# 8.03 Fall 2005
# Problem Set 3 Solutions

## Solution 3.1: Take home experiment # 2

### Coupled oscillators, resonance, and normal modes

1. How does the tension effect the time it takes for the energy to shift from one mass to the other?

   As I, Igor Sylvester, increased the tension in the string, the time it took for the energy to shift from one mass to the other increased and vice-versa. We can explain this by noting that the string connecting the two pendulums provides the coupling to the system. As the tension in the string is increased, the coupling in the system decreases. For example, consider the limiting case in which the tension of the string is infinitely large. Then, the the string would be straight and would not move.

2. Explore the effect of successively larger de-tunings in the 2-pendulum system.

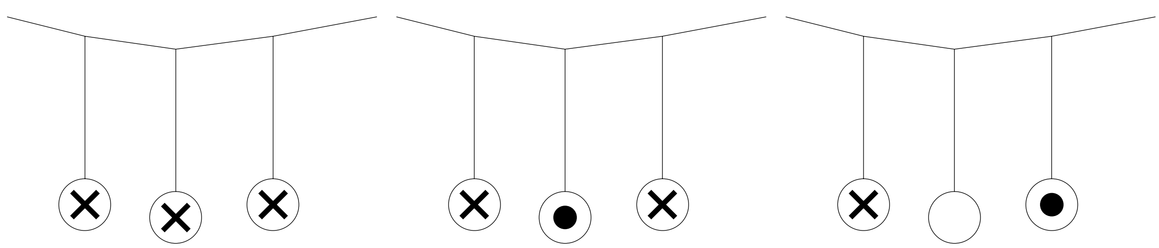
   When making one pendulum longer than the other, I observed that energy did flow from one pendulum to the other. However, the amplitude of the oscillation of the longer pendulum did not go to zero. Furthermore, the amplitude of the shorter pendulum did go to zero but it reached a smaller maximum than in the perfectly tuned configuration.

3. Were you successful at exciting the two normal modes associated with the motion of the two weights perpendicular to the string?

   Yes, I was. It was easy to excite the mode in which both pendulums are in phase; I held the two masses and let them go. However, it was more challenging to excite the mode in which the pendulums are out of phase because it was difficult to give the two masses equal initial amplitudes.

4. Sketch the three modes of the 3-pendulum. Did you find them?

   In the following figure, the cross means motion into the page and the dot means motion out of the page. Of course, we could reverse this conversion. What really matters is the relative phase of the pendulums. A lack of a cross or dot means that the mass stays still.

   I tried to excite the first (leftmost) mode by holding the three bobs at a fixed amplitude and releasing them simultaneously. For the second (middle) mode, I held the bob in the middle and released it. Then I tried to hold the other two bobs at an amplitude equal to that of the middle oscillating bob. I then released the two bobs exactly when the middle bob was out of phase and at its maximum amplitude. Finally, I tried to excite the last (rightmost) normal mode by holding the two outermost bobs each in one of my hands at the same amplitude and out of phase and releasing them simultaneously.

5. What happens when any one of the pendulums is initially excited if the length of one individual pendulum are changed?

   When I made the length of one pendulum shorter then most of the energy stayed in the shorter pendulum. In other words, the amplitudes of oscillations of the other two pendulums (of equal length) were small compared to that of the shorter pendulum. This effect was independent of the pendulum that I made shorter (middle or one of the outermost). I observed similar results when I made one pendulum longer.

6. How about making all pendulums' lengths different?

   When I excited the shorter pendulum, I observed that some of its energy shifted back and forth to the longer pendulum. The amplitude of oscillation of the middle-length pendulum was very small. Similarly, when I excited the middle-length pendulum, some of the energy shifted back and forth to the longer pendulum. The amplitude of oscillation of the shorter pendulum was small. Finally, when I excited the longer pendulum, very little energy shifted to the other shorter pendulums. In conclusion, the longer pendulum was more sensitive to excitation from the coupling in the system. The other two shorter pendulums were less sensitive to excitation.

