8.03 Fall 2004
Problem Set 1 Solutions

Solution 1.1: Manipulation of complex vectors

Part (a)

$$\begin{aligned}(4-\sqrt{5}j)^3 &= 4^3 - 3\cdot 4^2\cdot\sqrt{5}j + 3\cdot 4\cdot(\sqrt{5}j)^2 - (\sqrt{5}j)^3 \\ &= 64 + 5\sqrt{5}j - 48\sqrt{5}j - 60 \\ &= 4 - 43\sqrt{5}j\end{aligned}$$

Magnitude

$$|(4-\sqrt{5})^3| = \sqrt{4^2 + (43\sqrt{5})^2} = \sqrt{16+9245} = \sqrt{9261} = 96.23 \tag{1}$$

Direction

$$\arctan\left(\frac{-43\sqrt{5}}{4}\right) = -87.62^o$$

We show below a graphical representation. Raising the complex vector Z to the power 3 means that the new angle is 3 times larger than that of Z, and the length of the new vector is the length $|Z|^3$. The length of the vector Z^3 is not to scale ($|Z|^3 \approx 96$).

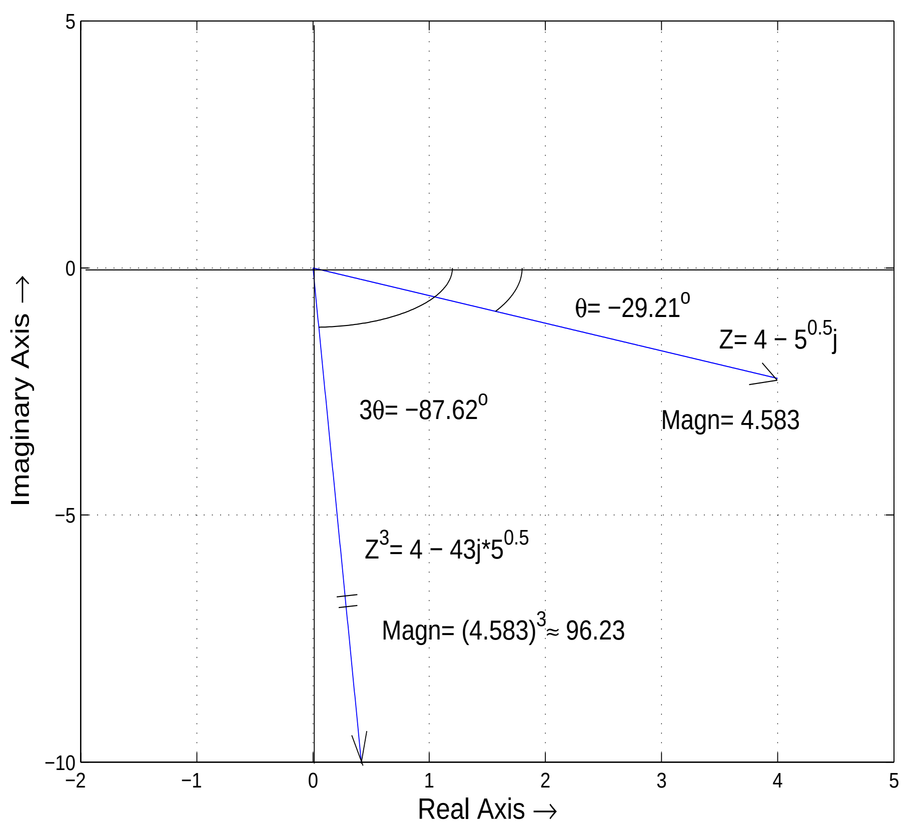

FIG. 1: 1.1(a) Vector Rotation

Part (b)

$$\begin{aligned}\frac{Ae^{j(\omega t+\pi/2)}}{4+5j} &= \frac{A(\cos(\omega t+\pi/2)+j\sin(\omega t+\pi/2))}{4+5j} = \frac{A(\cos(\omega t+\pi/2)+j\sin(\omega t+\pi/2))}{4+5j}\times\frac{4-5j}{4-5j} \\ &= \frac{A[4\cos(\omega t+\pi/2)+5\sin(\omega t+\pi/2)+j[4\sin(\omega t+\pi/2)-5\cos(\omega t+\pi/2)]]}{4^2+5^2}\end{aligned}$$

