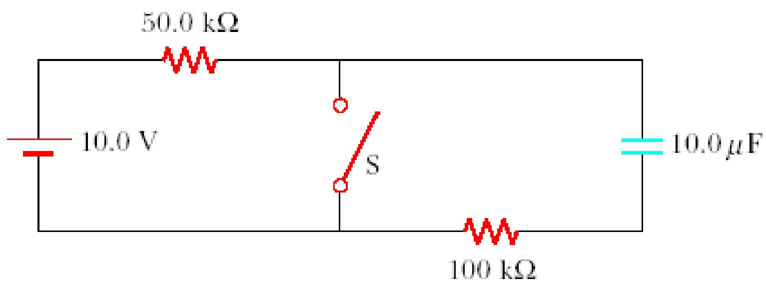
DC Circuits with Capacitors
Challenge Problem Solutions

Problem 1:

In the circuit shown, the switch S has been closed for a long time. At time t=0 the switch is opened. It remains open for "a long time" T, at which point it is closed again. Write an equation for (a) the voltage drop across the 100 kΩ resistor and (b) the charge stored on the capacitor as a function of time.



Problem 1 Solution:

(a) The capacitor begins uncharged. When the switch is opened at t=0 we have an RC circuit with R = 150 kΩ and C = 10.0 μF, so τ = RC = 1.50 s. The final voltage (after an infinite time) on the capacitor will be the battery voltage (10.0V) so we can write the equation for voltage on the capacitor during charging as:

$$V_C = V_F\left(1-e^{-t/\tau}\right) = 10.0\text{ V}\left(1-e^{-t/1.50\text{ s}}\right)\quad [\text{for } t<T]$$

The current is then

$$I = dQ/dt = CdV_C/dt = \frac{CV_F}{\tau}\left(e^{-t/\tau}\right) = \frac{(10\mu F)(10.0\text{ V})}{1.50\text{ s}}\left(e^{-t/1.50\text{ s}}\right)\quad [\text{for } t<T]$$

So the voltage drop across the 100 kΩ resistor is

$$V_R = RI = \frac{(100k\Omega)(10\mu F)(10.0\text{ V})}{1.50\text{ s}}\left(e^{-t/1.50\text{ s}}\right)\quad [\text{for } t<T]$$

During discharge the capacitor starts at its value at $t = T$ (which we can get with the equation above) and then drives through the 100 kΩ resistor and the switch. The time constant is thus now only 1.00 s. So the voltage goes like:

$$V_C = V_0 e^{-(t-T)/\tau} = 10.0\text{ V}\left(1-e^{-T/1.50\text{ s}}\right)e^{-(t-T)/1.00\text{ s}}\quad [\text{for } t\geq T]$$

Of course, we were asked for charge, not voltage, for which we use $Q = CV$.

The current through the 100 kΩ resistor is now

$$I = -C dV_C / dt = \frac{CV_0}{\tau} e^{-(t-T)/\tau} = \frac{(100\mu F)(10.0\text{ V})}{1.00\text{ s}} \left(1 - e^{-T/1.50\text{ s}}\right) e^{-(t-T)/1.00\text{ s}} \quad [\text{for } t \geq T]$$

So the voltage drop across the 100 kΩ resistor is

$$V_R = RI = \frac{RCV_0}{\tau} e^{-(t-T)/\tau} = \frac{(100k\Omega)(100\mu F)(10.0\text{ V})}{1.00\text{ s}} \left(1 - e^{-T/1.50\text{ s}}\right) e^{-(t-T)/1.00\text{ s}} \quad [\text{for } t \geq T].$$

Problem 2:

You know that the power supplied by a battery is given by P = VI (the battery voltage times the current it is supplying). You also know (from the Friday problem solving) that a resistor dissipates power (turns it into heat) at a rate given by $P = I^2R$.

Consider a simple RC circuit (battery, resistor R, capacitor C). Determine an expression for the energy stored in the capacitor by integrating the difference between the power supplied by the battery and that consumed by the resistor. Should the energy be related to the current through the capacitor or the potential across it?

