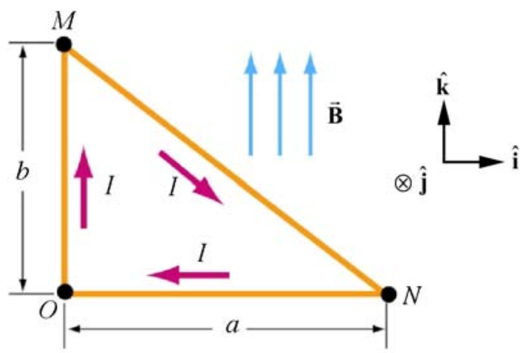
Magnetic Dipoles
Challenge Problem Solutions

Problem 1:

Circle the correct answer.

Consider a triangular loop of wire with sides a and b. The loop carries a current I in the direction shown, and is placed in a uniform magnetic field that has magnitude B and points in the same direction as the current in side OM of the loop.



At the moment shown in the figure the torque on the current loop

a) points in the $-\hat{\mathbf{i}}$-direction and has magnitude $IabB/2$.

b) points in the $+\hat{\mathbf{i}}$-direction and has magnitude $IabB/2$.

c) points in the $-\hat{\mathbf{j}}$-direction and has magnitude $IabB/2$.

d) points in the $+\hat{\mathbf{j}}$-direction and has magnitude $IabB/2$.

e) points in the $-\hat{\mathbf{i}}$-direction and has magnitude $IabB$.

f) points in the $+\hat{\mathbf{i}}$-direction and has magnitude $IabB$.

g) points in the $-\hat{\mathbf{j}}$-direction and has magnitude $IabB$.

h) points in the $+\hat{\mathbf{j}}$-direction and has magnitude $IabB$.

i) None of the above.

Problem 1 Solution:

b. The magnetic dipole moment vector is $\vec{\mu} = Iab/2\,\hat{j}$. The torque on the current loop is then $\vec{\tau} = \vec{\mu}\times\vec{B} = (Iab/2)\,\hat{j}\times B\hat{k} = (IabB/2)\hat{i}$.

Problem 2:

A wire ring lying in the *xy*-plane with its center at the origin carries a counterclockwise current I. There is a uniform magnetic field $\vec{\mathbf{B}} = B\hat{\mathbf{i}}$ in the $+x$-direction. The magnetic moment vector $\vec{\boldsymbol{\mu}}$ is perpendicular to the plane of the loop and has magnitude $\mu = IA$ and the direction is given by right-hand-rule with respect to the direction of the current. What is the torque on the loop?

Problem 2 Solution: The torque on a current loop in a uniform field is given by

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}},$$

where $\mu = IA$ and the vector $\vec{\boldsymbol{\mu}}$ is perpendicular to the plane of the loop and right-handed with respect to the direction of current flow. The magnetic dipole moment is given by

$$\vec{\mu} = I\vec{\mathbf{A}} = I(\pi R^2 \hat{\mathbf{k}}) = \pi I R^2 \hat{\mathbf{k}}.$$

Therefore,

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}} = \left(\pi I R^2 \hat{\mathbf{k}}\right) \times \left(B\hat{\mathbf{i}}\right) = \pi I R^2 B \hat{\mathbf{j}}.$$

Instead of using the above formula, we can calculate the torque directly as follows. Choose a small section of the loop of length $ds = R d\theta$. Then the vector describing the current-carrying element is given by

$$I d\vec{\mathbf{s}} = IR d\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$$

The force $d\vec{\mathbf{F}}$ that acts on this current element is

$$\begin{aligned} d\vec{\mathbf{F}} &= I d\vec{\mathbf{s}} \times \vec{\mathbf{B}} \\ &= I\, R d\theta(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}) \times (B\hat{\mathbf{i}}) \\ &= -IRB\cos\theta\, d\theta\,\hat{\mathbf{k}} \end{aligned}$$

The force acting on the loop can be found by integrating the above expression.