## Solution 3.2: (French 5-10) Coupled Oscillators using two springs

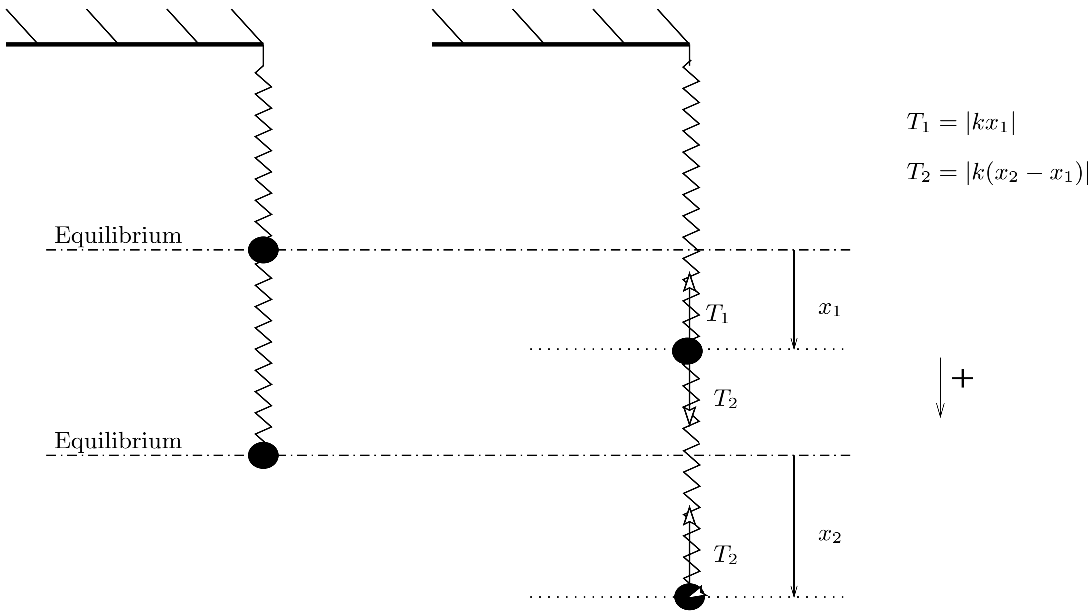



FIG. 1: Coupled oscillator using two springs

Let the displacement from the equilibrium positions for masses $m_1$ and $m_2$ be $x_1$ and $x_2$ respectively. Then the tensions in the two strings are $T_1 = kx_1$ and $T_2 = k(x_2 - x_1)$ respectively. Now

$$m_1\ddot{x_1} = +k(x_2 - x_1) - kx_1$$

$$m_2\ddot{x_2} = -k(x_2 - x_1)$$

Substituting $m_1 = m_2 = m$ and $\omega_s^2 = k/m$ we get

$$\ddot{x_1} = \omega_s^2(x_2 - 2x_1)$$

$$\ddot{x_2} = \omega_s^2(x_1 - x_2) \quad (1)$$

Let

$$x_1 = C_1\cos(\omega t) \qquad x_2 = C_2\cos(\omega t)$$

Now using these in Eq. 1

$$-\omega^2 C_1 \ + \ 2\omega_s^2 C_1 = \omega_s^2 C_2$$

$$-\omega^2 C_2 \ + \ \omega_s^2 C_2 = \omega_s^2 C_1 \tag{2}$$

**Method I: Without using Cramer's Rule**

¿From Eq. 2 we get

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega^2} = \frac{\omega_s^2 - \omega^2}{\omega_s^2}$$

$$\omega_s^4 = 2\omega_s^4 - 3\omega_s^2\omega^2 + \omega^4$$

$$\omega^4 - 3\omega_s^2\omega^2 + \omega_s^4 = 0 \tag{3}$$

$$\omega^2 = \frac{3\omega_s^2 \pm \sqrt{9\omega_s^4 - 4\omega_s^4}}{2} = \left(3 \pm \sqrt{5}\right)\frac{\omega_s^2}{2}$$

$$\omega^2 = \left(3 \pm \sqrt{5}\right)\frac{k}{2m} \tag{4}$$

$$\omega_+ = \sqrt{(3+\sqrt{5})\frac{k}{2m}} \qquad \omega_- = \sqrt{(3-\sqrt{5})\frac{k}{2m}}$$