Problem 2 Solution:

We know that the current that flows in the circuit decays exponentially:

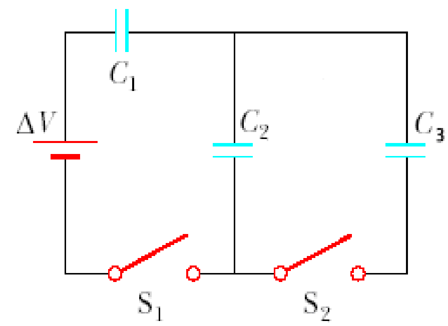
$$I = I_0 e^{-t/\tau} = \frac{\varepsilon}{R} e^{-t/RC} \; .$$

We can integrate the power supplied by the battery minus the power consumed by the resistor then to get:

$$\begin{aligned} U_{\mathrm{C}} &= \int_{t'=0}^{t} P_{\mathrm{B}}(t') - P_{\mathrm{R}}(t')\, dt' = \int_{t'=0}^{t} \frac{\varepsilon}{R} e^{-t'/RC} \cdot \varepsilon - \left(\frac{\varepsilon}{R} e^{-t'/RC} \right)^2 R dt' \\ &= \frac{\varepsilon^2}{R} \int_{t'=0}^{t} e^{-t'/\tau} - e^{-2t'/\tau} dt' = \frac{\varepsilon^2}{R} \left[-\tau e^{-t'/\tau} + \frac{\tau}{2} e^{-2t'/\tau} \right]_0^t = \frac{\varepsilon^2}{R} \frac{\tau}{2} \left[e^{-2t'/\tau} - 2e^{-t'/\tau} \right]_0^t \; . \\ &= \frac{1}{2} C\varepsilon^2 \cdot \left[e^{-2t/\tau} - 2e^{-t/\tau} + 1 \right] = \frac{1}{2} C \left[\varepsilon \left(1 - e^{-t/\tau} \right) \right]^2 = \boxed{\frac{1}{2} C V_C^2} \end{aligned}$$

That is, the energy stored in the capacitor depends on the voltage across the capacitor (which makes sense, as that is a feature of the capacitor, while the current through it depends more on what resistor it happens to be hooked to).

Problem 3:



In the circuit shown at right $C_1 = 2.0\ \mu$F, $C_2 = 6.0\ \mu$F, $C_3 = 3.0\ \mu$F and $\Delta V = 10.0\,\mathrm{V}$. Initially all capacitors are uncharged and the switches are open. At time $t = 0$ switch S_2 is closed. At time $t = T$ switch S_2 is then opened, proceeded nearly immediately by the closing of S_1. Finally at $t = 2T$ switch S_1 is opened, proceeded nearly immediately by the closing of S_2. Calculate the following:

(a) the charge on C_2 for $0 < t < T$ (after S_2 is closed)

(b) the charge on C_1 for $T < t < 2T$

(c) the final charge on each capacitor (for $t > 2T$)

Problem 3 Solution:

(a)As long as S1 is open the battery is out of the circuit and hence none of the capacitors will have any charge on them.

(b)When S_1 is closed, the battery is in series with C_1 and C_2. The charge on them will thus be equal, and equal to the charge that an equivalent capacitor would have.

$$C_{eqiv} = \left(C_1^{-1} + C_2^{-1}\right)^{-1} = \left(\frac{1}{2.0\,\mu\mathrm{F}} + \frac{1}{6.0\,\mu\mathrm{F}}\right)^{-1} = 1.5\,\mu\mathrm{F}$$

$$Q_2\left(T < t < 2T\right) = Q_{equiv} = C_{equiv}\Delta V_{equiv} = (1.5\,\mu\mathrm{F})(10.0\ \mathrm{V}) = \boxed{15\ \mu\mathrm{C}}$$

(c)When S_1 is opened, the battery and C_1 are removed from the circuit. This means that the charge on C1 is fixed at the value it was at, $\boxed{Q_1 = 15\ \mu\mathrm{C}}$.

The charge on C_2 will be shared with C_3, so that their potentials will be the same (since they are now in parallel). So:

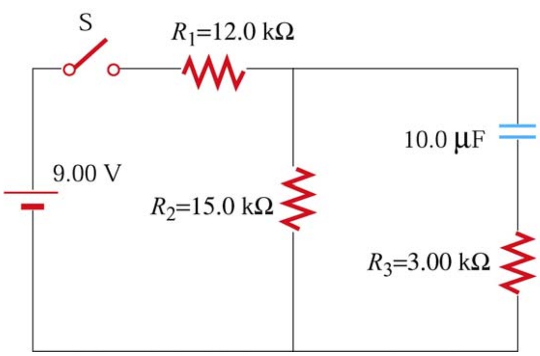
$$V_2 = Q_2/C_2 = V_3 = Q_3/C_3\,; \qquad Q_2 + Q_3 = Q_2\left(t = 2T^-\right)$$

$$\frac{Q_2}{C_2} = \frac{Q_3}{C_3} = \frac{Q_2\left(t = 2T^-\right) - Q_2}{C_3} \Rightarrow Q_2 C_3 = C_2\left(Q_2\left(t = 2T^-\right) - Q_2\right) \Rightarrow$$

$$Q_2 = \frac{C_2 Q_2\left(t = 2T^-\right)}{C_2 + C_3} = \frac{6.0\ \mu\mathrm{F}\cdot 15\ \mu\mathrm{C}}{6.0\ \mu\mathrm{F}+3.0\ \mu\mathrm{F}} = \boxed{10\ \mu\mathrm{C} = Q_2} \Rightarrow \boxed{Q_3 = 5\ \mu\mathrm{C}}$$