$$\begin{aligned} \vec{\mathbf{F}} &= \oint d\vec{\mathbf{F}} = \int_0^{2\pi} (-IRB\cos\theta) d\theta\,\hat{\mathbf{k}} \\ &= -IRB\left[\sin\theta\right]_0^{2\pi} \hat{\mathbf{k}} = 0 \end{aligned}$$

We expect this because the magnetic field is uniform and the force ona current loop in a uniform magnetic field is zero. Therefore we can choose any point to calculate the torque about. Let $\vec{\mathbf{r}}$ be the vector from the center of the loop to the element $Id\vec{\mathbf{s}}$. That is, $\vec{\mathbf{r}} = R\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)$. The torque $d\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times d\vec{\mathbf{F}}$ acting on the current element is then

$$
\begin{aligned}
d\vec{\boldsymbol{\tau}} &= \vec{\mathbf{r}} \times d\vec{\mathbf{F}} \\
&= R\left(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right) \times \left(-IRBd\theta\cos\theta\hat{\mathbf{k}}\right) \\
&= -IR^2Bd\theta\cos\theta\left(\sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}}\right)
\end{aligned}
$$

Integrate $d\vec{\boldsymbol{\tau}}$ over the loop to find the total torque $\vec{\boldsymbol{\tau}}$.

$$
\begin{aligned}
\vec{\boldsymbol{\tau}} &= \int d\vec{\boldsymbol{\tau}} \\
&= \int_0^{2\pi} -IR^2Bd\theta\cos\theta\left(\sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}}\right) \\
&= -IR^2B\int_0^{2\pi} (\sin\theta\cos\theta\hat{\mathbf{i}} - \cos^2\theta\hat{\mathbf{j}})d\theta \\
&= \pi IR^2B\hat{\mathbf{j}}
\end{aligned}
$$

This agrees with our result above.

Problem 3:

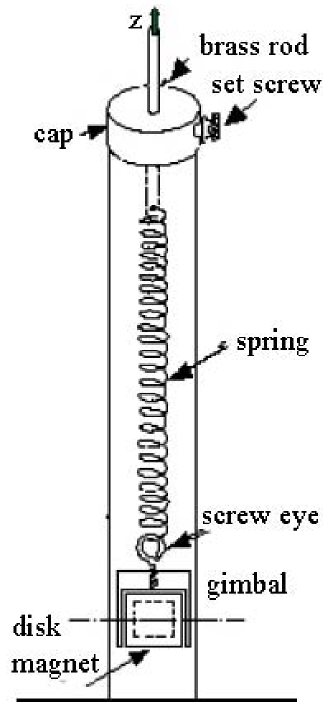
1. Force on a Dipole in the Helmholtz Apparatus

The magnetic field along the axis of a coil is given by

$$B_z(z) = \frac{N\,\mu_0\,I\,R^2}{2}\frac{1}{(z^2+R^2)^{3/2}}$$

where z is measured from the center of the coil.

Consider a disk magnet (a dipole) suspended on a spring, which we will use to observe forces on dipoles due to different magnetic field configurations.



(a) Assuming we energize only the top coil (current running counter-clockwise in the coil, creating the field quoted above), and assuming that the dipole is always well aligned with the field and on axis, what is the force on the dipole as a function of position? (HINT: In this situation $F_z = \mu_z \; dB_z/dz$)

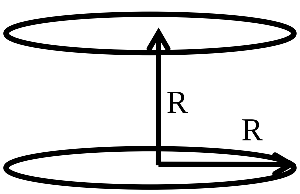
(b) The disk magnet (together with its support) has mass *m*, the spring has spring constant *k* and the magnet has magnetic moment μ. With the current on, we lift the brass rod until the disk magnet is sitting a distance z_0 above the top of the coil. Now the current is turned off. Does the disk magnet move up or down? Find the displacement Δz to the new equilibrium position of the disk magnet.

(c) At what height(s) is the force on the dipole the largest?

(d) What is the force where the field is the largest?