$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}} = \frac{\sqrt{5}+1}{\sqrt{5}-1} \tag{5}$$

For $\omega_+ = \sqrt{(3+\sqrt{5})k/2m}$

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega_+^2} = \frac{2\omega_s^2}{4\omega_s^2 - (3+\sqrt{5})\omega_s^2} = \frac{2}{1-\sqrt{5}} \tag{6}$$

For $\omega_- = \sqrt{(3-\sqrt{5})k/2m}$

$$\frac{C_1}{C_2} = \frac{\omega_s^2}{2\omega_s^2 - \omega_-^2} = \frac{2\omega_s^2}{4\omega_s^2 - (3-\sqrt{5})\omega_s^2} = \frac{2}{1+\sqrt{5}} \tag{7}$$

**Method II: Using Cramer's Rule**

On collecting coefficients of $C_1$ and $C_2$ in Eq. 2 we get

$$(2\omega_s^2 - \omega^2)C_1 - \omega_s^2 C_2 = 0$$

$$-\omega_s^2 C_1 + (\omega_s^2 - \omega^2)C_2 = 0$$

$$C_1 = \frac{\begin{vmatrix} 0 & -\omega_s^2 \\ 0 & \omega_s^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix}}$$

$$C_2 = \frac{\begin{vmatrix} 2\omega_s^2 - \omega^2 & 0 \\ -\omega_s^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix}}$$

Non-zero solutions for $C_1$ and $C_2$ only possible if

$$\begin{vmatrix} 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 - \omega^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \omega^4 - 3\omega_s^2\omega^2 + \omega_s^4 = 0$$

This is the same as Eq. 3. From here on the solution is identical to Method I

## Solution 3.3: (French 5-11) Coupled spring and pendulum

### Part (a)

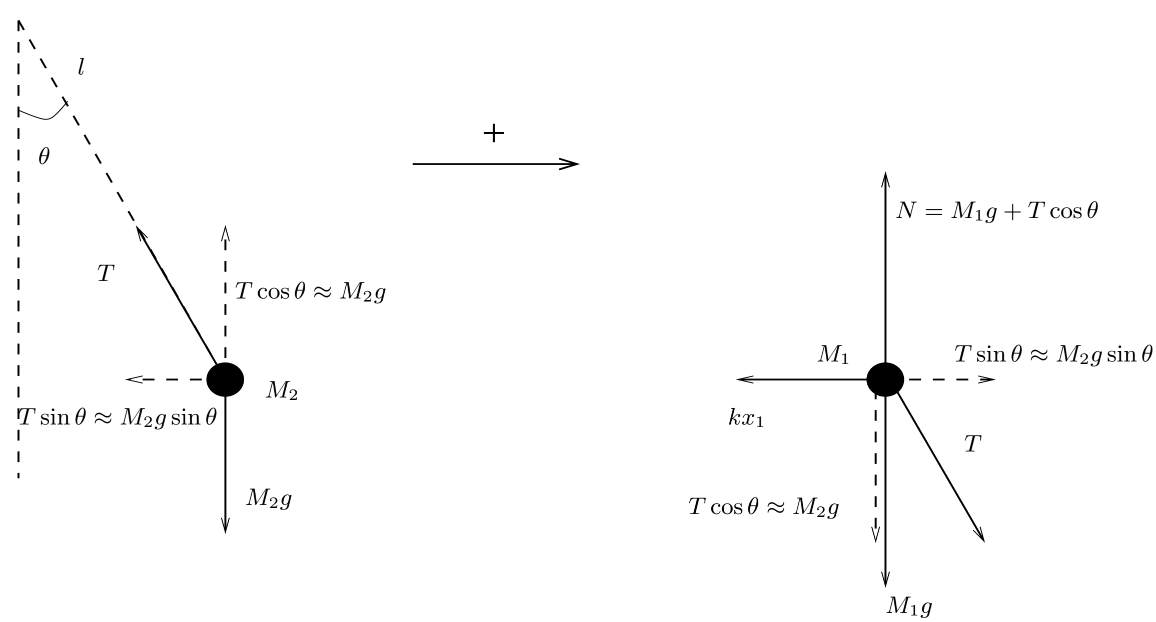



FIG. 2: Isolation diagrams for masses $M_1$ and $M_2$

The tension in the string is $T \approx M_2 g$. The equation of motion for mass $M_2$ in the $x$-direction is as follows

