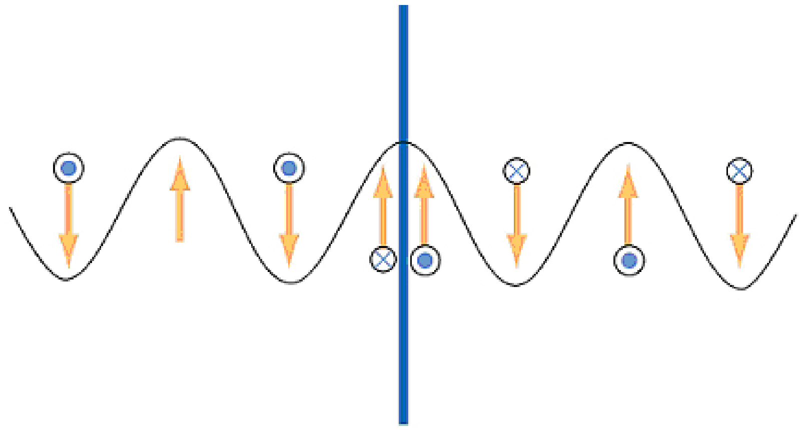
Energy and Momentum in EM Waves
Challenge Problems and Solutions

Problem 1:



A plane E&M wave is generated by shaking a sheet of positive charge up and down. At the time shown

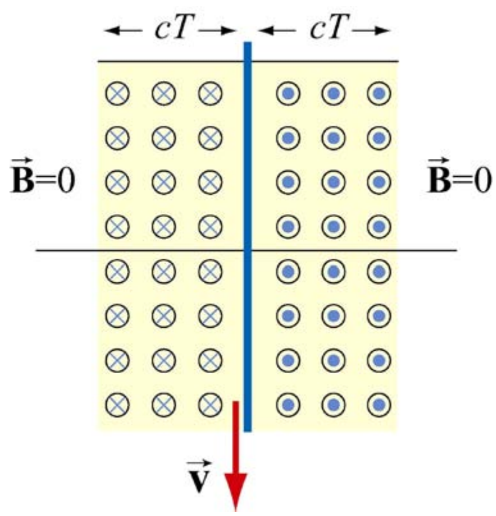
1. The E field is oscillating into and out of the page and the sheet is moving up
2. The E field is oscillating into and out of the page and the sheet is moving down
3. The B field is oscillating into and out of the page and the sheet is moving up
4. The B field is oscillating into and out of the page and the sheet is moving down
5. I don't know (this answer is worth 1 point)

Problem 1 Solution:

4. The B field is oscillating into and out of the page and the sheet is moving down

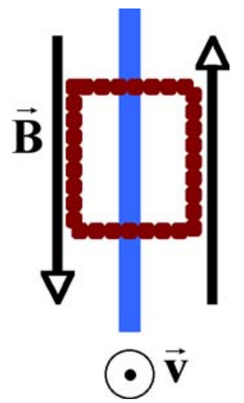
Problem 2:
The charged sheet at right has a uniform charge density σ and is being pulled downward at a velocity $\vec{\mathbf{v}}$.

1) What is the B field that is generated?
2) If the sheet position oscillates as $y(t) = y_0 \sin(\omega t)$, what are the $\vec{\mathbf{E}}(x,t)$ and $\vec{\mathbf{B}}(x,t)$?



Problem 2 Solution:
1) What is the B field that is generated?

Its always best to redraw so that the magnetic field lies in the plane of the page:



So we need to do Ampere's law around the loop. The current is just the moving charge density:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = 2Bl = \mu_o I_{\text{enc}} = \mu_o \sigma v l \qquad \Rightarrow \qquad \boxed{B = \mu_o \sigma v/2}$$

2) If the sheet position oscillates as $y(t) = y_0 \sin(\omega t)$, what are the $\vec{\mathbf{E}}(x,t)$ and $\vec{\mathbf{B}}(x,t)$?

$$y(t) = y_0 \sin(\omega t) \qquad \Rightarrow \qquad v = y_0 \omega \cos(\omega t)$$

$$\vec{\mathbf{B}} = \frac{\mu_o \sigma}{2} y_0 \omega \cos\left(\frac{\omega}{c} x - \omega t\right) \hat{\mathbf{k}}$$