(e) Our coils have a radius R = 7 cm and N = 168 turns, and the experiment is done with I = 1 A in the coil. The spring constant k ~ 1 N/m, and μ ~ 0.5 A m^2. The mass m ~ 5 g is in the shape of a cylinder ~ 0.5 cm in diameter and ~ 1 cm long. If we place the magnet at the location where the spring is stretched the furthest when the field is on, at about what height will the magnet sit after the field is turned off?

2. Motion of a Dipole in a Helmholtz Field



In Part I of this experiment we will place the disk magnet (a dipole with moment μ) at the center of the Helmholtz Apparatus (in Helmholtz mode). We will start with the disk magnet aligned along the x-axis (perpendicular to the central z-axis of the coils), and then energize the coils with a current of 1 A.

Recall that a Helmholtz coil consists of two coils of radius R and N turns each, separated by a distance R, as pictured above. The field from each coil is given at the beginning of the previous problem.

(a) The disk magnet will experience a torque. Will it also experience a force? Explain why or why not.

Problem 3 Solutions:
1. Force on a Dipole in the Helmholtz Apparatus

(a)

$$F_z = \mu_z \, dB_z / dz = \mu_z \, z \frac{N \mu_0 I R^2}{2} \frac{d}{dz}(z^2 + R^2)^{-3/2}$$

$$= \boxed{\mu_z \frac{N \mu_0 I R^2}{2} \left(\frac{-3z}{(z^2 + R^2)^{5/2}} \right)}$$

That is, towards the coil center

(b) When the current is on, the downward the gravitational force and the downward magnetic force are balanced by the spring force which stretches the spring by an amount $\Delta l = l_i - l_0$. When the equilibrium has been reached, Newton's Second Law becomes

$$F_{\text{spring}} + mg - k(l_i - l_0) = 0$$

The magnetic force is

$$F_B\left(z = z_0\right) = \mu \frac{N \mu_0 I R^2}{2} \left(\frac{-3z_0}{(z_0^2 + R^2)^{5/2}} \right)$$

Therefore

$$\mu \frac{N \mu_0 I R^2}{2} \left(\frac{-3z_0}{(z_0^2 + R^2)^{5/2}} \right) + mg - k(l_i - l_0) = 0 \qquad (0.1)$$

When the current is off, and a new equilibrium position has been attained, the object moves upward and the spring is now stretched an amount $l_f - l_0$, therefore Newton's Second Law becomes

$$mg - k(l_f - l_0) = 0 \, .$$

Solving for $mg = k(l_f - l_0)$ and substituting into Eq. (0.1) yields

$$\mu \frac{N \mu_0 I R^2}{2} \left(\frac{-3z_0}{(z_0^2 + R^2)^{5/2}} \right) + k(l_f - l_0) - k(l_i - l_0) = 0$$

Thus

$$\mu \frac{N \mu_0 I R^2}{2} \left(\frac{-3z_0}{(z_0^2 + R^2)^{5/2}} \right) = k(l_i - l_f)$$

The spring moves upwards a distance

$$l_i - l_f = \mu \frac{N\,\mu_0\, I\, R^2}{2k}\left(\frac{-3z_0}{(z_0^2+R^2)^{5/2}}\right) = \frac{F_{mag}}{k}. \tag{0.2}$$

(c) To find this we just maximize the force function (find zeros of its derivative):

$$\frac{dF_B}{dz} = \frac{d}{dz}\left[\mu \frac{N\,\mu_0\, I\, R^2}{2}\left(\frac{-3z}{(z^2+R^2)^{5/2}}\right)\right] = 0$$

$$\Rightarrow \qquad 0 = \frac{d}{dz}\left(z\left(z^2+R^2\right)^{-5/2}\right) = \left(z^2+R^2\right)^{-5/2} - 5z^2\left(z^2+R^2\right)^{-7/2}$$

$$\Rightarrow \qquad 0 = z^2 + R^2 - 5z^2 \qquad \Rightarrow \qquad \boxed{z = \pm\frac{R}{2}}$$

(d) Where the field is largest the force must be zero. You can either think "That's where an aligned dipole would like to be" or "Maximum field means derivative of field is zero means no force."