Real Part

$$\frac{A}{41}[4\cos(\omega t+\pi/2)+5\sin(\omega t+\pi/2)] \tag{2}$$

Imaginary Part

$$j\frac{A}{41}[(4\sin(\omega t+\pi/2)-5\cos(\omega t+\pi/2)] \tag{3}$$

Part (c)

Remember $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$$\begin{aligned}Z_1 &= j^j = [e^{j(\pi/2\pm 2n\pi)}]^j = e^{j(\pi/2\pm 2n\pi)\times j} = e^{j^2(\pi/2\pm 2n\pi)} \\ &= e^{-(\pi/2\pm 2n\pi)} \\ &\simeq 0.208\ ,\ 3.88\times 10^{-4}\ ,\ 1.11\times 10^{2}\ ...\quad (n=0,1...)\end{aligned} \tag{4}$$

Note: All values are real!

$$\begin{aligned}Z_2 &= j^{8.03} = [e^{j(\pi/2\pm 2n\pi)}]^{8.03} = e^{j([8.03\times(\pi/2\pm 2n\pi)]} \\ &= \cos\left[8.03\times(\frac{\pi}{2}\pm 2n\pi)\right] + j\sin\left[8.03\times(\frac{\pi}{2}\pm 2n\pi)\right] \\ &= 0.999+0.047j\ ,\ 0.9724+0.233j\ ,\ 0.990-0.141j\ ...\quad (n=0,1...)\end{aligned}$$

Solution 1.2: SHM of y as a function of x

$$\begin{aligned}y &= A\cos(kx)+B\sin(kx) \\ \Rightarrow\quad \frac{dy}{dx} &= -Ak\sin(kx)+Bk\cos(kx) \\ \Rightarrow\quad \frac{d^2y}{dx^2} &= -Ak^2\cos(kx)-Bk^2\sin(kx) = -k^2[A\cos(kx)+B\sin(kx)] \\ \frac{d^2y}{dx^2} &= -k^2y\end{aligned}$$

Hence the given differential equation has $y = A\cos(kx) + B\sin(kx)$ as its solution.

Now to express the equation in the desired form, we divide and multiply it by $\sqrt{A^2+B^2}$. When we substitute $\cos(\alpha) = A/\sqrt{A^2+B^2}$ and $\sin(\alpha) = -B/\sqrt{A^2+B^2}$, the equation takes the desired form:

$$\begin{aligned}y &= \frac{A\cos(kx)+B\sin(kx)}{\sqrt{A^2+B^2}}\times\sqrt{A^2+B^2} \\ &= \sqrt{A^2+B^2}[\cos(\alpha)\cos(kx)-\sin(\alpha)\sin(kx)] \\ &= \sqrt{A^2+B^2}\cos(kx+\alpha)\end{aligned}$$

$$y = \sqrt{A^2+B^2}\cos(kx+\alpha) = \sqrt{A^2+B^2}Re[e^{j(kx+\alpha)}] = Re[(\sqrt{A^2+B^2}e^{j\alpha})e^{jkx}] \tag{5}$$

where

$$C = \sqrt{A^2+B^2} \qquad \alpha = \tan^{-1}-(\frac{B}{A})$$

Solution 1.3: Oscillating springs

Part (a)

The mass at the end of the spring oscillates with an amplitude of 5 cm and at a frequency of 1 Hz, hence the values of A and ω are:

$$\begin{aligned} A &= 5\ cm \\ \omega &= 2\pi f = 2\pi \times 1 = 2\pi\ rad/s \end{aligned}$$

