Module 28:
Maxwell's Equations and Electromagnetic Waves

Module 28: Outline

Maxwell's Equations

Electromagnetic Radiation

Plane Waves

Standing Waves

Energy Flow

Maxwell’s Equations

Maxwell's Equations

1. $\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0}$ (with $Q_{in} \rightarrow 0$) (Gauss's Law)

2. $\oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$ (Magnetic Gauss's Law)

3. $\oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt}$ (Faraday's Law)

4. $\oint_C \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$ (with $\mu_0 I_{enc} \rightarrow 0$) (Ampere-Maxwell Law)

What about free space (no charge or current)?

Electromagnetism Review

- **E fields are associated with:**

  (1) electric charges (Gauss's Law)

  (2) time changing B fields (Faraday's Law)

- **B fields are associated with**

  (3a) moving electric charges (Ampere-Maxwell Law)

  (3b) time changing E fields (Maxwell's Addition (Ampere-Maxwell Law)

- **Conservation of magnetic flux**

  (4) No magnetic charge (Gauss's Law for Magnetism)

Electromagnetism Review

- **Conservation of Charge**

$$\oiint_{\substack{closed\\surface}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = -\frac{d}{dt} \iiint_{\substack{volume\\enclosed}} \rho \, dV$$

- **E and B fields exert forces on (moving) electric charges**

$$\vec{\mathbf{F}}_q = q(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}})$$

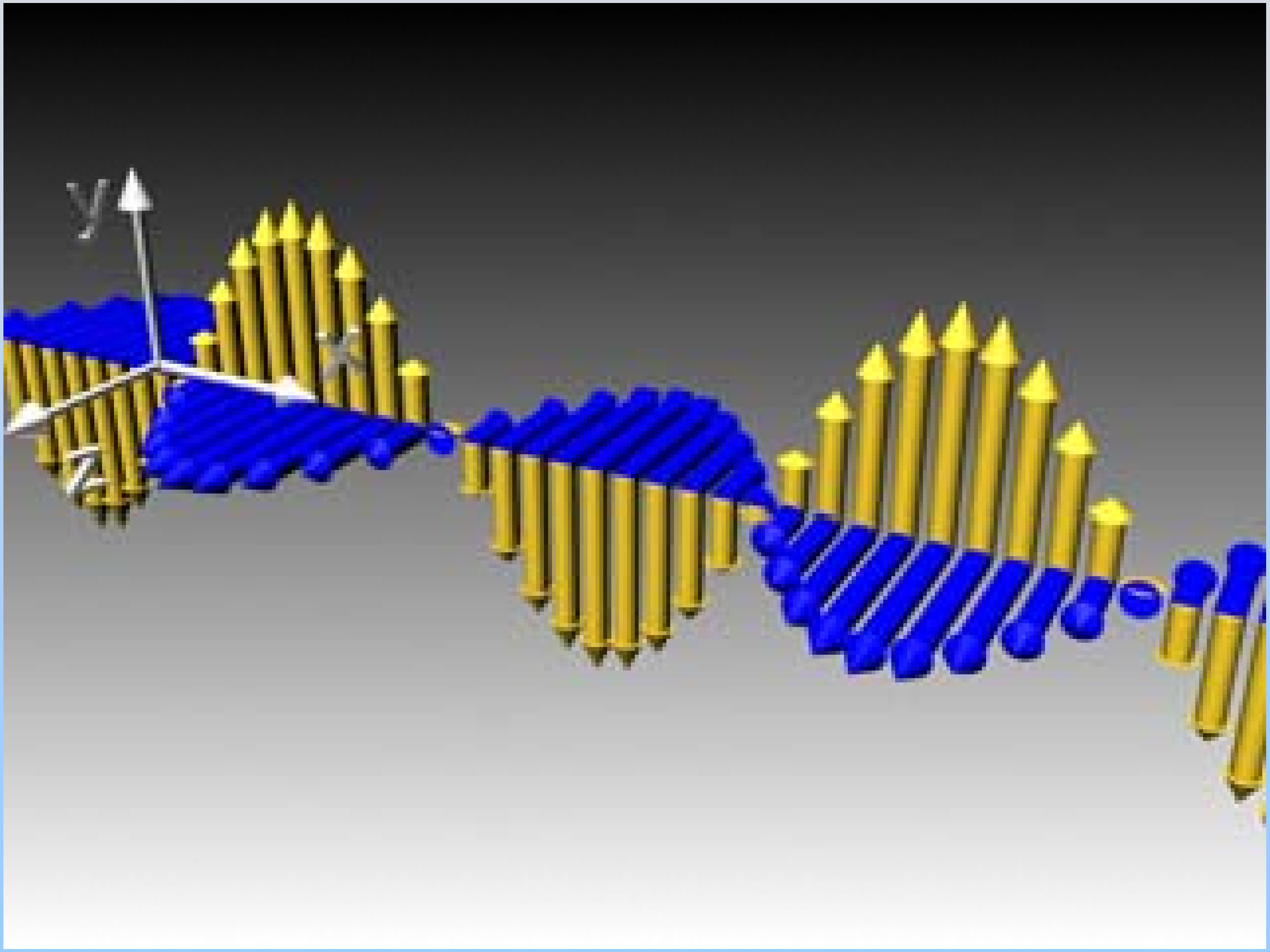
- **Energy stored in Electric and Magnetic Fields**

$$U_E = \iiint_{all\ space} u_E \, dV = \iiint_{all\ space} \frac{\varepsilon_0}{2} E^2 \, dV$$

$$U_B = \iiint_{all\ space} u_B \, dV = \iiint_{all\ space} \frac{1}{2\mu_0} B^2 \, dV$$

Electromagnetic Waves

Electromagnetic Radiation: Plane Waves



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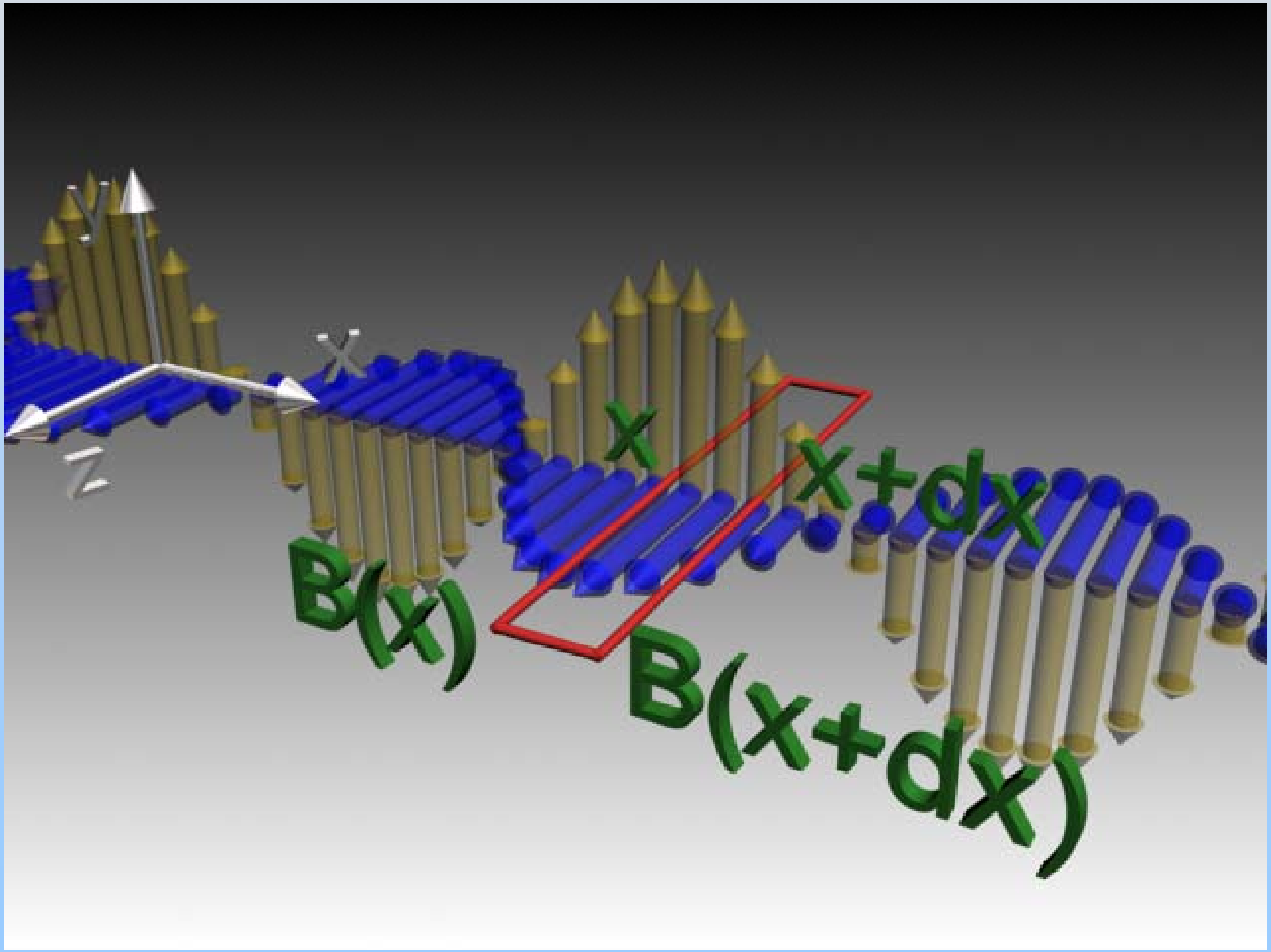
How Do Maxwell's Equations Lead to EM Waves?

