Capacitors and Dielectrics
Challenge Problem Solutions

Problem 1:

A parallel plate capacitor has capacitance C. It is connected to a battery of EMF ε until fully charged, and then disconnected. The plates are then pulled apart an extra distance d, during which the measured potential difference between them changed by a factor of 4. Below are a series of questions about how other quantities changed. Although they are related you do not need to rely on the answers to early questions in order to correctly answer the later ones.

a) Did the potential difference increase or decrease by a factor of 4?

INCREASE DECREASE

b) By what factor did the electric field change due to this increase in distance? Make sure that you indicate whether the field increased or decreased.

c) By what factor did the energy stored in the electric field change? Make sure that you indicate whether the energy increased or decreased.

d) A dielectric of dielectric constant κ is now inserted to completely fill the volume between the plates. Now by what factor does the energy stored in the electric field change? Does it increase or decrease?

e) What is the volume of the dielectric necessary to fill the region between the plates? (Make sure that you give your answer only in terms of variables defined in the statement of this problem, fundamental constants and numbers)

Problem 1 Solutions:

(a)

$\boxed{\text{INCREASE}}$

(b)

Since the charge cannot change (the battery is disconnected) the electric field cannot change either. $\boxed{\text{No Change!}}$

(c)

The electric field is constant but the volume in which the field exists increased, so the energy must have increased. But by how much? The energy $U = \frac{1}{2}QV$. The charge doesn't change, the potential increased by a factor of 4, so the energy:

Increased by a factor of 4

(d)

Inserting a dielectric decreases the electric field by a factor of κ so it decreases the potential by a factor of κ as well. So now, by using the same energy formula $U = \frac{1}{2}QV$,

Energy decreases by a factor of κ

(e)

How in the world do we know the volume? We must be able to figure out the cross-sectional area and the distance between the plates. The first relationship we have is from knowing the capacitance:

$$C = \frac{\varepsilon_0 A}{x}$$

where x is the original distance between the plates. Make sure you don't use the more typical variable d here because that is used for the distance the plates are pulled apart.

Next, the original voltage $V_0 = E\,x$, which increases by a factor of 4 when the plates are moved apart by a distance d, that is, 4 $V_0 = E\,(x+d)$. From these two equations we can solve for x:

$$4V_0 = 4Ex = E(x+d) \quad \Rightarrow \quad x = d/3$$

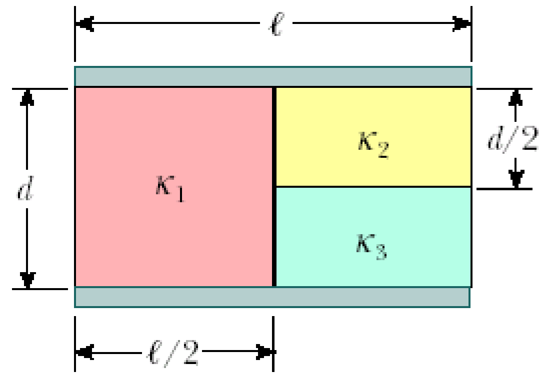
Now, we can use the capacitance to get the area, and multiply that by the distance between the plates (now $x + d$) to get the volume:

$$\text{Volume} = A(x+d) = \frac{xC}{\varepsilon_0}(x+d) = \frac{dC}{3\varepsilon_0}\left(\frac{d}{3}+d\right) = \boxed{\frac{4d^2C}{9\varepsilon_0}}$$

Problem 2:

(a) Consider a plane-parallel capacitor completely filled with a dielectric material of dielectric constant κ . What is the capacitance of this system?

(b) A parallel-plate capacitor is constructed by filling the space between two square plates with blocks of three dielectric materials, as in the figure below. You may assume that $\ell >> d$. Find an expression for the capacitance of the device in terms of the plate area A and d, $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$, and $\boldsymbol{\kappa}_3$.