(e) To make the spring stretch the furthest we must be at the location of the largest force, a distance z_0=R/2 above the coil (from c). From above the spring will relax by:

$$-\Delta z = \mu \frac{N\,\mu_0\, I\, R^2}{2k}\left(\frac{3z_0}{(z_0^2+R^2)^{5/2}}\right) = \mu \frac{N\,\mu_0\, I}{2kR^2}\left(\frac{3\cdot\frac{1}{2}}{(\frac{1}{4}+1)^{5/2}}\right)$$

$$\approx \left(0.5\text{ A m}^2\right)\frac{(168)\left(4\pi\times10^{-7}\text{ T m/A}\right)(1\text{ A})}{2(1\text{ N/m})(7\text{ cm})^2}\left(\frac{3\cdot 32}{2\cdot 5^{5/2}}\right) = 9.2\text{ mm}$$

So the final position is a distance $3.5\text{ cm} + 9.2\text{ mm} \approx 4.4\text{ cm}$ above the center of the coil

2. Motion of a Dipole in a Helmholtz Field

The torque will create an angular acceleration:

$$\tau = \mu B \sin(\theta) = I\alpha = I\ddot{\theta},$$

which will lead to angular motion. This is a pretty ugly differential equation to solve, but we can make a stab at it in two different types of approximations. The first is to assume a constant torque, probably not the maximum torque, but maybe half of the maximum. Then we have:

$$\Delta\theta = \tfrac{1}{2}\alpha t^2 = \tfrac{1}{2}\frac{\tfrac{1}{2}\mu B}{I}t^2 .$$

We can calculate the field at the center:

$$B_{\text{Helmholtz}} = 2\cdot\frac{N\,\mu_0\,I\,R^2}{2}\frac{1}{\left(\left(R/2\right)^2 + R^2\right)^{3/2}} = \frac{N\,\mu_0\,I}{R\left(\left(1/2\right)^2+1\right)^{3/2}}$$

$$\approx \frac{(168)\left(4\pi\times10^{-7}\text{ T m A}^{-1}\right)(1\text{ A})8}{(7\text{ cm})5^{3/2}} = 2.2\text{ mT} = 22\text{ Gauss}$$

The moment of inertia of a cylinder is $I = \tfrac{1}{2}mR^2 \approx \tfrac{1}{2}(5\text{ g})(0.25\text{ cm})^2 = 1.6\times10^{-8}\text{ kg m}^2$

$$t \approx \sqrt{\frac{4I\Delta\theta}{\mu B}} \approx \sqrt{\frac{4\left(1.6\times10^{-8}\text{ kg m}^2\right)(\pi/2)}{\left(0.5\text{ A m}^2\right)\left(2.2\times10^{-3}\text{ T}\right)}} \approx 9\text{ ms}$$

Another way of estimating the time is by approximating the motion as simple harmonic (which it definitely isn't because Δθ is so big). Then the time is a quarter of a period, which is

$$t = \frac{T}{4} = \frac{2\pi}{4\omega} = \frac{\pi}{2}\sqrt{\frac{I}{\mu B}} \approx \frac{\pi}{2}\sqrt{\frac{\left(1.6\times10^{-8}\text{ kg m}^2\right)}{\left(0.5\text{ A m}^2\right)\left(2.2\times10^{-3}\text{ T}\right)}} = 6\text{ ms}$$

Note that this should really be a lower bound because it is for small oscillations.

Once you get larger oscillations the period starts increasing – the real period for a non-simple harmonic oscillator with amplitude θ is

$$T_{\pi/2} = T_{SHM}\cdot\left(1+\left(\frac{1!!}{2!!}\right)^2\sin^2\left(\frac{\theta}{4}\right)+\left(\frac{3!!}{4!!}\right)^2\sin^4\left(\frac{\theta}{4}\right)+\ldots\right) \approx T_{SHM}\cdot 1.18$$

Meaning the time to rotate will be about 7 ms, so our approximations were both pretty good. In any case, it will be too fast for us to see the motion, instead we'll just see the end result.