Wave Equation

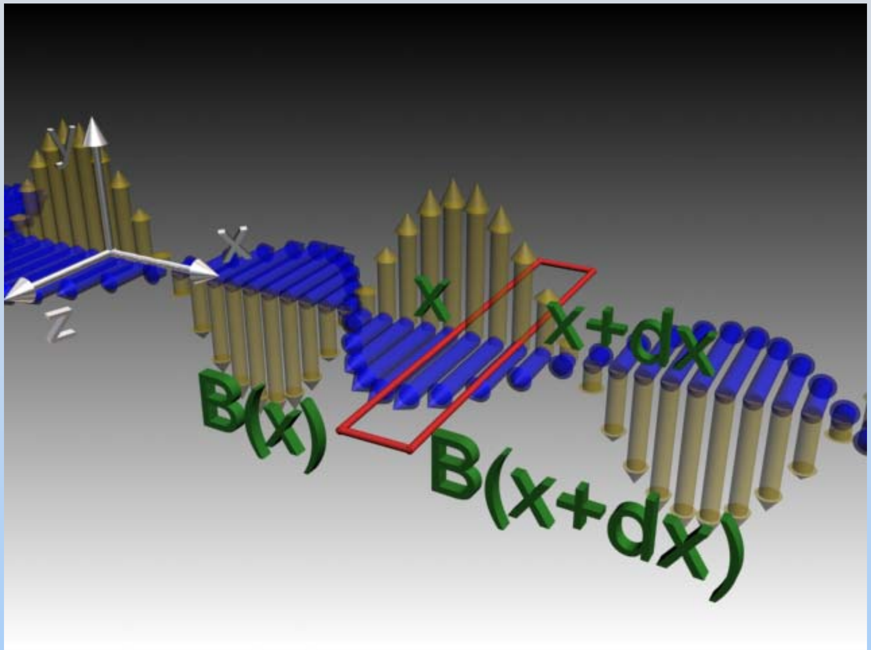
Start with Ampere-Maxwell Eq: $\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$



Wave Equation

Start with Ampere-Maxwell Eq: $\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$

Apply it to red rectangle:



$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B_z(x,t)l - B_z(x+dx,t)l$$

$$\mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \mu_0 \varepsilon_0 \left(l\, dx \frac{\partial E_y}{\partial t} \right)$$

$$-\frac{B_z(x+dx,t) - B_z(x,t)}{dx} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

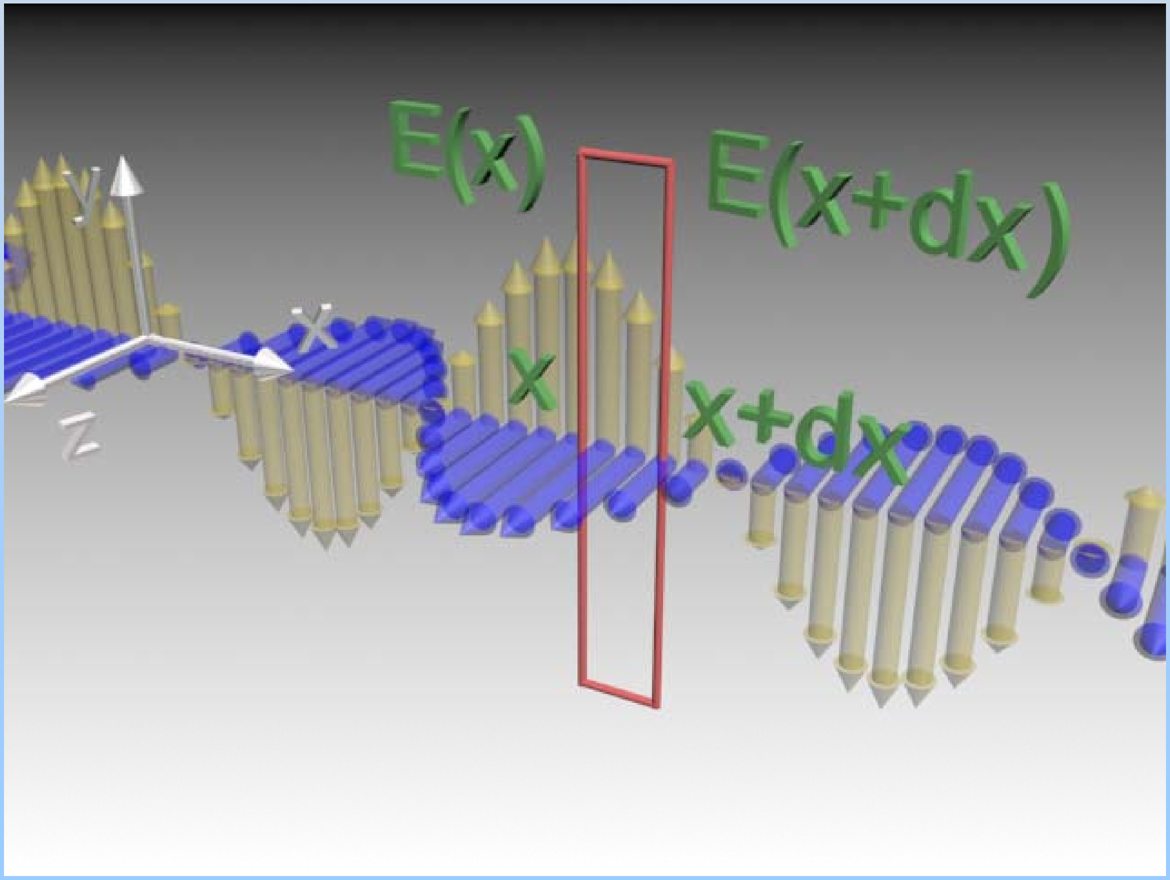
So in the limit that *dx* is very small:

$$\boxed{-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}}$$

Problem: Wave Equation

Use Faraday's Law: $$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

and apply it to red rectangle to find the partial differential equation

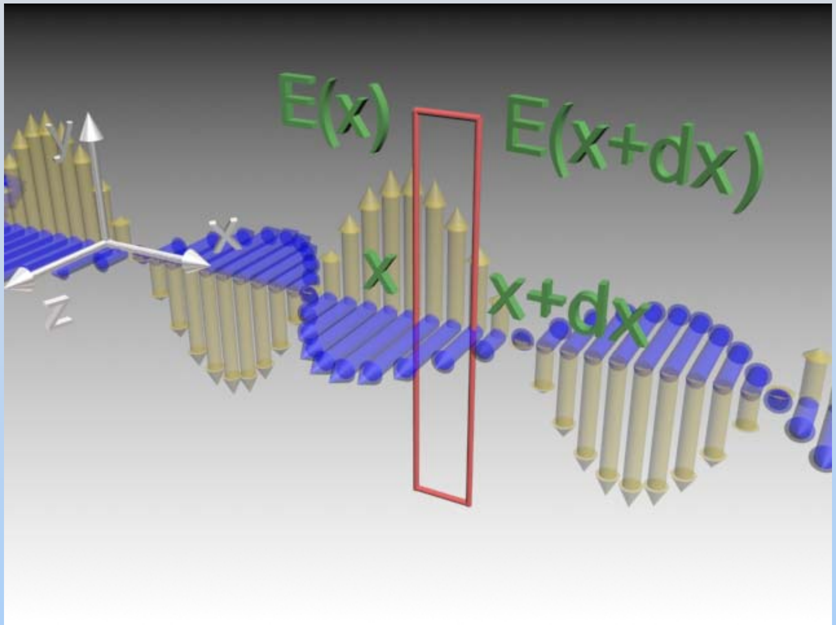


$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$$

Problem: Wave Equation

Use Faraday's Law: $$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

and apply it to red rectangle:



$$\oint_C \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = E_y(x+dx,t)l - E_y(x,t)l$$

$$-\frac{d}{dt}\int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = -l\,dx\frac{\partial B_z}{\partial t}$$

$$\frac{E_y(x+dx,t) - E_y(x,t)}{dx} = -\frac{\partial B_z}{\partial t}$$

So in the limit that *dx* is very small:

$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}}$$

1D Wave Equation for E

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad -\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Take *x*-derivative of 1st and use the 2nd equation

$$\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) = \underline{\frac{\partial^2 E_y}{\partial x^2}} = \frac{\partial}{\partial x}\left(-\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = \underline{\mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}}$$

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}}$$

1D Wave Equation for E

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

This is an equation for a wave. Let: $E_y = f\left(x - vt\right)$

$$\left.\begin{aligned} \frac{\partial^2 E_y}{\partial x^2} &= f''\left(x - vt\right) \\ \frac{\partial^2 E_y}{\partial t^2} &= v^2 f''\left(x - vt\right) \end{aligned}\right\} \quad v^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Problem: 1D Wave Eq. for B

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Take appropriate derivatives of the above equations and show that

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}}$$

Problem: 1D Wave Eq. for B

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Take *x*-derivative of 1st and use the 2nd equation

$$\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial t}\right) = \underline{\frac{\partial^2 B_z}{\partial t^2}} = \frac{\partial}{\partial t}\left(-\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial t}\right) = \underline{\frac{1}{\mu_0 \varepsilon_0}\frac{\partial^2 B_z}{\partial x^2}}$$

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}}$$

Electromagnetic Wave Equations

Both E & B travel like waves:

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$

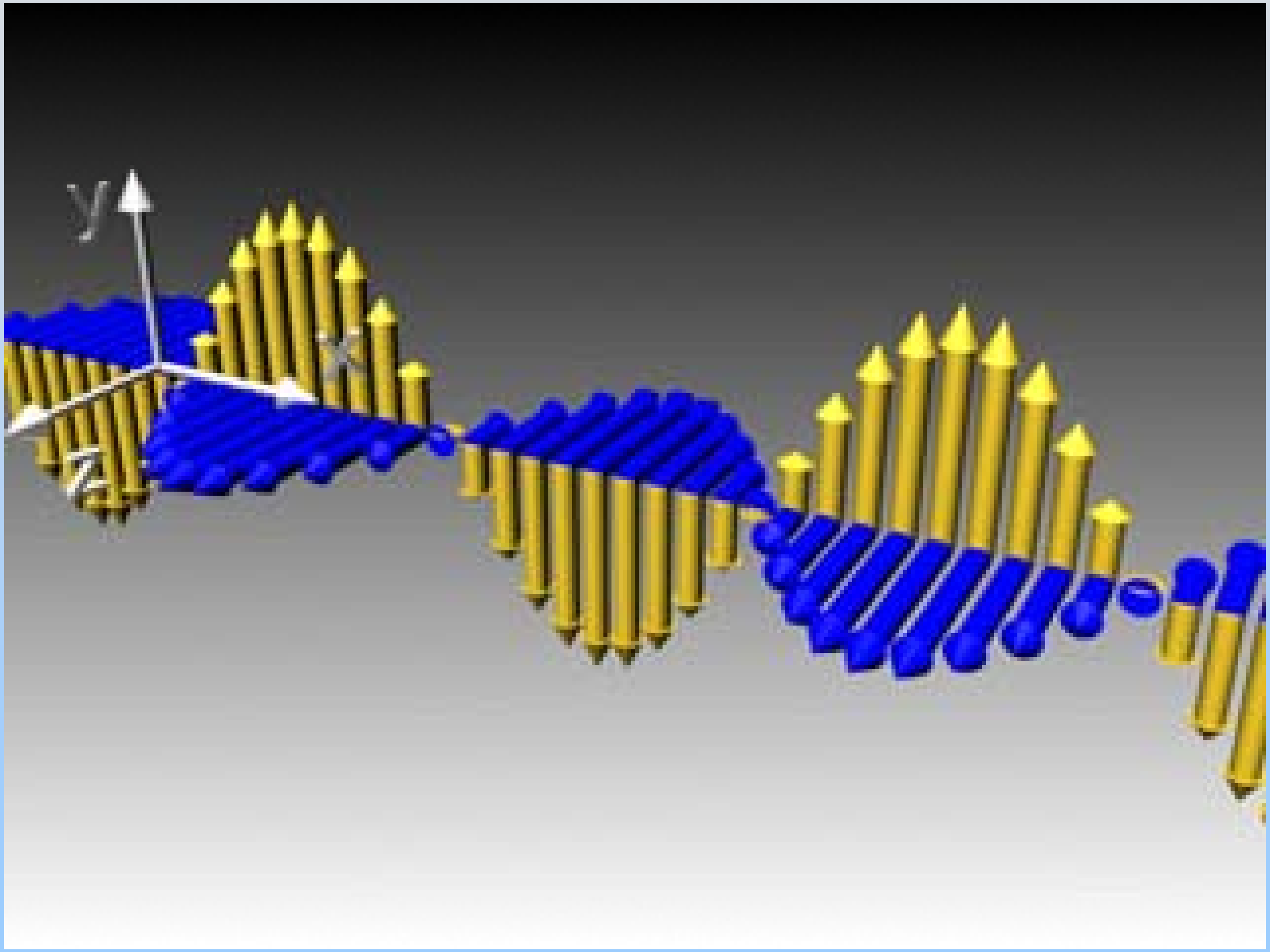
But there are strict relations between them:

$$\frac{\partial B_z}{\partial t} = - \frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = - \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Here, E_y and B_z are “the same,” traveling along x axis