$$\begin{aligned} M_2\ddot{x}_2 &= -M_2 g\sin(\theta) \\ M_2\ddot{x}_2 &= -M_2\frac{g}{l}(x_2-x_1) \end{aligned}$$

and for mass $M_1$ is

$$M_1\ddot{x}_1 = -kx_1 + M_2\frac{g}{l}(x_2-x_1)$$

**Part (b) & (c)**

Substituting $\omega_s^2 = k/M_2$, $\omega_p^2 = g/l$ and $M_1 = M_2 = M$ we get

$$\ddot{x}_2 + \omega_p^2 x_2 - \omega_p^2 x_1 = 0$$

$$\ddot{x}_1 + (\omega_s^2+\omega_p^2)x_1 - \omega_p^2 x_2 = 0$$

Let

$$x_1 = C_1\cos(\omega t) \qquad x_2 = C_2\cos(\omega t)$$

Substituting this in eq. 16:

$$-\omega^2 C_2 + \omega_p^2 C_2 = \omega_p^2 C_1$$

$$-\omega^2 C_1 + (\omega_s^2+\omega_s^2)C_1 = \omega_p^2 C_2 \tag{8}$$

**Method I: Without using Cramer's Rule**

$$\begin{aligned} \frac{C_1}{C_2} &= \frac{-\omega^2+\omega_p^2}{\omega_p^2} = \frac{\omega_p^2}{-\omega^2+\omega_p^2+\omega_s^2} \\ \omega_p^4 &= \omega^4 - \omega^2\omega_s^2 - 2\omega^2\omega_p^2 + \omega_p^2\omega_s^2 + \omega_p^4 \end{aligned}$$

$$\omega^4 - (2\omega_p^2+\omega_s^2)\omega^2 + \omega_p^2\omega_s^2 = 0 \tag{9}$$

$$\begin{aligned} \omega^2 &= \frac{2\omega_p^2+\omega_s^2}{2} \pm \frac{1}{2}\sqrt{(2\omega_p^2+\omega_s^2)^2 - 4\omega_p^2\omega_s^2} \\ &= \frac{2\omega_p^2+\omega_s^2}{2} \pm \frac{1}{2}\sqrt{4\omega_p^4+\omega_s^4} \end{aligned} \tag{10}$$

$$\omega_\pm = \left[\frac{2\omega_p^2+\omega_s^2}{2} \pm \frac{1}{2}\left(4\omega_p^4+\omega_s^4\right)^{1/2}\right]^{1/2} \tag{11}$$

For $\omega_+$

$$\frac{C_1}{C_2} = \frac{-\omega_+^2+\omega_p^2}{\omega_p^2} = \frac{-\omega_s^2 - \sqrt{4\omega_p^4+\omega_s^4}}{2\omega_p^2} \tag{12}$$

For $\omega_-$

$$\frac{C_1}{C_2} = \frac{-\omega_-^2 + \omega_p^2}{\omega_p^2} = \frac{-\omega_s^2 + \sqrt{4\omega_p^4 + \omega_s^4)}}{2\omega_p^2} \tag{13}$$

**Method II: Using Cramer's Rule**

On collecting coefficients of $C_1$ and $C_2$ in Eq. 8 we get

$$\omega_p^2 C_1 + (\omega^2 - \omega_p^2)C_2 = 0$$

$$(-\omega^2 + \omega_p^2 + \omega_s^2)C_1 - \omega_p^2 C_2 = 0$$

$$C_1 = \frac{\begin{vmatrix} 0 & \omega^2 - \omega_p^2 \\ 0 & -\omega_p^2 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$

$$C_2 = \frac{\begin{vmatrix} \omega_p^2 & 0 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & 0 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}}$$

Non-zero values of $C_1$ and $C_2$ only possible if

$$\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad -\omega_p^4 - (\omega^2 - \omega_p^2)(-\omega^2 + \omega_p^2 + \omega_s^2) = 0$$

$$\omega^4 - (\omega_s^2 + 2\omega_p^2)\omega^2 + \omega_p^2\omega_s^2 = 0$$

This is the same as Eq. 9. From here on the solution is identical to Method I.