Problem 4:

Consider the *RC* circuit shown in the figure. Suppose that the switch has been closed for a length of time sufficiently long enough for the capacitor to be fully charged.



(a) Find the steady-state current in each resistor.

(b) Find the charge Q on the capacitor.

(c) The switch is opened at $t = 0$. Write an equation for the current I_2 in R_2 as a function of time.

(d) Find the time that it takes for the charge on the capacitor to fall to 1/e of its initial value.

Problem 4 Solutions:

(a)Since the capacitor represents an open circuit, there is no current through R_3. Therefore, all the charges flowing through R_1 goes through R_2: hence $I_1 = I_2$ and $I_3 = 0$. Now, all you need to do is to find a current flowing through the two resistors in series.

(b)At equilibrium, the capacitor is fully charged and ΔV_{cap} is equal to the voltage drop across R_2 since there is no current through R_3 (and therefore the voltage drop across it is zero).

$$\Delta V_{\text{cap}} = I_2 R_2 = \frac{R_2}{R_1 + R_2}\varepsilon = \frac{15.0k\Omega}{12.0k\Omega + 15.0k\Omega}(9.00\text{V}) = 5.00\text{V}$$

Thus, the charge on the capacitor is given by

$$Q = C\Delta V_{\text{cap}} = C\varepsilon = (10.0\mu\text{F})(5.00\text{V}) = 50.0\mu\text{C} = 5.00\times 10^{-5}\,\text{C}$$

$$I_1 = I_2 = \frac{\varepsilon}{R_{12}} = \frac{\varepsilon}{R_1 + R_2} = \frac{9.00V}{12.0k\Omega + 15.0k\Omega} = 0.333\text{mA} = 3.33\times 10^{-4}\,\text{A}$$

(c)With the switch opened, the capacitor discharges through the resistors, R_2 and R_3. There is no emf in the circuit. You also need to notice R_1 is no longer a part of the closed circuit and there is no current through it. Now, you should follow the discussion in Section 7.6.2 of the *Course Notes* with $R = R_{23} = R_2 + R_3$ and $I = I_2 = -I_3$. You'll then obtain

(d)

$$\frac{I_2(t)}{I_2(0)} = \frac{0.278e^{-t/180\,\text{ms}}}{0.278e^{-0/180\,\text{ms}}} = e^{-t/180\,\text{ms}} = e^{-1}$$

Thus,

$$\frac{-t}{180\,\text{ms}} = -1 \text{ or } t = 180 \text{ ms}$$

which is called "time constant (τ).

$$q(t) = Qe^{-t/R_{23}C}$$

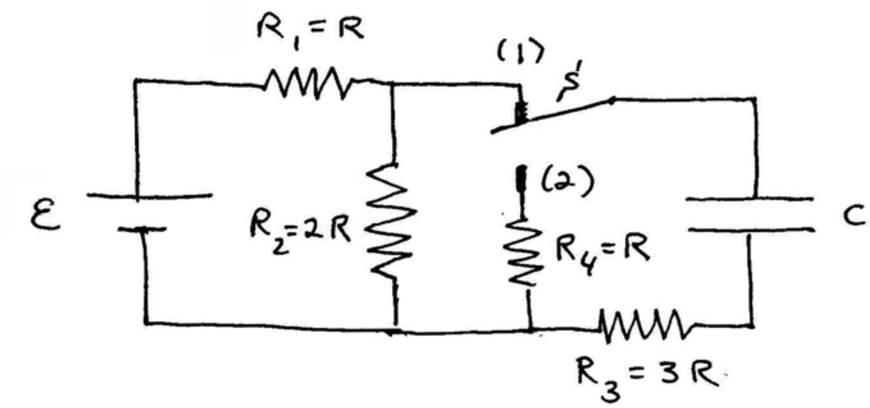
and

$$I_2(t) = -\frac{dq_2}{dt} = \left(\frac{Q}{R_{23}C}\right)e^{-t/R_{23}C} = \left(\frac{C\Delta V_{\text{cap}}}{(R_2+R_3)C}\right)e^{-t/(R_2+R_3)C} = 0.278\,e^{-t/180\text{ms}} \text{ milliamps}$$

Problem 5:

NOTE: YOU MUST SHOW WORK in order to get any credit for this problem. Make it clear to us that you understand what you are doing (use a few words!) .