Understanding Traveling Waves Solutions to Wave Equation

Electromagnetic Waves: Plane Waves



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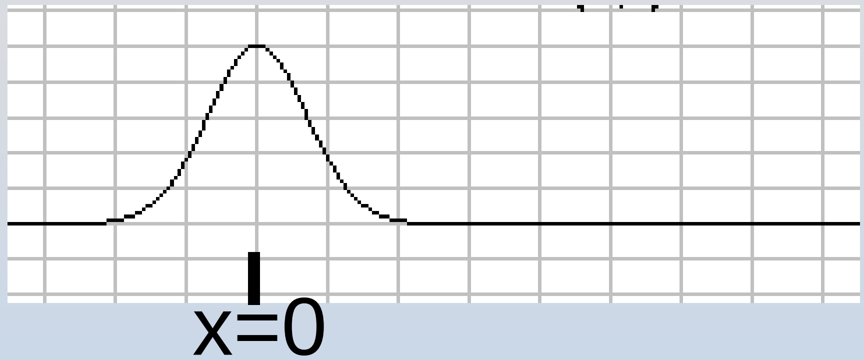
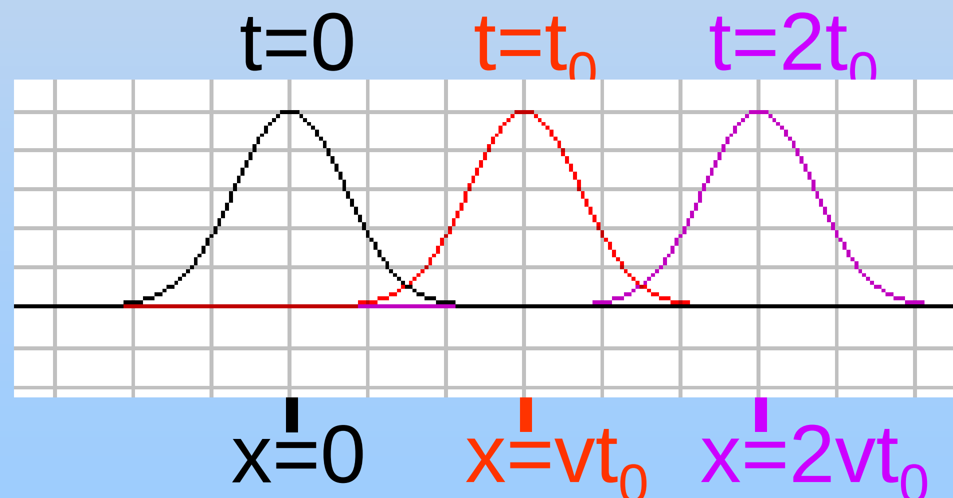
Traveling Waves

Consider f(x) =



What is g(x,t) = f(x-vt)?



f(x-vt) is traveling wave moving to the right!

Traveling Sine Wave

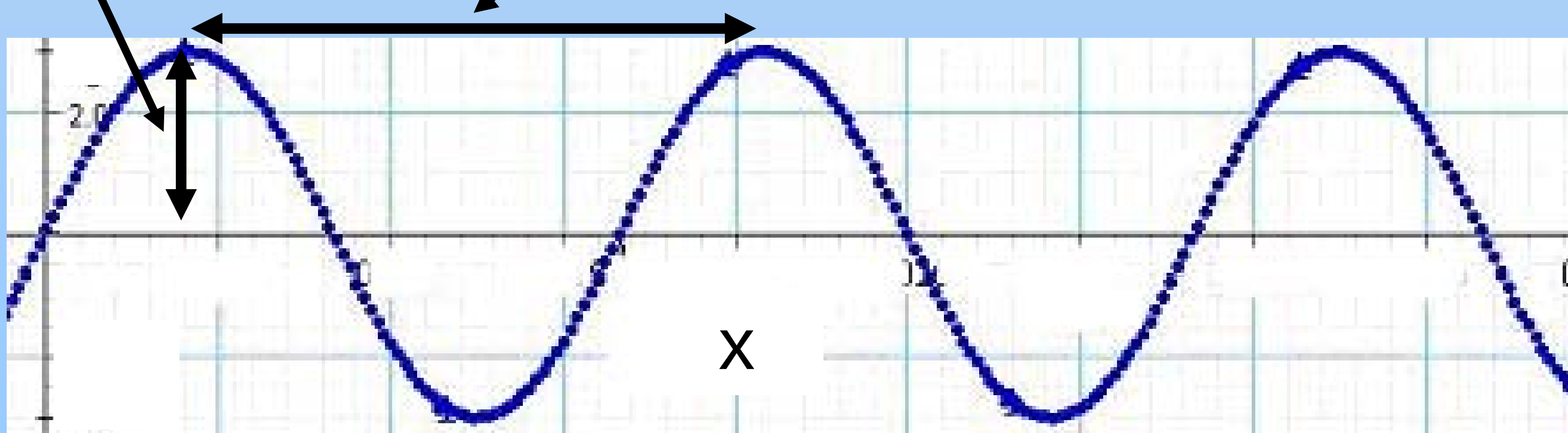
What is g(x,t) = f(x+vt)? Travels to left at velocity v

$$y = y_0 \sin(k(x+vt)) = y_0 \sin(kx+kvt)$$

Look at $t = 0$: $g(x,0) = y = y_0 \sin(kx)$:

Amplitude (y_0)

$$\text{wavelength} = \frac{2\pi}{\text{wave number}} \rightarrow \lambda = \frac{2\pi}{k}$$



Traveling Sine Wave

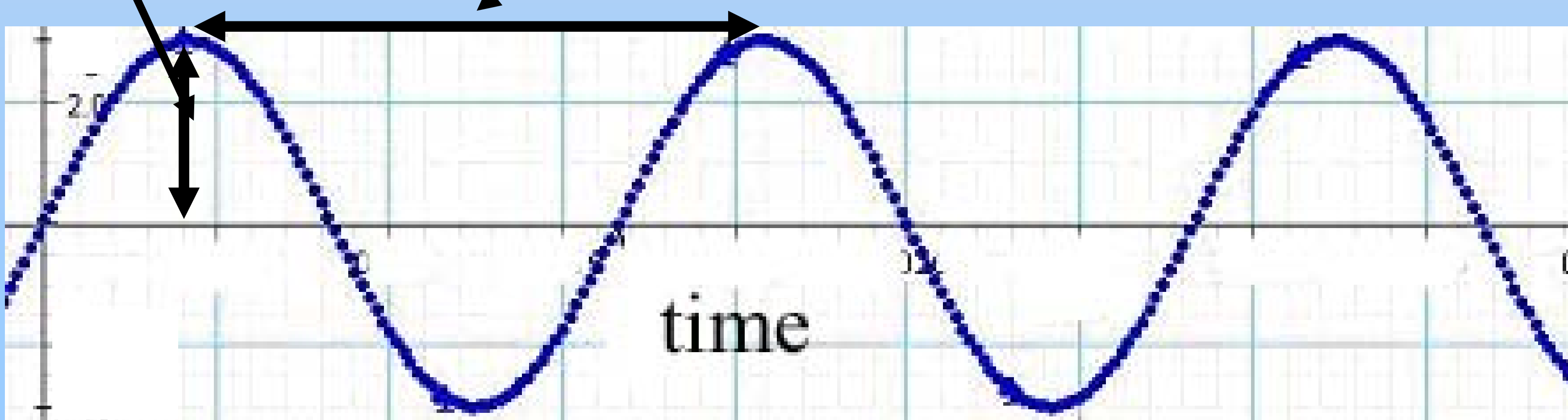
$$g(x,t) = y_0 \sin k\left(x + vt\right)$$

Look at x=0: $g(0,t) = y_0 \sin(kvt) \equiv y_0 \sin(\omega t)$

$$\text{period} = \frac{1}{\text{frequency}} \rightarrow T = \frac{1}{f}$$

Amplitude (y_0)

$$\text{period} = \frac{2\pi}{\text{angular frequency}} \rightarrow T = \frac{2\pi}{\omega}$$