## Solution 3.4: (Bekefi & Barrett 1.16) Coupled oscillators using three springs

Side (a) of Fig. 3 shows the system at rest and side (b) shows it at some random time $t$. Displacements from Equilibrium are $x_1$ and $x_2$. Now $y_1 = d_1 + d_2 + x_1$ and $y_2 = d_2 + x_2$

**Part (a)**

The equations of motion are:

$$m\ddot{x}_1 = -2kx_1 - k(x_1 - x_2) \quad \Rightarrow \quad \ddot{x}_1 + 3\omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

$$m\ddot{x}_2 = +k(x_1 - x_2) \quad \Rightarrow \quad \ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0 \tag{14}$$

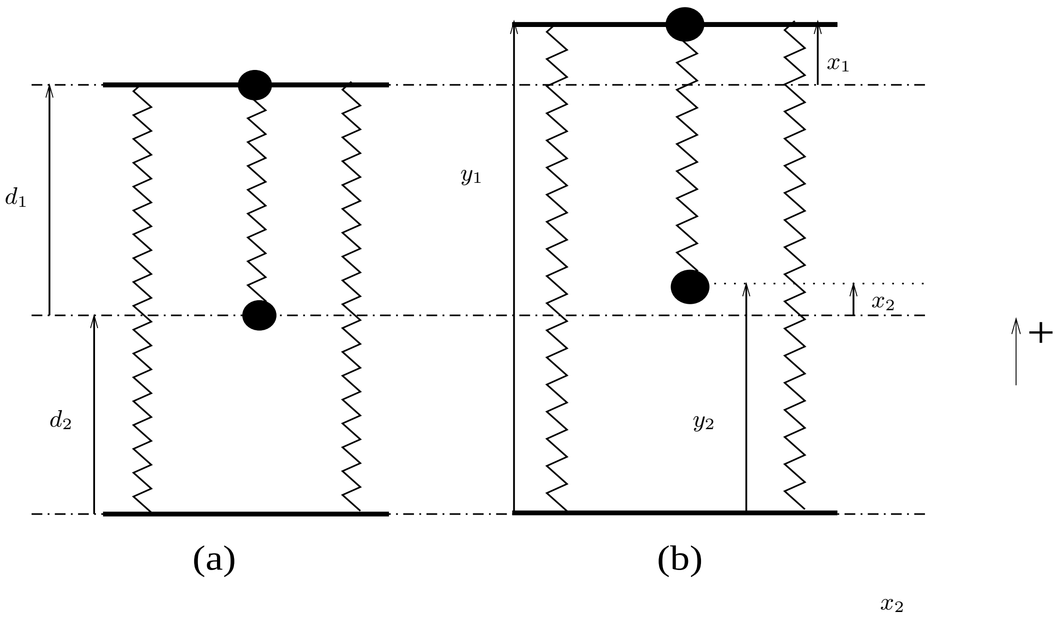



FIG. 3: Coupled oscillator using three springs

where $\omega_0^2 = k/m$

**Part (b)**

Substituting $x_1 = A\cos(\omega t)$ and $x_2 = B\cos(\omega t)$ in equations of motion from Eq. 14

$$A(3\omega_0^2 - \omega^2) = B\omega_0^2 \qquad A\omega_0^2 = B(\omega_0^2 - \omega^2)$$

$$\begin{aligned}
\frac{A}{B} &= \frac{\omega_0^2}{3\omega_0^2 - \omega^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2} \\
\omega_0^4 &= 3\omega_0^4 - 4\omega^2\omega_0^2 + \omega^4 \\
\omega^4 &- 4\omega_0^2\omega^2 + 2\omega_0^4 = 0 \\
\omega_\pm^2 &= \omega_0^2(2 \pm \sqrt{2})
\end{aligned}$$