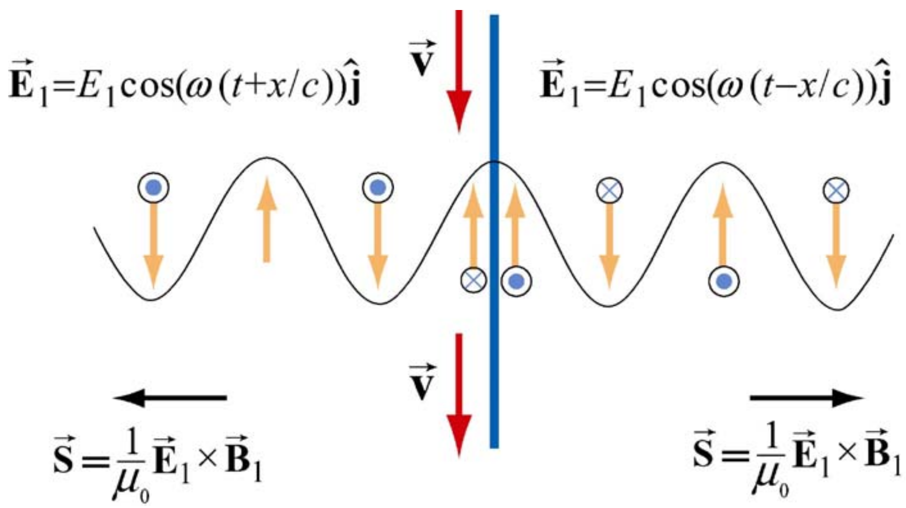
Sheet moves in y, wave travels in x

$$\vec{\mathbf{E}} = \frac{\mu_o \sigma c}{2} y_0 \omega \cos\left(\frac{\omega}{c} x - \omega t\right) \hat{\mathbf{j}}$$

Problem 3:

For the wave pictured below, where you calculated that $B_1 = \mu_o \sigma v/2$:

1) What is total power per unit area radiated away?
2) Where is that energy coming from?
3) Calculate power generated to see efficiency



Problem 3 Solution:

1) What is total power per unit area radiated away?

$$\frac{P_{\text{total}}}{\text{Area}} = \underbrace{2S}_{\text{two sides}} = 2\frac{E_1 B_1}{\mu_o} = 2\frac{cB_1^2}{\mu_o} = 2\frac{c\left(\mu_o \sigma v/2\right)^2}{\mu_o} = \frac{\mu_o c \sigma^2 v^2}{2}$$

2) Where is that energy coming from?

It is coming from the moving sheet.

3) Calculate power generated to see efficiency

The electric field exerts a force on the charges, and they are moving, so

$$\frac{P}{A} = \frac{Fv}{A} = \frac{qEv}{a} = \sigma E v = \sigma c B v = \sigma c v\left(\mu_o \sigma v/2\right) = \frac{\mu_o c \sigma^2 v^2}{2}$$

This is the same as the power radiated, so this is 100% efficient!

Problem 4:

Amazing Communication

Cell phones are pretty cool. So is the deep space network used to communicate with the voyager spacecraft (http://voyager.jpl.nasa.gov/news/profiles_dsn.html), which are currently about 80 AU away (1 AU = 150 million km). Let's think about both.

a) About how much power does a cell phone use? Think about how often you need to charge your cell phone and how much energy could realistically be stored in it.

b) Assuming that most of the power from the cellphone is used in signal transmission (which is becoming a progressively worse assumption, but use it anyway), and knowing the average size of a cell phone cell (26 km^2), what kind of signal strength (power per unit area) is needed at the receiver in order to still "have signal?"

c) Let's compare that with a radio, to see if its in the same ballpark. FCC regulations prevent broadcasts at powers above 100 kW. How far away from a radio transmitter can you still hear the station? How much power density are you then receiving?

d) What kind of electric fields do these power densities correspond to?

e) The voyager spacecraft have 20 Watt transmitters (3 meter dishes broadcasting at 2.3 GHz). The dishes are aimed at the Earth. How wide an angular dispersion could they have such that there is still enough power at the Earth to receive the signal?