Traveling Sine Wave

Wavelength: λ

Frequency : f

$$y = y_0 \sin(kx - \omega t)$$

Wave Number: $k = \dfrac{2\pi}{\lambda}$

Angular Frequency: $\omega = 2\pi f$

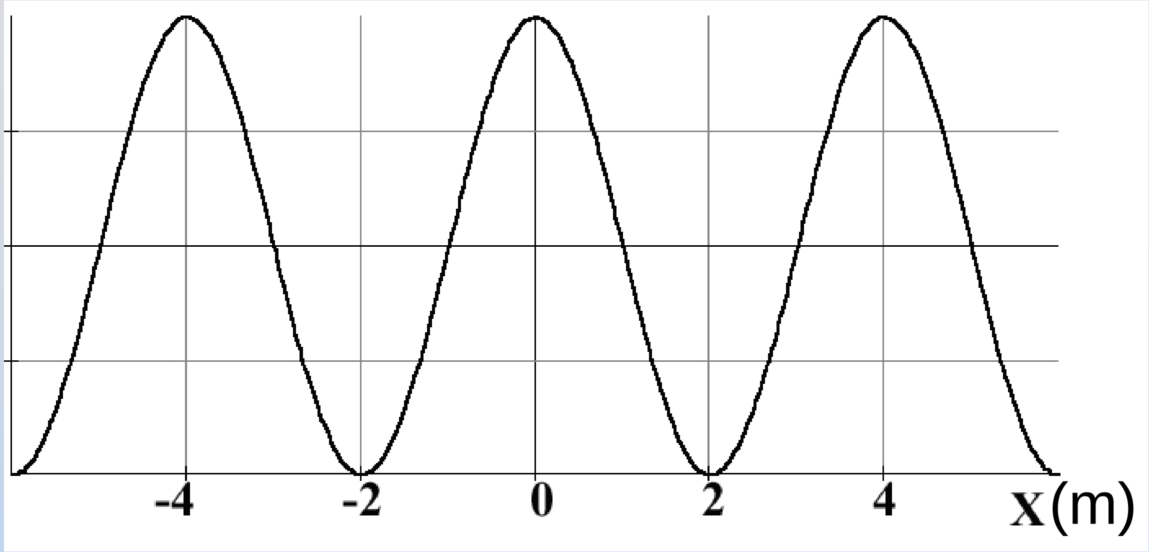
Period: $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

Speed of Propagation: $v = \dfrac{\omega}{k} = \lambda f$

Direction of Propagation: $+x$

Concept Question Question: Wave

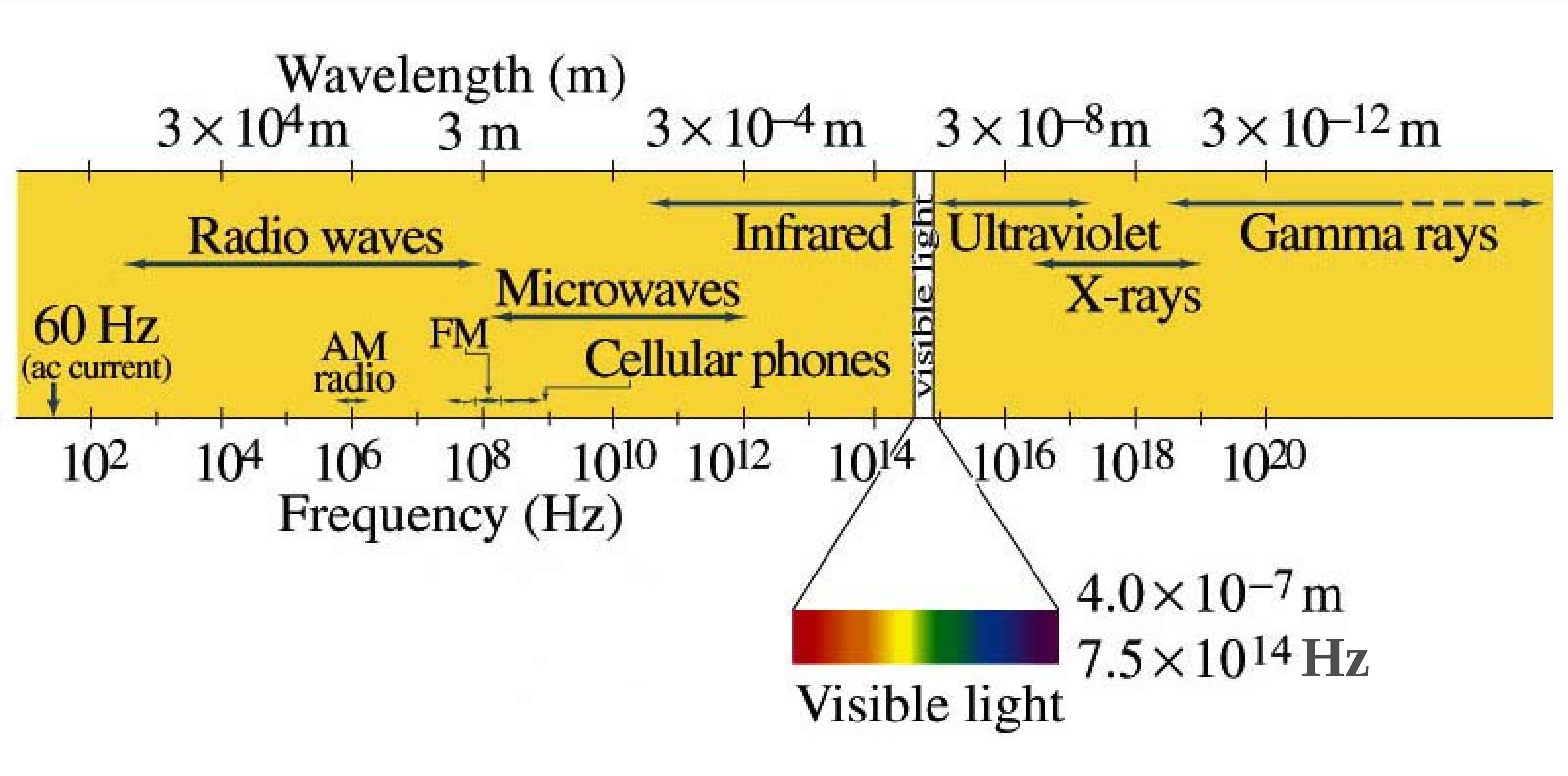
Concept Question: Wave



The graph shows a plot of the function $y = \cos(k\,x)$. The value of k is

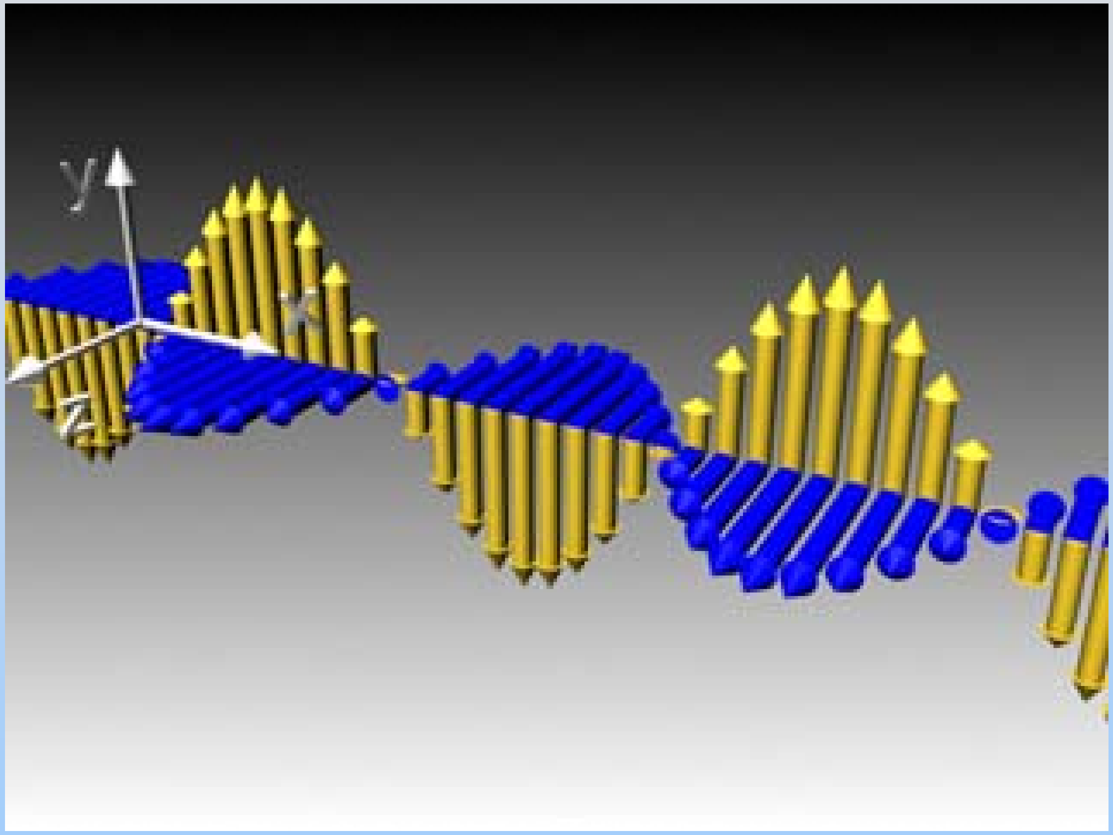
1. ½ m^{-1}
2. ¼ m^{-1}
3. π m^{-1}
4. $\pi/2$ m^{-1}
5. I don’t know