We are given that at time $t = 0$ the mass is at the position $x = 0$. Using this and substituting the values from above in the equation $x = A\cos(\omega t + \alpha)$ we get

$$0 = 5\cos(\alpha) \quad \Rightarrow \quad \cos(\alpha) = 0$$
$$\Downarrow$$
$$\alpha = \pm\frac{\pi}{2} \tag{6}$$

Hence the possible equations of motion for the mass as a function of time are

$$x = 5\cos(2\pi t + \frac{\pi}{2})\ ;\ 5\cos(2\pi t - \frac{\pi}{2})\ cm \tag{7}$$

where the values of the required constants are $A = 5$ cm, $\omega = 2\pi$ rad/s, and $\alpha = \pm\frac{\pi}{2}$.

Part (b)

$$\begin{aligned} x &= A\cos(\omega t + \alpha) \\ \Rightarrow \quad dx/dt &= -A\omega\sin(\omega t + \alpha) \\ \Rightarrow \quad d^2x/dt^2 &= -A\omega^2\cos(\omega t + \alpha) = -\omega^2 x \end{aligned}$$

Substituting values for A, ω, and α from part (a); and putting $t = \frac{8}{3}$ sec, we get

$$\begin{aligned} x &= 5\cos[(2\pi \times \frac{8}{3}) \pm \frac{\pi}{2}] = 5\cos(\frac{16\pi}{3} \pm \frac{\pi}{2}) \\ &= 5\cos(\frac{35\pi}{6}), 5\cos(\frac{29\pi}{6}) = 5\cos(\frac{11\pi}{6}), 5\cos(\frac{5\pi}{6}) \\ &= \pm\frac{5\sqrt{3}}{2}\ cm = \pm 4.330\ cm \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{dx}{dt} &= -5 \times 2\pi\sin[(2\pi \times \frac{8}{3}) \pm \frac{\pi}{2}] = -10\pi\sin(\frac{16\pi}{3} \pm \frac{\pi}{2}) \\ &= \pm 5\pi\ cm/s = \pm 15.708\ cm/s \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{d^2x}{dt^2} &= 5 \times (2\pi)^2\cos[(2\pi \times \frac{8}{3}) \pm \frac{\pi}{2}] = 20\pi^2\cos(\frac{16\pi}{3} \pm \frac{\pi}{2}) \\ &= \pm 10\sqrt{3}\pi^2\ cm/s^2 = \pm 170.95\ cm/s^2 \end{aligned} \tag{10}$$

Solution 1.4: Floating Cylinder

Part (a)

The diameter of the floating cylinder is d and it has l of its length submerged in water. The volume of water displaced by the submerged part of the cylinder in equilibrium condition is $\pi d^2 l/4$. Let the density of water be ρ_w and the density of cylinder be ρ_{cyl}. Hence the mass of the cylinder is:

$$M_{cyl} = \rho_w V_{displaced} = \rho_w \pi \frac{d^2 l}{4} = \rho_{cyl} \pi \frac{d^2 L}{4}$$

When the cylinder is submerged by an additional length x from its equilibrium position, the restoring force acting on it is as follows

$$F_{restoring} = -\frac{\rho_w g \pi d^2}{4} x \qquad \Rightarrow \qquad M_{cyl}\ddot{x} = -\frac{\rho_w g \pi d^2}{4} x$$

$$0 = \ddot{x} + \frac{\rho_w g \pi d^2 x}{4 M_{cyl}} \tag{11}$$

$$\Rightarrow \quad \omega^2 = \rho_w \frac{g \pi d^2}{4 M_{cyl}} = \frac{g}{l}$$

$$x(t) = A\cos(\omega t + \alpha) = A\cos(\sqrt{\frac{g}{l}} t + \alpha) \tag{12}$$