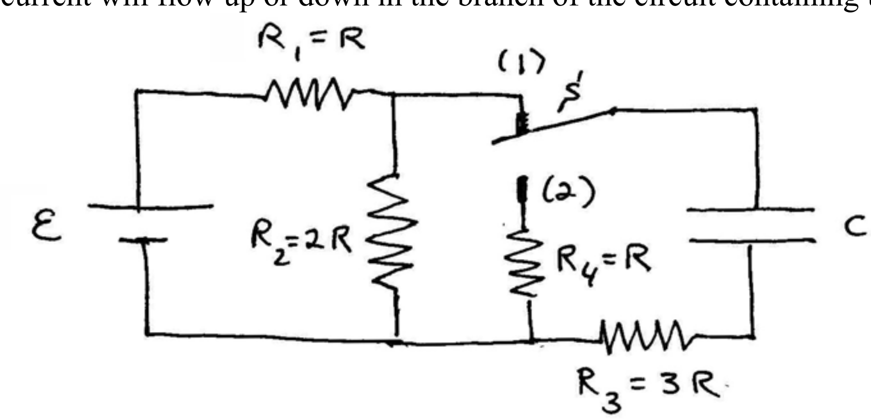
Consider the following circuit shown in the figure below. All questions can be answered without solving any differential equations.



a) Find the current through each of the four resistors, with resistances R_1, R_2, R_3, and R_4, a long time after the switch S has been in position (1).

b) Find the absolute value of the potential difference $|V_C|$ across the capacitor a long time after the switch S has been in position (1).

c) At $t = 0$ the switch is moved to position (2). What current will flow out of the capacitor at the instant the switch is moved to position (2)? Indicate whether the current will flow up or down in the branch of the circuit containing the capacitor.



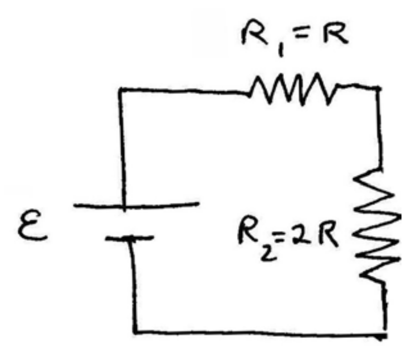
d) Make a graph of current vs. time for the current that flows out of the capacitor after the switch is moved to position (2) at $t = 0$. Indicate the value of the current at time $t = 0$ on your graph.

e) Find an expression for how long it takes the current that flows out of the capacitor to reach a value equal to e^{-1} of the value of that current when the switch is moved to position (2) at $t = 0$. (You can answer this question without solving a differential equation.)

f) After a long period in position (2), the switch is thrown to position (1) again. Immediately after the switch has been thrown to position (1), find the current through the battery.

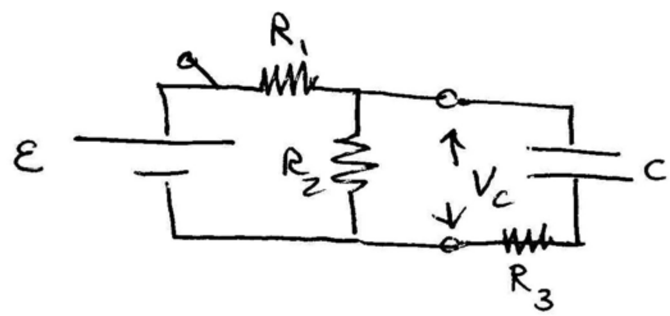
Problem 5 Solutions:

(a) No current flows through resistor R_4. A long time after the switch as been closed no current flows through the branch of the circuit containing the capacitor and resistor R_3. So the circuit looks like the circuit diagram below.