For $\omega_1 = \omega_0(2 - \sqrt{2})^{1/2}$

$$\frac{B}{A} = 1 + \sqrt{2}$$

For $\omega_2 = \omega_0(2 + \sqrt{2})^{1/2}$

$$\frac{B}{A} = 1 - \sqrt{2}$$

Hence the general solutions are:

$$\begin{aligned}
y_1(t) &= d_1 + d_2 + x_1(t) = d_1 + d_2 + A\cos(\omega_1 t + \alpha) + B\cos(\omega_2 t + \beta) \\
y_2(t) &= d_2 + x_2(t) = d_1 + d_2 + (1+\sqrt{2})A\cos(\omega_1 t + \alpha) + (1-\sqrt{2})B\cos(\omega_2 t + \beta)
\end{aligned} \tag{15}$$

**Part (c)**

Side (a) of Fig. 4 shows the normal mode with higher frequency $\omega_2$ such that $x_2(t) = (1 - \sqrt{2})x_1(t)$. Side (b) shows the normal mode with lower frequency $\omega_1$ such that $x_2(t) = (1 + \sqrt{2})x_1(t)$.

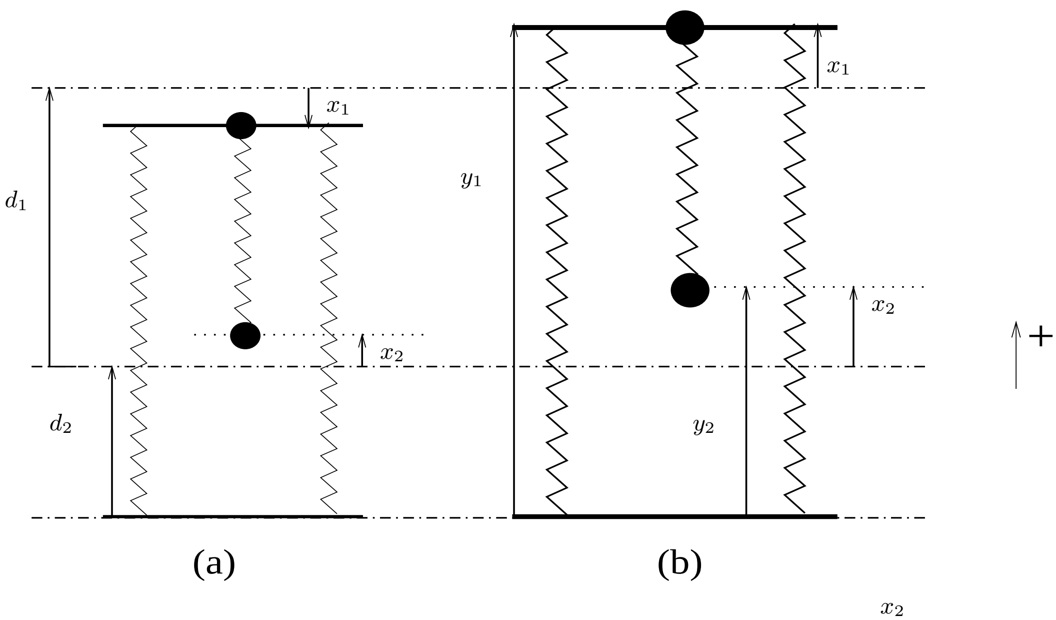



FIG. 4: Normal modes of the three spring oscillator system

## Solution 3.5: Driven coupled oscillator

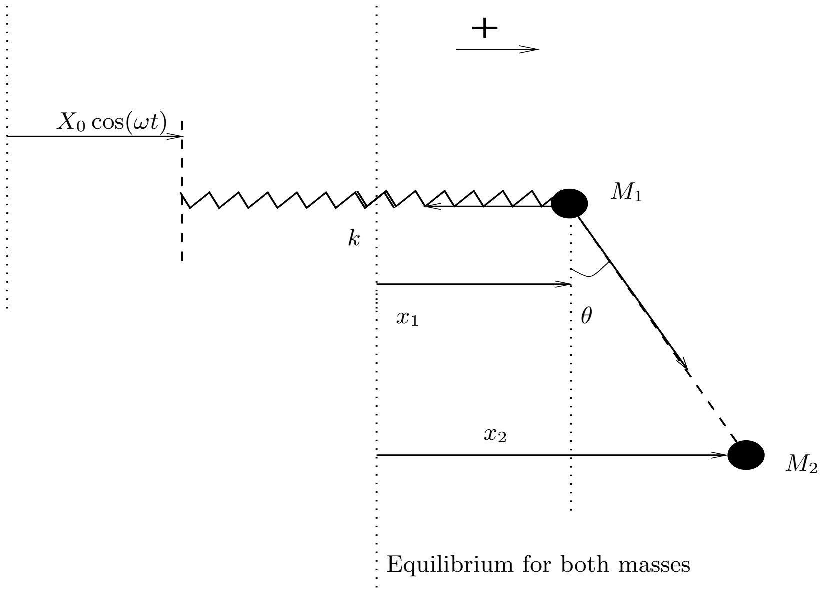



FIG. 5: Driven coupled oscillator

**Part (a)**

The equation of motion for mass $M_2$ is unchanged

$$M_2\ddot{x_2} \;=\; -M_2 g\sin(\theta)$$

and for mass $M_1$ is

$$M_1\ddot{x_1} \;=\; -k[x_1 - X(t)] + M_2\frac{g}{l}(x_2 - x_1)$$

$$M_1\ddot{x_1} \;+\; kx_1 + M_2\frac{g}{l}(x_1 - x_2) = kX_0\cos(\omega t)$$

