b$, $\omega_{1,0} < \omega_{0,1}$. Thus the curve for $m = 1$ and $n = 0$ lies below the one for $m = 0$ and $n = 1$.



Problem 8.5 (Bekefi & Barrett 5.7) — Resonance cavity

We are given

$$\vec{E} = E_0 \sin(k_x x)\sin(k_y y)\sin(\omega t)\hat{z}$$

Since the cavity is a conductor, the tangential component of the electric field must vanish at the walls, i.e.

$$E_z(x=0) = E_z(x=a) = E_z(y=0) = E_z(y=a) = 0.$$

Hence, $k_x = n\pi/a$, $k_y = m\pi/a$ and $k_z = 0$.

The z-component of the wave equation gives

$$\begin{aligned} \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \\ k_x^2 + k_y^2 &= \frac{1}{c^2}\omega^2 \\ \Rightarrow \omega_{m,n} &= \frac{\pi c}{a}\sqrt{n^2+m^2}, \end{aligned}$$

where m and n are integers such that $m \geq 1$ and $n \geq 1$.

(a) The first solution is $n = m = 1$. Then, $\omega_1 = \pi c\sqrt{2}/a$ so $\lambda_1 = \sqrt{2}a$.

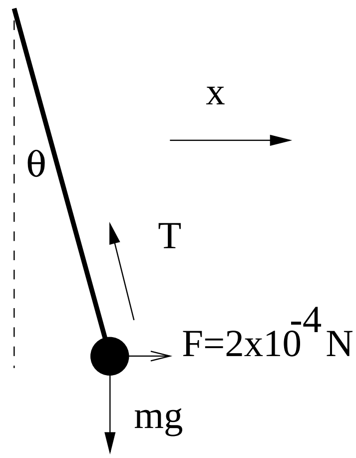
(b) $n = 2$ and $m = 1$ or $n = 1$ and $m = 2$. Then, $\omega_2 = \pi c\sqrt{5}/a$ so $\lambda_2 = 2a/\sqrt{5}$.

Problem 8.6 — Radiation pressure

In case of absoption, the radiation pressure is S/c (c.f. B&B p.244), where S is the magnitude of the Poynting vector (W/m^2). In case of reflection, the pressure is $2S/c$. Thus, the force on the mirror is $2SA/c$, where A is the cross-sectional area of the laser beam, $SA = 30$ kW. Thus, the force is $2(3\times10^3)/(3\times10^8) = 2\times10^{-4}$ N. From the figure, Newton's law gives

$$T\sin\theta \approx mg\frac{x}{L} \approx 2\times10^{-4}.$$

Hence, $x = 2$ mm.


x
θ
T
F=2x10-4 N
mg