Hence the angular frequency of the oscillations is $\omega = \sqrt{g/l}$ rad/s and the frequency in cycles per second is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \quad Hz \tag{13}$$

Part (b)

The equation of motion is of the form $x(t) = B\cos(\omega t + \alpha)$. We assume *up* to be the positive and *down* to be the negative direction. At $t = 0$, $x = -B$ thus

$$x(0) = -B = B\cos(\sqrt{\frac{g}{l}} \times 0 + \alpha)$$

$$\alpha = \cos^{-1}(-1) = \pi$$

The velocity of the mass is

$$\dot{x}(t) = -B\sqrt{\frac{g}{l}}\sin(\sqrt{\frac{g}{l}} t + \pi) = B\sqrt{\frac{g}{l}}\sin(\sqrt{\frac{g}{l}} t) \tag{14}$$

The plot will look as shown in Fig. 2. Amplitude of velocity is $V_{max} = B\sqrt{g/l}$.

Solution 1.5: A damped oscillating spring

Mass of the object is $m = 0.2$ kg and the spring constant of the suspending spring is $k = 80$ N/m. The resistive force providing the damping force has the value of $-bv$, where v is velocity in m/s.

Part (a)

Let the oscillations of the spring be along the x axis. The spring force and damping force acting on the mass are:

$$F_{restoring} = -kx$$

$$F_{damping} = -bv = -b\dot{x}$$

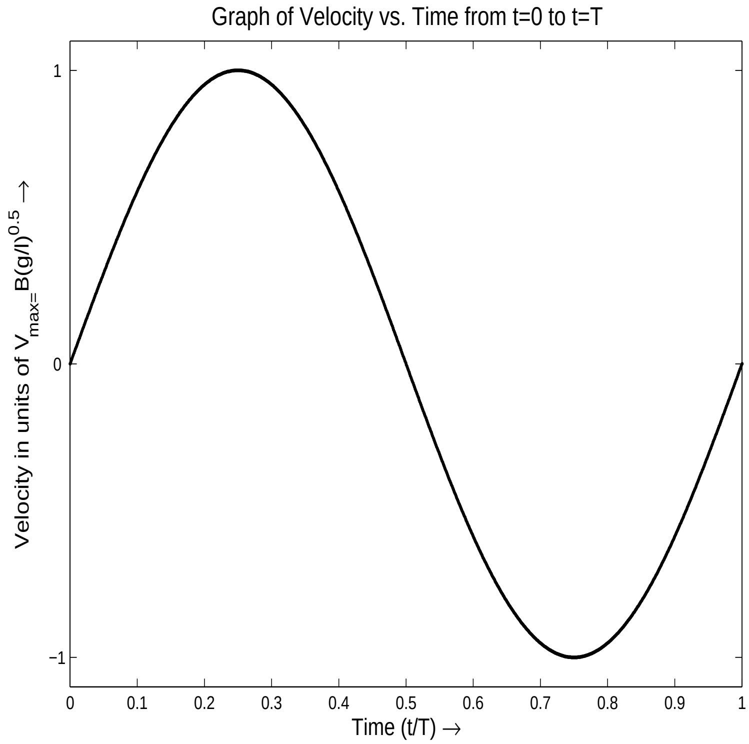

FIG. 2: 1.4(b) Graph of velocity versus time

Newton's 2^{nd} law:

$$m\ddot{x} = F_{net} = F_{restoring} + F_{damping} = -kx - b\dot{x}$$

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$$

Hence the differential equation describing the motion of the mass is:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \tag{15}$$

or

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0 \tag{16}$$

where

$$\gamma = \frac{b}{m} \qquad \omega_0^2 = \frac{k}{m}$$

Part (b)