**Part (b)**

Substituting $\omega_s^2 = k/M_2$, $\omega_p^2 = g/l$ and $M_1 = M_2 = M$ we get

$$\ddot{x_2} + \omega_p^2 x_2 - \omega_p^2 x_1 \;=\; 0$$

$$\ddot{x_1} + (\omega_s^2 + \omega_p^2)x_1 - \omega_p^2 x_2 \;=\; \omega_s^2 X_0 \tag{16}$$

Let

$$x_1 = C_1\cos(\omega t) \qquad x_2 = C_2\cos(\omega t)$$

Now using these in Eq. 16

$$\omega_p^2 C_1 + (\omega^2 - \omega_p^2)C_2 \;=\; 0$$

$$(-\omega^2 + \omega_s^2 + \omega_s^2)C_1 - \omega_p^2 C_2 \;=\; \omega_s^2 X_0 \tag{17}$$

$$C_1 = \frac{\begin{vmatrix} 0 & \omega^2 - \omega_p^2 \\ \omega_s^2 X_0 & -\omega_p^2 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}} = \frac{kX_0(g - l\omega^2)}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

$$C_2 = \frac{\begin{vmatrix} \omega_p^2 & 0 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & \omega_s^2 X_0 \end{vmatrix}}{\begin{vmatrix} \omega_p^2 & \omega^2 - \omega_p^2 \\ -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_p^2 \end{vmatrix}} = \frac{kgX_0}{Ml\omega^4 - (2Mg + kl)\omega^2 + kg}$$

These are the steady state solutions. The general solution is a linear combination between the transient solution and the steady state solutions. Notice that the transient solution has four adjustable parameters which follow from the initial conditions.