Electromagnetic Waves



Remember: $\lambda f = c$

Electromagnetic Waves: Plane Waves



Link to movie

Watch 2 Ways:

1) Sine wave traveling to right (+x)
2) Collection of out of phase oscillators (watch one position)

Don’t confuse vectors with heights – they are magnitudes of E (gold) and B (blue)

Direction of Propagation

$$\vec{\mathbf{E}} = \hat{\mathbf{E}} E_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t); \quad \vec{\mathbf{B}} = \hat{\mathbf{B}} B_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t)$$

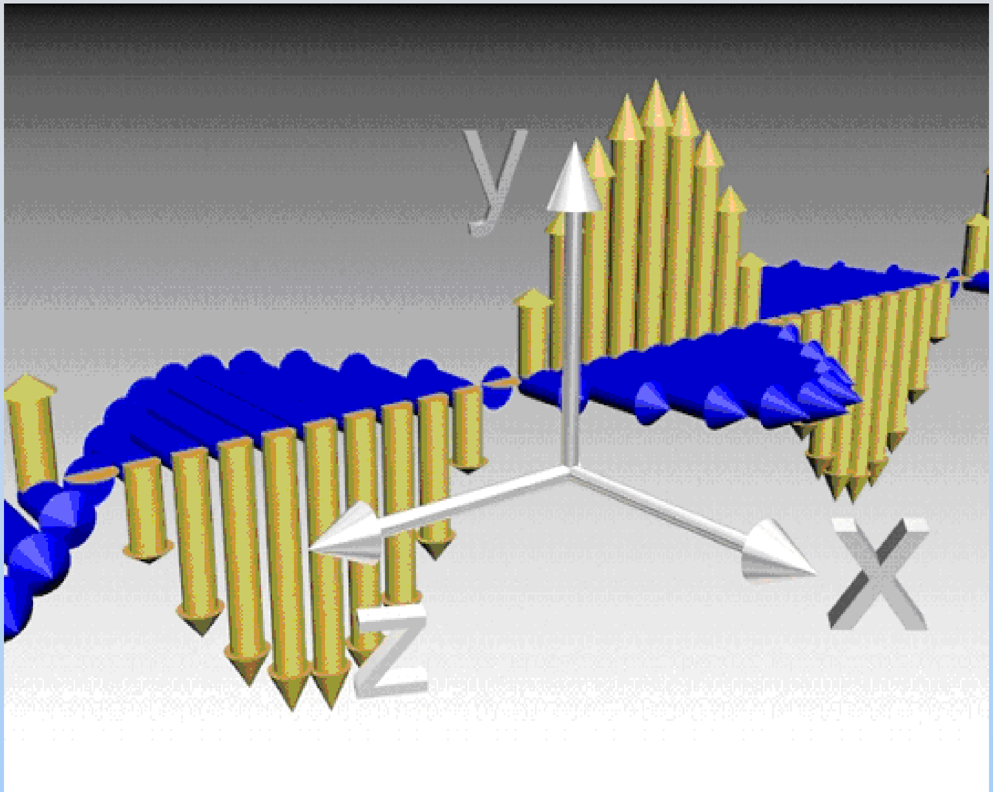
$$\boxed{\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{p}}}$$

$\hat{\mathbf{E}}$	$\hat{\mathbf{B}}$	$\hat{\mathbf{p}}$	$(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}})$
$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	z
$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	x
$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	y
$\hat{\mathbf{j}}$	$\hat{\mathbf{i}}$	$-\hat{\mathbf{k}}$	$-z$
$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$-x$
$\hat{\mathbf{i}}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$	$-y$

Concept Question Question: Direction of Propagation

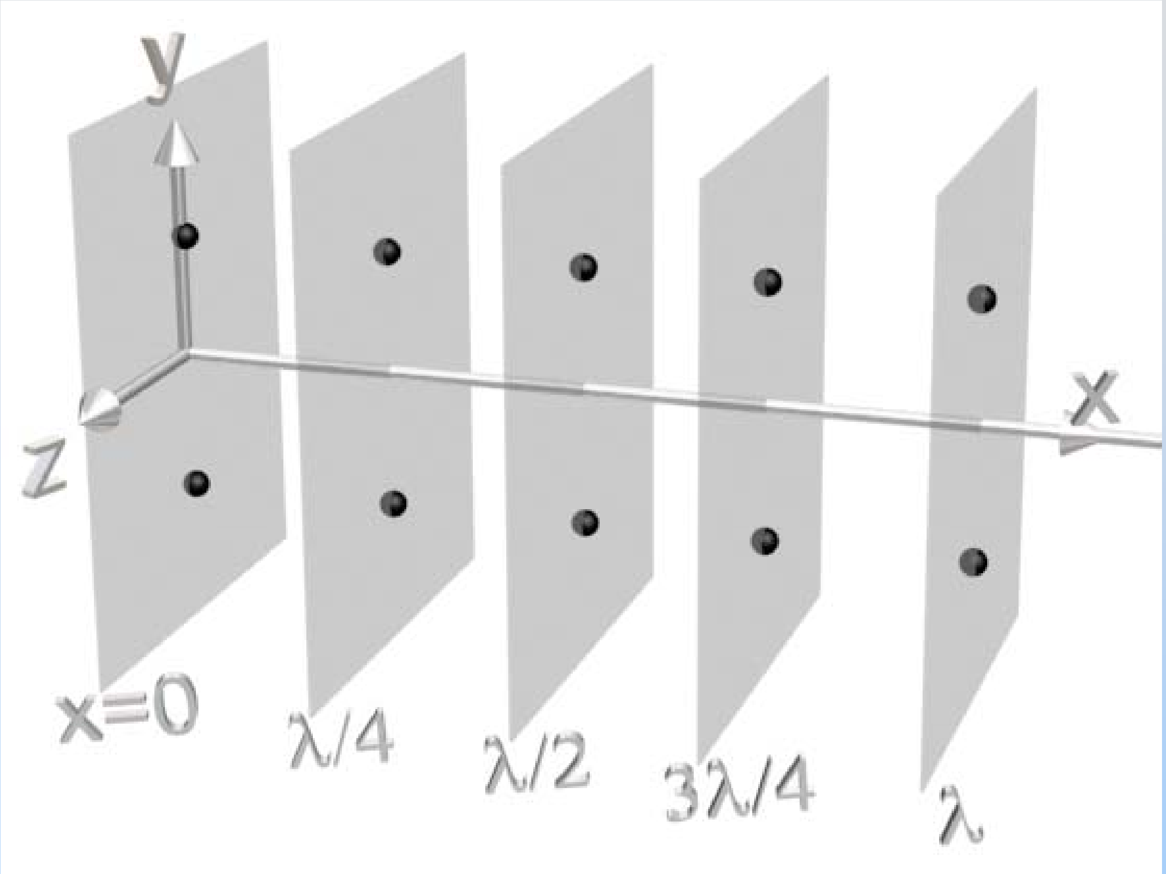
Concept Question: Direction of Propagation