We are given that the damped frequency is $\omega = 0.995\omega_0$. Now using the Eq. 3-34 from French, the value of the damped frequency in terms of the undamped frequency and the damping parameter is:

$$\begin{aligned}
\omega^2 &= \omega_0^2 - \frac{\gamma^2}{4} \\
&\Downarrow \\
(0.995\omega_0)^2 = 0.99{\omega_0}^2 &= \omega_0^2 - \frac{\gamma^2}{4} \\
&\Downarrow \\
\frac{\gamma^2}{4} = \frac{b^2}{4m^2} &= 0.01\omega_0^2 = 0.01\frac{k}{m} \\
&\Downarrow \\
b &= \sqrt{0.04km} = 0.2\sqrt{km}
\end{aligned}$$

substituting the values given in the problem, we find $b = 0.8$ Ns/m =0.8 kg/s .

Part (c)

$$\begin{aligned}
\omega_0 &= \sqrt{\frac{k}{m}} = 20\ rad/s \\
\gamma &= \frac{b}{m} = 4\ rad/s \\
Q &= \frac{\omega_0}{\gamma} = 5
\end{aligned} \tag{17}$$

Four complete cycles imply that the time $t = 8\pi/\omega$. Eq. 3-35 (French) gives us the envelope for the damped oscillatory motion as a function of time

$$\begin{aligned}
A(t) &= A_0 \exp(-\frac{\gamma t}{2}) = A_0 \exp(-\frac{4 \cdot 4\pi}{0.995 \cdot 20}) \\
&= A_0 \exp(-0.804\pi) \\
\frac{A(t)}{A_0} &= \exp(-0.804\pi) = 0.08
\end{aligned} \tag{18}$$

The factor by which the amplitude is reduced after four complete cycles is 0.08.

Part (d)

The equation defining the decay of energy of the system is:

$$E(t) = E_0 e^{-\gamma t}$$

substituting values from above, we get

$$\frac{E(t)}{E_0} = \exp(-\gamma t) = \exp(-1.608\pi) = 0.0064 \tag{19}$$

The factor by which the energy is reduced after four complete cycles is 6.4×10^{-3}; this is the square of the ratio of the amplitudes.

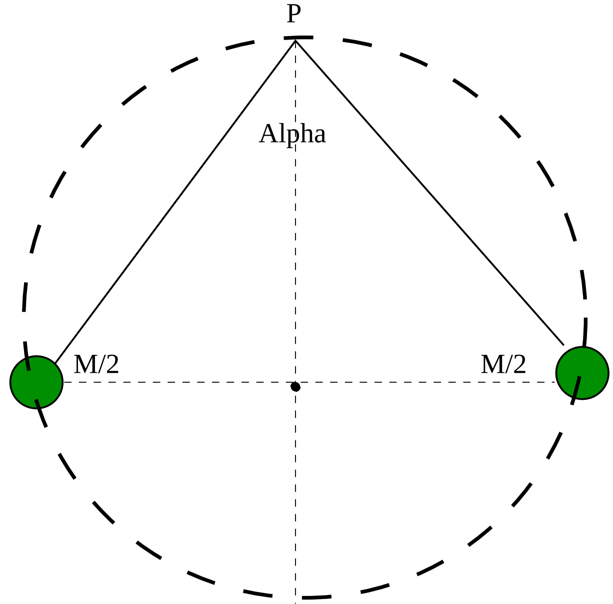

FIG. 3: 1.8 Simple Two Mass Pendulum

Solution 1.6: A physical pendulum

Part (a)