Problem 2 Solutions:

(a)The capacitance is

$$C = \frac{\kappa \varepsilon_0 A/2}{d} = \kappa C_0$$

(b) The capacitor can be regarded as being consisted of three capacitors, $C_1 = \frac{\kappa_1 \varepsilon_0 \, A/2}{d}$, $C_2 = \frac{\kappa_2 \varepsilon_0 \, A/2}{d}$ and $C_3 = \frac{\kappa_3 \varepsilon_0 \, A/2}{d}$, with C_2 and C_3 connected in series, and the combination connected in parallel with C_1. Thus, the equivalent capacitance is

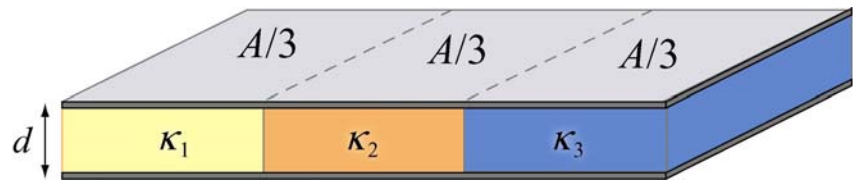
$$C = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3} \right)^{-1} = C_1 + \frac{C_2 C_3}{C_2 + C_3} = \frac{\kappa_1 \varepsilon_0 \, A/2}{d} + \frac{\varepsilon_0 A}{d} \left(\frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3} \right)$$

$$= \frac{\varepsilon_0 A}{d} \left(\frac{\kappa_1}{2} + \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3} \right)$$

Problem 3:

(a) Consider a plane-parallel capacitor completely filled with a dielectric material of dielectric constant κ . What is the capacitance of this system?

(b) A parallel-plate capacitor of area A and spacing d is filled with three dielectrics as shown in the figure below. Each occupies 1/3 of the volume. What is the capacitance of this system? [*Hint:* Consider an equivalent system to be three parallel capacitors, and justify this assumption.] Show that you obtain the proper limits as the dielectric constants approach unity, $\kappa_i \rightarrow 1$.]



(c) Suppose the capacitor is filled as shown in the figure below. What is its capacitance? Use Gauss's law to find the field in each dielectric, and then calculate ΔV across the entire capacitor. Again, check your answer as the dielectric constants approach unity, $\kappa_i \rightarrow 1$. Could you have assumed that this system is equivalent to three capacitors in series?

Problem 3 Solutions:

(a) & (b) Since the potential difference on each part of the capacitor is the same, we may treat the system as being composed of three capacitors in parallel. Thus, the capacitance of the system is

$$C = C_1 + C_2 + C_3$$

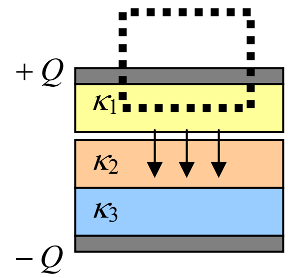
With

$$C_i = \frac{\kappa_i \varepsilon_0 (A/3)}{d}, \quad i = 1,2,3$$

we obtain

(c) We choose the Gaussian surface to be a cylinder whose bottom face is one of the three dielectric slabs. For example the figure shows the case where the bottom face is in dielectric 1. Since there is no electric field outside the capacitor, we have (using Gauss's law with a dielectric)

$$\oiint_S \kappa \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_0} = \kappa_i E_i A = \frac{\sigma A}{\varepsilon_0}$$



so $E_i = \dfrac{\sigma}{\kappa_i \varepsilon_0}$ where, $i = 1,2,3$ for the three different regions.

Therefore,

$$\Delta V = E_1\left(\frac{d}{3}\right) + E_2\left(\frac{d}{3}\right) + E_3\left(\frac{d}{3}\right) = \frac{Qd}{3\varepsilon_0 A}\left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \frac{1}{\kappa_3}\right)$$

and

$$C = \frac{Q}{\Delta V} = \frac{Q}{\Delta V} = \frac{3\varepsilon_0 A}{d}\left(\frac{\kappa_1\kappa_2\kappa_3}{\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1}\right)$$

This gives the correct limit when all the dielectric constants are unity.

$$C = \frac{\kappa_1\varepsilon_0 (A/3)}{d} + \frac{\kappa_2\varepsilon_0 (A/3)}{d} + \frac{\kappa_3\varepsilon_0 (A/3)}{d} = \frac{\varepsilon_0 A}{3d}(\kappa_1 + \kappa_2 + \kappa_3)$$

This gives the correct limit when all the dielectric constants are unity.