**Part (c)**

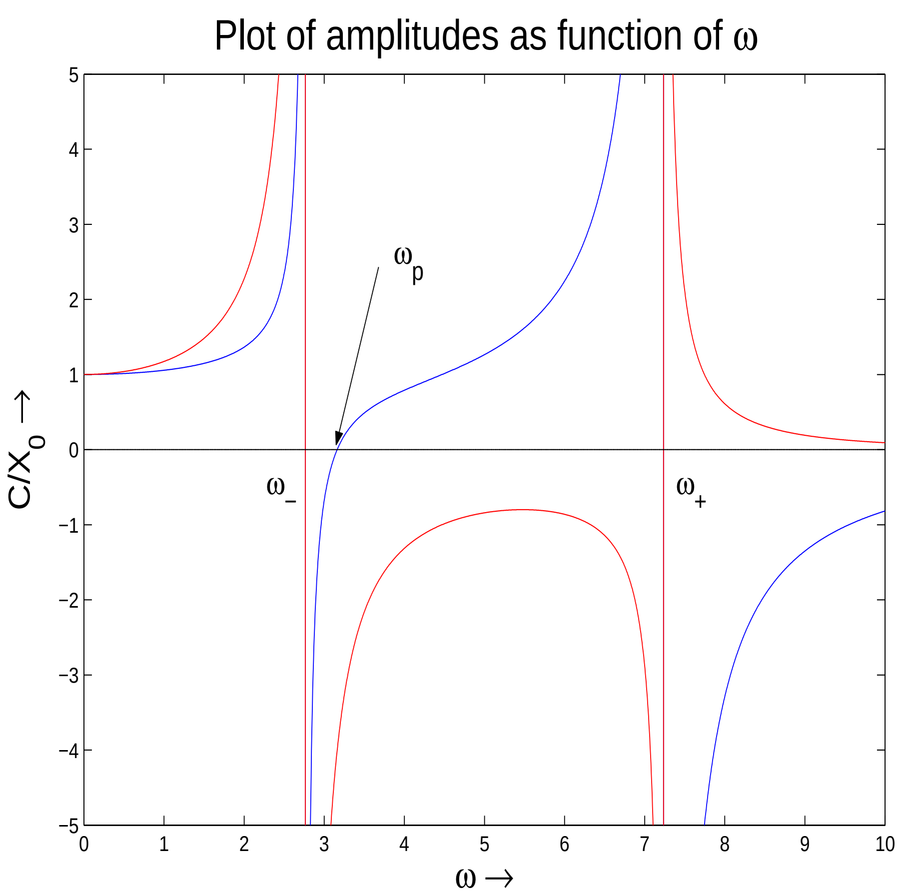



FIG. 6: Plot of amplitudes as a function of $\omega$ [ $C_1$:Blue line & $C_2$:Red Line ]

Fig. 6 shows the plot of amplitudes $C_1$ and $C_2$ as a function of the frequency. Note: At $\omega = 0$, the amplitudes are $C_1 = C_2 = X_0$. This figure is unrealistic. It was derived (i) under the small angle approximation and (ii) for zero damping. Thus the very large amplitude for $C_1$ and $C_2$ as shown are meaningless. If you add sufficient damping, and if you cannot make the small angle approximations, because the angles are large, the problem becomes substantially more complicated. But it can be solved numerically. You will then find meaningful values for the amplitudes. A more insightful way to express (and plot) the amplitude of the pendulum would be to do this in terms of the angle $\theta$, rather than $C_2$.

**Part (d)**

We can note from the functional form of $C_2$ that it cannot have the value zero (except for $\omega \to \infty$). However, $C_1$ will be zero when

$$g - l\omega^2 = 0$$

$$\omega = \omega_p = \sqrt{\frac{g}{l}} \qquad\qquad C_1 \to 0$$

This is the resonance frequency of the pendulum. Thus, at this frequency, the two horizontal forces on the upper mass, $kX_o \cos(\omega t)$ and $T\sin(\theta)$, cancel. Since $\sin(\theta) < 1$, $kX_o$ must always be smaller than $T$. At first sight this inequality seems a bit bizarre, as, according to our derivation, the frequency at which the upper mass stands still, is independent of the spring constant $k$. Also, keep in mind that we never had to make any assumption regarding $k$ in our derivation (the inequality must have been met automatically without our realizing it).

You SHOULD also ask yourself the question:

**How on Earth can the pendulum swing if the mass attached to the spring does not move at all; what is driving the pendulum?**

The answer is simple: **it is not possible!** It is only possible in our dream-world of zero damping. In the presence of damping, no matter how little, the peculiar state is unstable. This can easily be seen as follows.

Assume that the system is in that state. That means that at any moment in time the net horizontal force on the pendulum mass is zero. Thus the vectorial sum of the spring force and $T\sin(\theta)$ must be ZERO. However, if the mass on the spring is not moving, the pendulum is no longer driven, and thus its amplitude will decay, and the net force on the mass on the spring is no longer zero, and thus that mass will start to move. Thus the peculiar state is unstable. You will be able to go through that “special” state by varying omega, but you cannot ”stop” there. However, I demonstrated in lectures (9/28) using 3 different driven systems, that you can get very close to those “special” states, and that is already amazing (and very non-intuitive).

*Walter*