To solve this problem, we first consider the simpler case of a two mass rigid pendulum, both of whose masses are equidistant from the pivot point at P. All three points lie on a circle of diamater D and subtend an angle α at the pivot, as shown in Fig. 3. In this system let the distance of each mass from the pivot point be l. The moment of inertia of the two masses together is $I_p = Ml^2/2 + Ml^2/2 = Ml^2$. At equilibrium the position of each mass is $l\cos(\frac{1}{2}\alpha) = l^2/D$ below P. The gravitational potential energy of the system, after being displaced over a small angle θ is $U \approx Mg\frac{l^2}{D}\frac{\theta^2}{2}$

$$
\begin{aligned}
E &\approx \frac{1}{2}Ml^2\dot{\theta}^2 + \frac{1}{2}g\frac{Ml^2}{D}\theta^2 \\
\frac{dE}{dt} &= Ml^2\dot{\theta}\ddot{\theta} + Mg\frac{l^2}{D}\theta\dot{\theta} = 0 \\
0 &= \ddot{\theta} + \frac{g}{D}\theta \\
&\Downarrow \\
T &= 2\pi\sqrt{\frac{D}{g}}
\end{aligned}
$$

Hence the period is independent of the mass M and angle α. It only depends on the diameter D of the circle.

So now considering the circular arc system whose period we have to calculate, we see that we can see it as a collection of many such two-mass pendulums. Since the period of all those pendulums is the same $T = 2\pi\sqrt{\frac{D}{g}}$, the period of the arc is also

$$T = 2\pi\sqrt{\frac{D}{g}} = 2\pi\sqrt{\frac{2R}{g}} \tag{20}$$

Part (b)

The period of the oscillations is independent of the length of the arc and the 120^o angle. Hence when we complete the arc to form the hoop, the period of the hoop is same as the period of the small angle oscillations of the arc.

Solution 1.7: Damped oscillator and initial conditions

Part (a)

The solution for the case of critical damping ($\gamma/2 = \omega_0$) is of the form $s = (A + Bt)e^{-\gamma t/2}$. We know that $s(t = 0) = 0$ and $\dot{s}(t = 0) = v_0$. So

$$\begin{aligned} s(0) &= Ae^0 = 0 \\ \Rightarrow \quad A &= 0 \\ \\ \dot{s}(0) &= Be^{-\gamma t/2} - \frac{\gamma}{2}(A + Bt)e^{-\gamma t/2} = -\frac{\gamma}{2}A + B = v_0 \\ \Rightarrow \quad B &= v_0 \end{aligned}$$

Hence the time evolution of the displacement of the pen is

$$s(t) = v_0 te^{-\gamma t/2} = v_0 te^{-\omega_0 t} \tag{21}$$

$s(t)$ does *not* change sign before it settles to its equilibrium position as $s = 0$.

A plot of the evolution of a critically damped system is shown in Fig. 4.

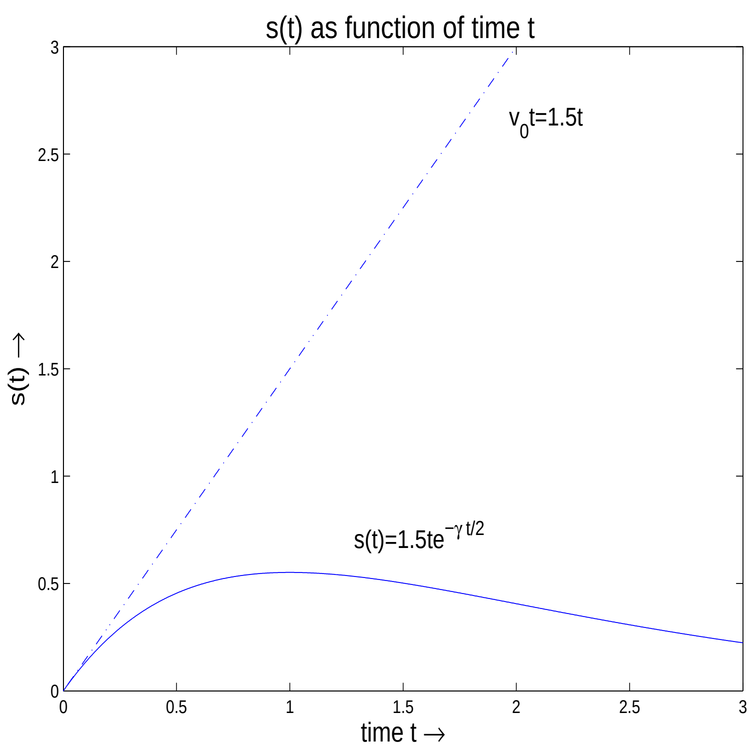

FIG. 4: 1.9(a) Plot of s(t) in the critically damped system

Part (b)