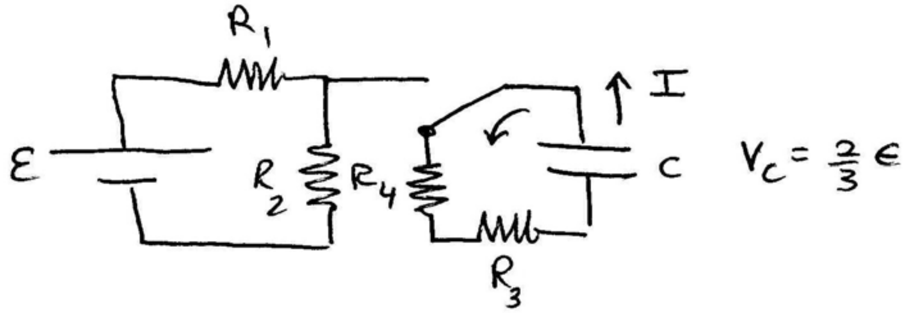
In this single loop circuit with equivalent resistance $R_{eq} = R_1 + R_2 = 3R$, the current is the same through both resistors with resistances R_1, R_2, and is given by $I = \varepsilon/(3R)$.

(b) The potential across the capacitor is the same as the potential across resistor 2, (see figure below)



$$|V_C| = IR_2 = \frac{\varepsilon}{3R} 2R = \frac{2\varepsilon}{3}.$$

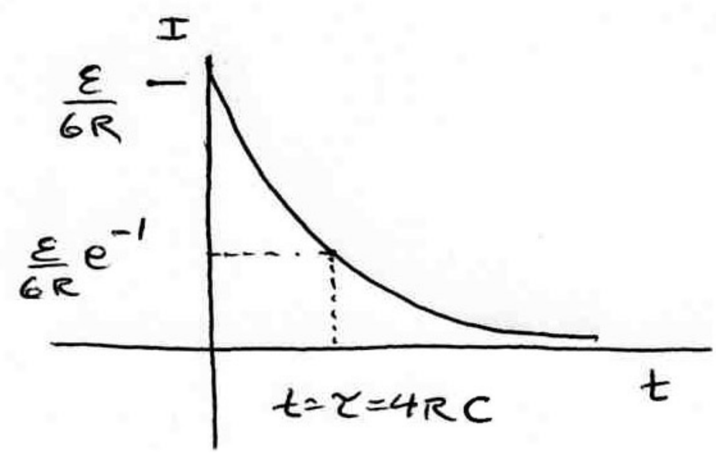
(c) When the switch is moved to position 2 the circuit looks like the circuit diagram shown below.



The current flows counterclockwise (up from the capacitor). Because $|V_C| = (2/3)\varepsilon$ and the equivalent resistance is $R_{eq} = R_4 + R_3 = 4R$, the current is

$$I = \frac{|V_C|}{R_{eq}} = \frac{2\varepsilon}{3(4R)} = \frac{\varepsilon}{6R}.$$

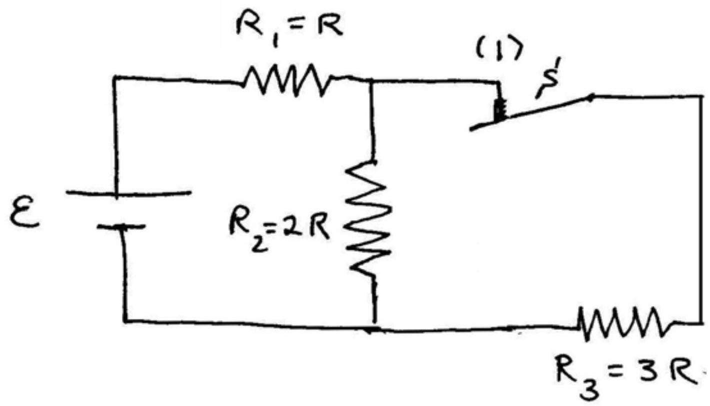
d)



(e) Because the equivalence resistance is $R_{eq} = 4R$, the time constant is

$$\tau = R_{eq}C = 4RC.$$

(f) After a long period in position (2) the capacitor is now uncharged. Immediately after the switch has been thrown to position (1), the capacitor can be replaced by a wire, and the circuit now looks like



Resistors 2 and 3 are now in parallel with equivalent resistance

$$(R_{eq})_{par} = \frac{R_2 R_3}{R_2 + R_3} = \frac{(2R)(3R)}{(2R + 3R)} = \frac{6R}{5} \ .$$

Because resistor 1 is in series with the parallel pair of resistors 2 and 3, the equivalent resistance of the resistor network is

$$R_{eq} = R_1 + \frac{R_2 R_3}{R_2 + R_3} = R + \frac{6R}{5} = \frac{11R}{5} .$$

Therefore the current through the battery is

$$I = \varepsilon / R_{eq} = 5\varepsilon / 11R \ .$$