The figure shows the E (yellow) and B (blue) fields of a plane wave. This wave is propagating in the



1. +x direction
2. –x direction
3. +z direction
4. –z direction
5. I don't know

Problem: Plane Waves



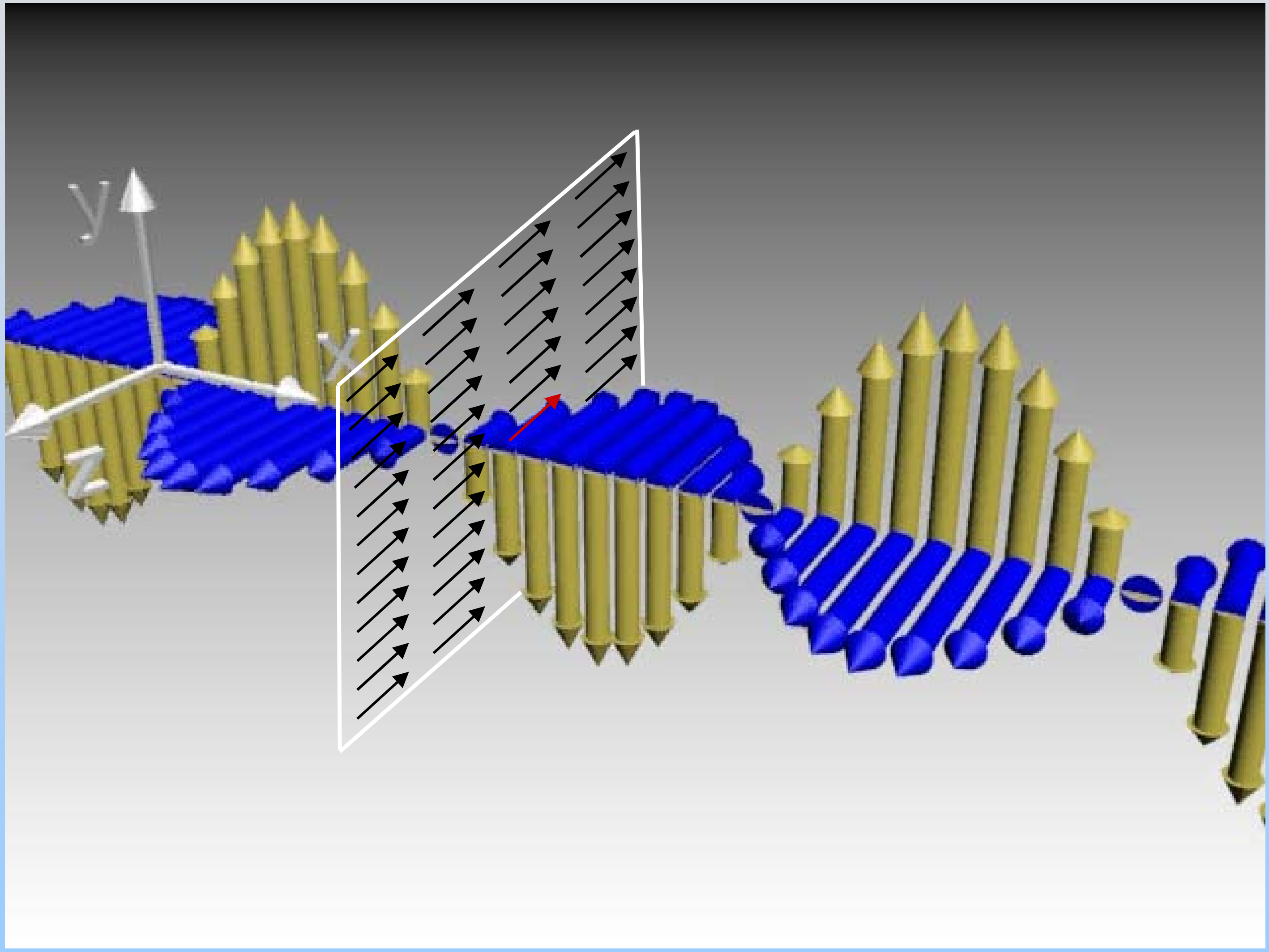
1) Plot E, B at each of the ten points pictured for *t*=0

2) Why is this a "plane wave?"

$$\vec{\mathbf{E}}(x,y,z,t) = E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{j}}$$

$$\vec{\mathbf{B}}(x,y,z,t) = \frac{1}{c}E_{y,0}\sin\left(\frac{2\pi}{\lambda}(x-ct)\right)\hat{\mathbf{k}}$$

Electromagnetic Radiation: Plane Waves



Electromagnetic Waves

Both E & B travel like waves:

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}$$

But there are strict relations between them:

$$\frac{\partial B_z}{\partial t} = - \frac{\partial E_y}{\partial x} \qquad \frac{\partial B_z}{\partial x} = - \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Here, E_y and B_z are “the same,” traveling along x axis

Amplitudes of E & B

$$\text{Let } E_y = E_0 f\left(x - vt\right); B_z = B_0 f\left(x - vt\right)$$

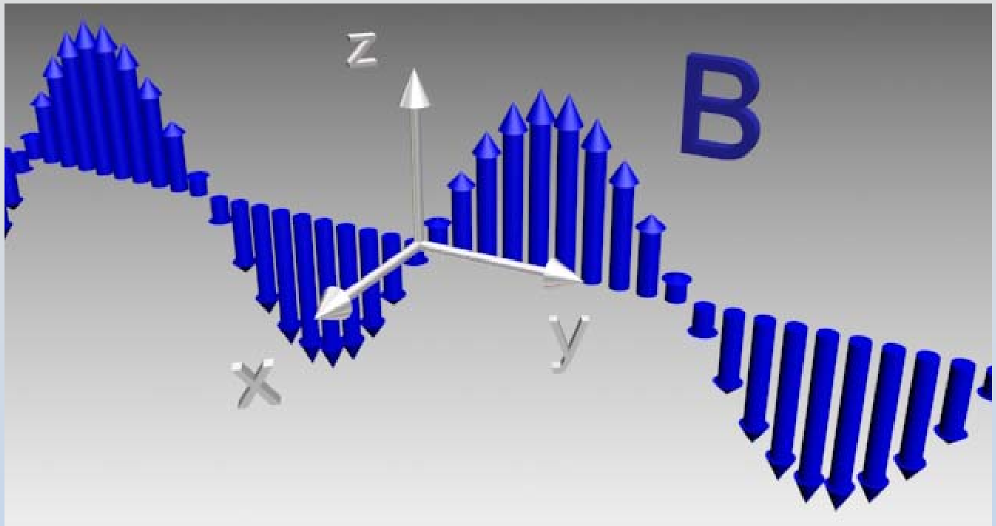
$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \Rightarrow -vB_0 f'\left(x - vt\right) = -E_0 f'\left(x - vt\right)$$

$$\boxed{\Rightarrow vB_0 = E_0}$$

E_y and B_z are "the same," just different amplitudes

Concept Question Questions: Traveling Wave

Concept Question: Traveling Wave



The B field of a plane EM wave is $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{k}} B_0 \sin(ky - \omega t)$

The electric field of this wave is given by

1. $\vec{\mathbf{E}}(z,t) = \hat{\mathbf{j}} E_0 \sin(ky - \omega t)$
2. $\vec{\mathbf{E}}(z,t) = -\hat{\mathbf{j}} E_0 \sin(ky - \omega t)$
3. $\vec{\mathbf{E}}(z,t) = \hat{\mathbf{i}} E_0 \sin(ky - \omega t)$
4. $\vec{\mathbf{E}}(z,t) = -\hat{\mathbf{i}} E_0 \sin(ky - \omega t)$
5. I don't know

Concept Question EM Wave

The E field of a plane wave is:

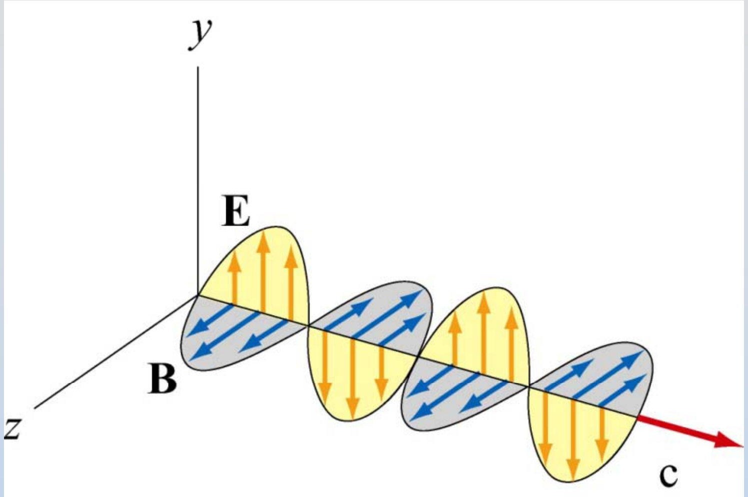
$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{j}} E_0 \sin(kz + \omega t)$$

The magnetic field of this wave is given by:

1. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
2. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{i}} B_0 \sin(kz + \omega t)$
3. $\vec{\mathbf{B}}(z,t) = \hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
4. $\vec{\mathbf{B}}(z,t) = -\hat{\mathbf{k}} B_0 \sin(kz + \omega t)$
5. I don’t know

Summary: Traveling Electromagnetic Waves

Properties of EM Waves



Travel (through vacuum) with speed of light

$$v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \frac{m}{s}$$

At every point in the wave and any instant of time, E and B are in phase with one another, with

$$\frac{E}{B} = \frac{E_0}{B_0} = c$$

E and B fields perpendicular to one another, and to the direction of propagation (they are **transverse**):

$$\text{Direction of propagation} = \text{Direction of } \mathbf{E} \times \mathbf{B}$$

Traveling E & B Waves

Wavelength: λ

Frequency : f

$$\vec{\mathbf{E}} = \hat{\mathbf{E}} E_0 \sin(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t)$$

Wave Number: $k = \dfrac{2\pi}{\lambda}$

Angular Freq.: $\omega = 2\pi f$

Period: $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

Speed: $v = \dfrac{\omega}{k} = \lambda f$

Direction: $+\hat{\mathbf{k}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$

$$\frac{E}{B} = \frac{E_0}{B_0} = v$$

In vacuum...

$$= c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \frac{m}{s}$$

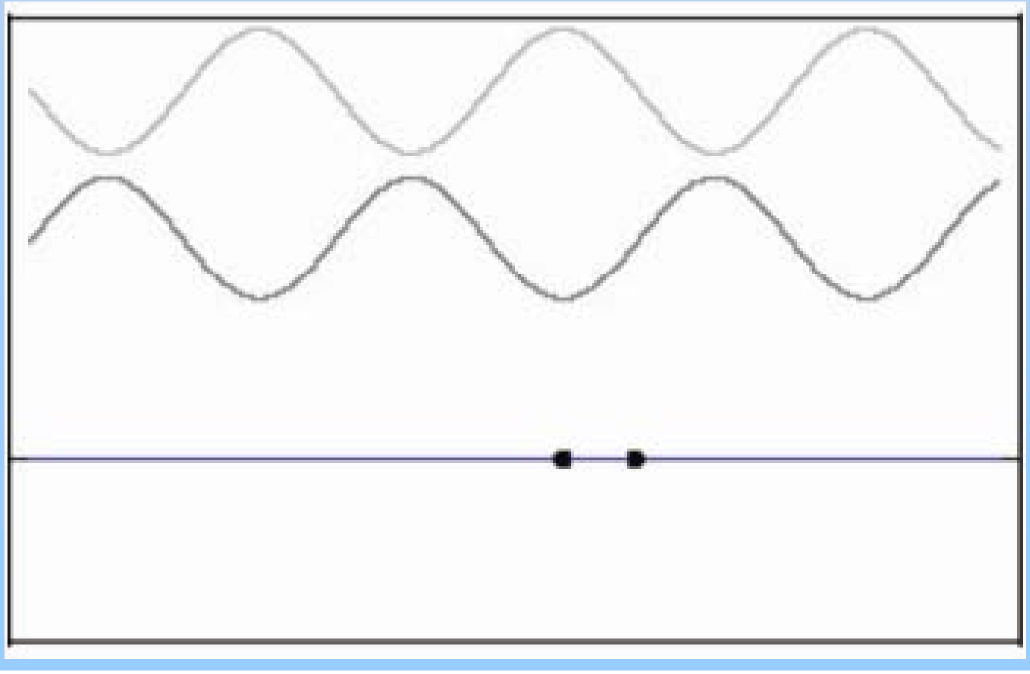
Standing Waves

Standing Waves

What happens if two waves headed in opposite directions are allowed to interfere?

$$E_1 = E_0 \sin(kx - \omega t) \qquad E_2 = E_0 \sin(kx + \omega t)$$

Superposition: $E = E_1 + E_2 = 2E_0 \sin(kx)\cos(\omega t)$

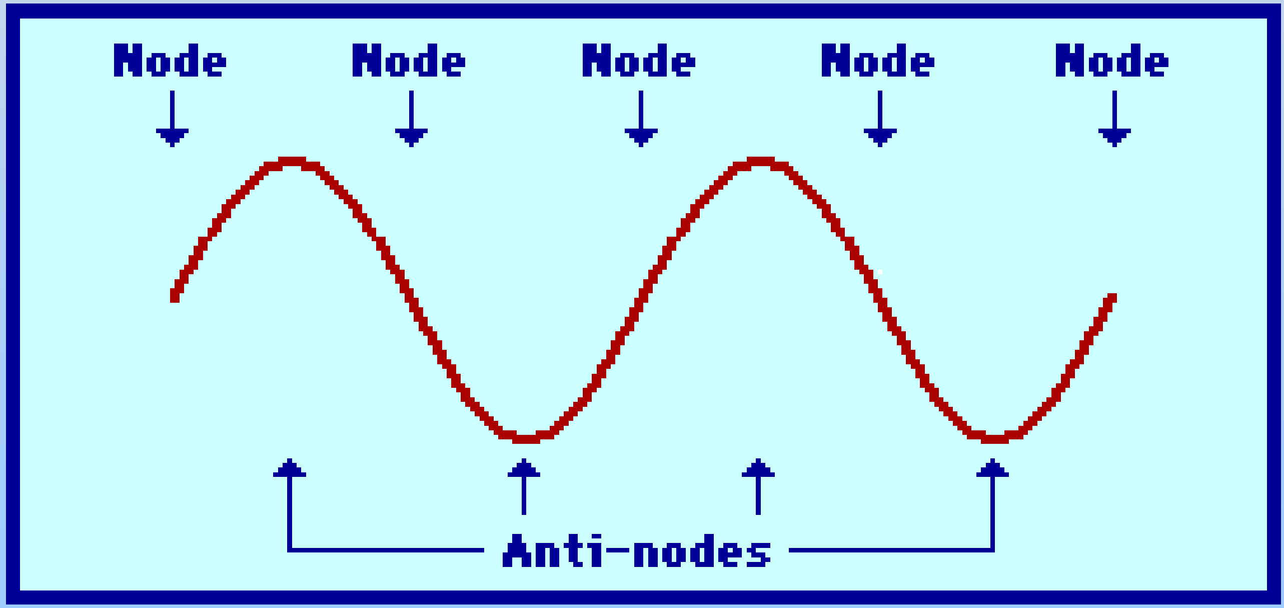


Standing Waves

Most commonly seen in resonating systems:

Musical Instruments, Microwave Ovens

$$E = 2E_0 \sin(kx)\cos(\omega t)$$



8.02SC Physics II: Electricity and Magnetism
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