The solution describing the evolution of an overdamped system is

$$s = A_1 e^{-(\gamma/2+\beta)t} + A_2 e^{-(\gamma/2-\beta)t}$$

Now

$$s(0) = A_1 + A_2 = s_0$$

$$\begin{aligned}
\dot{s}(0) &= -A_1(\frac{\gamma}{2}+\beta)e^{-(\gamma/2+\beta)t} - A_2(\frac{\gamma}{2}-\beta)e^{-(\gamma/2-\beta)t} \\
0 &= -A_1(\frac{\gamma}{2}+\beta) - A_2(\frac{\gamma}{2}-\beta) \\
0 &= (A_2 - s_0)(\frac{\gamma}{2}+\beta) - A_2(\frac{\gamma}{2}-\beta) \\
2A_2\beta &= s_0(\frac{\gamma}{2}+\beta) \\
\Rightarrow \quad A_2 &= s_0\frac{1}{2\beta}(\frac{\gamma}{2}+\beta) \\
\Rightarrow \quad A_1 &= s_0[1-\frac{1}{2\beta}(\frac{\gamma}{2}+\beta)]
\end{aligned}$$

Equation of motion is

$$s = s_0[1-\frac{1}{2\beta}(\frac{\gamma}{2}+\beta)]e^{-(\gamma/2+\beta)t} + s_0\frac{1}{2\beta}(\frac{\gamma}{2}+\beta)e^{-(\gamma/2-\beta)t} \tag{22}$$

where

$$\beta = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

Part (c)

Plot of s(t) for the given values is shown in Fig. 5

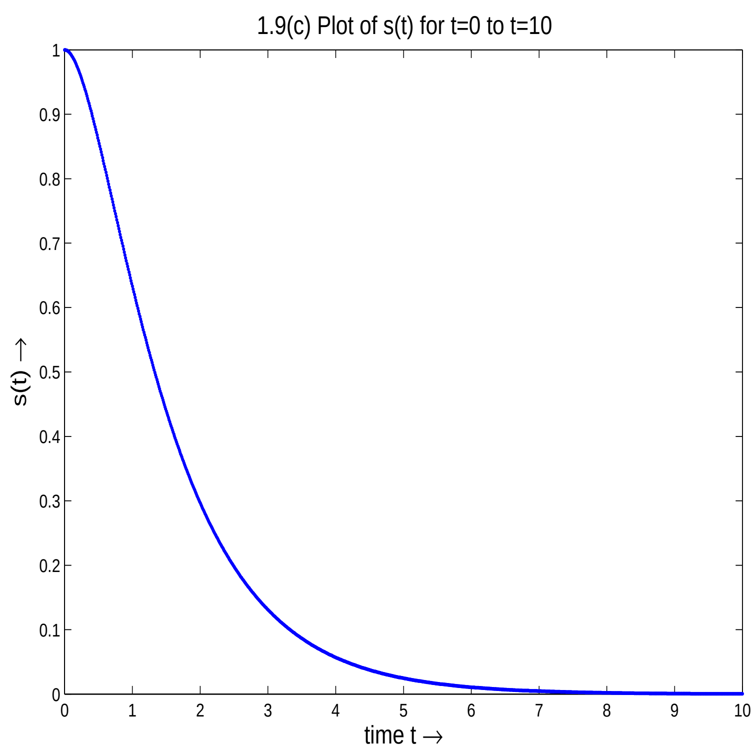

FIG. 5: 1.9(c) Plot of s(t) in the overdamped system