Problem 4 Solutions:

a) Typical new cell phones broadcast at about ¼ Watt. Older mobile phones (as well as boosters for cell phones that some people get for their cars) broadcast at 3 W.

b) We'll pretend that the cell is circular (in reality they try to make them hexagonal). That means that you are never further than about $r = \sqrt{A/\pi} \approx 3$ km from the base station. The signal strength at the base station could then be as weak as

$$S = P/A = (0.25 \text{ W})\big/ 4\pi(3 \text{ km})^2 = 2.2 \text{ nW/m}^2$$

c) You might be able to hear a station about 200 km away. Beyond 100 miles the curvature of the Earth begins to kill you (FM radio waves go in a straight line and will just go off into space,as opposed to AM and CB radio waves that can bounce off the ionosphere and explain why you can occasionally get good signals from around the globe using those technologies).

$$S = P/A = (100 \text{ kW})\big/ 4\pi(200 \text{ km})^2 = 200 \text{ nW/m}^2$$

So this is a little bigger than what we calculated in (b) but not outrageously out of line, especially considering that line of sight rather than power limitations are probably largely responsible for the loss of signal.

d) From the Poynting vector:

$$\bar{S} = \frac{1}{2\mu_o} E_0 B_0 = \frac{E_0^2}{2\mu_o c} \Rightarrow E_0 = \sqrt{2\mu_o c \bar{S}} \approx \begin{cases} \text{1 mV/m for cell phones} \\ \text{12 mV/m for FM radio} \end{cases}$$

e) It's pretty amazing that you can use a 20 Watt transmitter from so far and still get a signal, but they can aim really well. What is the possible angular spread? We don't want a power per unit area below about 1 nW/m^2, meaning the 20 W can spread out over 2 x 10^{10} m^2 (a circle of radius about 80 km), meaning an angular dispersion of:

$$\tan\theta \approx \theta \approx \frac{80 \text{ km}}{80 \text{ AU}} = \frac{1}{150 \times 10^6} \approx 7 \times 10^{-9} \text{ rad} = 0.4\,\mu\text{deg}$$

Pretty darn impressive, especially the part about aiming to that accuracy. I would guess that in reality they don't do that well and instead have a much better receiving antenna, meaning that the signal could be a few orders of magnitude weaker, but I can't find any info about that to confirm it. I do know, however, that the antenna for sending signals to spacecraft is big (70 m, 20 kW). Kind of sad that they don't have the power of a good radio station, but the antenna is MUCH more directional.

Problem 5:

1. Spark Gap Distance and Timing
Consider an LC spark gap transmitter. Transmitter. The time to charge the transmitter capacitor until it discharges depends on the resistance in the charging circuit (R = 4.5 MΩ), the capacitance (C = 33 pF) and the voltage required to initiate breakdown. Assume that the power supply supplies 800 V but that breakdown typically occurs at a voltage of about 500 V on the capacitor.

(a) Thinking of the tungsten electrodes as parallel plates, how far apart must they be in order generate a spark at 500 V?

(b) In reality, the electrodes aren't parallel plates, but rather cylinders with a fairly small radius of curvature. Given this, will the distance needed between the electrodes to generate sparking be smaller or larger than you calculated in (a)? Why?

(c) About how much time will it take for the power supply to charge the capacitor from empty to discharge?

2. Wavelength and Frequency of the Radiation
The spark-gap antenna is a quarter-wavelength antenna, radiating as described above. Using l = 31 mm for the length of one of the arms of the antenna, what is

(a) the wavelength of the emitted radiation?

(b) the frequency of the emitted radiation?

3. Reflections
Suppose you place the transmitter some distance in front of a perfectly conducting sheet, oriented so that the propagation direction of the waves hitting the reflector is perpendicular to the plane of the reflector (so that they'll reflect straight back out towards the transmitter). For example, place the transmitter at $z = -D$ with the antenna parallel to the x-axis, and have the reflector fill the z = 0 (xy-) plane.

(a) Write an equation for the electric field component of the radiation from the transmitter (the *incident* wave). Treat the field as plane wave, with a constant amplitude E_0 and angular frequency ω_0.

(b) What condition must the total electric field satisfy at the surface of the conductor (z = 0)?

(c) What is the direction of propagation of the reflected wave?

(d) Write an equation for the time dependent amplitude & direction of the reflected wave, making the same assumptions as above.

(e) Write an equation for the total amplitude of the electric field as a function of position, by adding (c) and (d).

(f) Nodes are locations (in this case planes) where the electric field is zero at all times. What is the distance between nodes along the z-axis?

(g) What is the numerical distance you thus expect for our transmitter (i.e. use 2a)?

Problem 5 Solutions:

1.

(a) We must reach the breakdown field of air. Assuming a parallel plate geometry the field between the plates is constant so:

$$d = V/E = (500 \text{ V})/(3\times10^{6} \text{ V/m}) \approx 0.2 \text{ mm}$$

(b) The electric field is enhanced with the smaller radius of curvature (sparks always happen near the sharpest points of a conductor) so the distance will actually be a little bigger.

(c) We just need to see how long it takes to raise the potential to 500 V:

$$V(t) = V_f\left(1 - e^{-t/\tau}\right) \Rightarrow t = -\tau \ln\left(1 - \frac{V_{\text{breakdown}}}{V_f}\right) = -(4.5 \text{ M}\Omega \cdot 33 \text{ pF})\ln\left(1 - \frac{500 \text{ V}}{800 \text{ V}}\right) = 150\ \mu\text{s}$$

2.

(a) The period is $T = 4l/c$, so the wavelength is $\lambda = cT = 4l = 4(31 \text{ mm}) \approx 12 \text{ cm}$

(b) The frequency is $f = c/\lambda = (3\times10^{8} \text{ m/s})/(12 \text{ cm}) = 2.4 \text{ GHz}$

3.

(a)

$$\vec{\mathbf{E}}_{\text{incident}} = E_0 \cos\left(\frac{\omega_0}{c}(z - ct)\right)\hat{\mathbf{i}}$$

(b) The electric field in a conductor is 0, so that is the condition *for all time* at z = 0.

(c) It reflects, so the direction of propagation flips – it is $-\hat{\mathbf{k}}$

(d)

$$\vec{\mathbf{E}}_{\text{reflected}} = -E_0 \cos\left(\frac{\omega_0}{c}(z+ct)\right)\hat{\mathbf{i}}$$

(e)

$$\begin{aligned}\vec{\mathbf{E}}_{\text{total}} &= \vec{\mathbf{E}}_{\text{incident}} + \vec{\mathbf{E}}_{\text{reflected}} = E_0\hat{\mathbf{i}}\left(\cos\left(\tfrac{\omega_0}{c}(z-ct)\right) - \cos\left(\tfrac{\omega_0}{c}(z+ct)\right)\right) \\ &= E_0\hat{\mathbf{i}}\left(\left(\cos\left(\tfrac{\omega_0 z}{c}\right)\cos(\omega_0 t) + \sin\left(\tfrac{\omega_0 z}{c}\right)\sin(\omega_0 t)\right) - \left(\cos\left(\tfrac{\omega_0 z}{c}\right)\cos(\omega_0 t) - \sin\left(\tfrac{\omega_0 z}{c}\right)\sin(\omega_0 t)\right)\right) \\ &= 2E_0 \sin\left(\tfrac{\omega_0}{c} z\right)\sin(\omega_0 t)\hat{\mathbf{i}}\end{aligned}$$

(f) The position dependence of the standing wave is $\sin\left(\dfrac{\omega_0}{c} z\right)$ which has a wavelength of

$\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{\omega_0/c} = \dfrac{2\pi c}{\omega_0}$ and hence nodes every half a wavelength, that is, every $\dfrac{\pi c}{\omega_0} = \dfrac{\lambda}{2}$

(g) The nodes are every half wavelength, separated by a distance of about 6 cm.

Problem 6:

Reflections of True Love

(a) A light bulb puts out 100 W of electromagnetic radiation. What is the time-average intensity of radiation from this light bulb at a distance of one meter from the bulb? What are the maximum values of electric and magnetic fields, E_0 and B_0, at this same distance from the bulb? Assume a plane wave.

(b) The face of your true love is one meter from this 100 W bulb. What maximum surface current must flow on your true love's face in order to reflect the light from the bulb into your adoring eyes? Assume that your true love's face is (what else?) perfect--perfectly smooth and perfectly reflecting--and that the incident light and reflected light are normal to the surface.

Problem 6 Solutions:

(a)

$$\langle I \rangle = \frac{\langle P \rangle}{4\pi r^2} = \frac{100\,W}{4(3.14)(1\,m)^2} = 7.96\ W/m^2$$

Since

$$\langle I \rangle = \frac{c}{2}\varepsilon_0 E_0^{\ 2}$$

$$E_{\max} = E_0 = \sqrt{\frac{2\langle I \rangle}{c\varepsilon_0}} = \sqrt{\frac{2(7.96\,W/m^2)}{(3.00\times 10^8\ m/s)(8.85\times 10^{-12}\ C^2/N\cdot m)}} = 77.4\,V/m$$

(b) The surface current is given by

$$\begin{aligned} K_{\max} &= \frac{2E_0}{c\mu_0} \\ &= \frac{2(77.4\,V/m)}{(3.00\times 10^8\ m/s)(4\pi\times 10^{-7}\ T\cdot m/A)} \\ &= 0.411\,A/m \\ &= 411\,mA/m \end{aligned}$$

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