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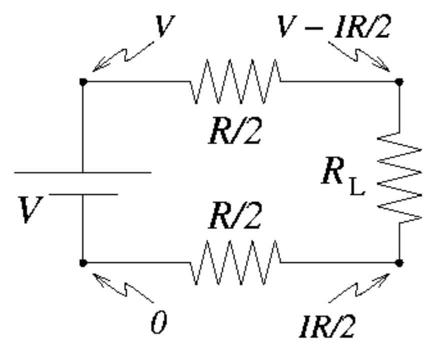
MIT 8.02 Spring 2002
Assignment #5 Solutions

Problem 5.1

High voltage power transmission.

The simple circuit diagram at right can be used in the analysis of this power transmission system. The power station acts as a "battery" of voltage V. Three "resistors", two of resistances $R/2$ (the power lines) and one of resistance R_{L} (the "load resistance" of the power consumers), are wired to the battery in series. Voltages are as shown at various points about the circuit. We find that the voltage delivered to the power consumers is $V - \Delta V = V - IR$, so therefore $\Delta V = IR$ is the total voltage drop along the power lines and R is the total resistance of the lines.



(Note: we have taken $V = 0$ at a particular point, but the choice is totally arbitrary. Voltage *differences* between points about the circuit are unaffected by this choice.)

(a) The resistivity of aluminum at +40°C and at −40°C can be found using Giancoli Equation (25-5) (p. 641):

$$\rho_T = \rho_0[1 + \alpha(T - T_0)] \quad .$$

Giancoli Table 25-1 (p. 640) gives $\rho_0 = 2.65 \times 10^{-8}\,\Omega \cdot \mathrm{m}$ and $\alpha = 0.00429°\mathrm{C}^{-1}$ for aluminum at $T_0 = 20°\mathrm{C}$. This gives us

$$\begin{aligned} \rho_{+40} &= (2.65 \times 10^{-8})[1 + (0.00429)(40 - 20)] = 2.88 \times 10^{-8}\,\Omega \cdot \mathrm{m} \quad , \\ \rho_{-40} &= (2.65 \times 10^{-8})[1 + (0.00429)(-40 - 20)] = 1.97 \times 10^{-8}\,\Omega \cdot \mathrm{m} \quad . \end{aligned}$$

Resistances can then be calculated using Giancoli Equation (25-3) (p. 640). First we convert units: 2×300 miles $= 9.66 \times 10^5$ m and $5\,\mathrm{cm}^2 = 5 \times 10^{-4}\,\mathrm{m}^2$.

$$\begin{aligned} R_{+40} &= (2.88 \times 10^{-8})(9.66 \times 10^5)/(5 \times 10^{-4}) = 56\,\Omega \quad , \\ R_{-40} &= (1.97 \times 10^{-8})(9.66 \times 10^5)/(5 \times 10^{-4}) = 38\,\Omega \quad . \end{aligned}$$

(b) By Ohm's law, the voltage drop along the lines is simply $\Delta V = IR$. $P = VI$ is the total power transmitted by the power station, so $I = P/V$, and we may write $\Delta V = RP/V$.

(c) Imposing $\Delta V = RP/V = (0.02)V$ gives us

$$V = \sqrt{RP/(0.02)} \quad .$$

Putting in $P = 2 \times 10^8\,\mathrm{W}$ and our two values for R, we find

$$V_{\mathrm{min},+40} = 7.5 \times 10^5\,\mathrm{V} \ , \qquad V_{\mathrm{min},-40} = 6.2 \times 10^5\,\mathrm{V} \ .$$

(d) The power dissipated in the lines will be

$$\Delta P = \Delta V I = (0.02) V I = (0.02) P = 4\,\mathrm{MW}$$

for both temperatures.

(e) Giancoli Equation (28-2) (p. 711) gives the force per unit length between two parallel current-carrying wires. Both of our wires carry the same current I, so (with $d = 8\,\mathrm{m}$ and $l = 25\,\mathrm{m}$)

$$F = \frac{\mu_0}{2\pi} \frac{I^2 l}{d} \ .$$

Using $I^2 = (0.02)P/R$ and plugging in known values gives

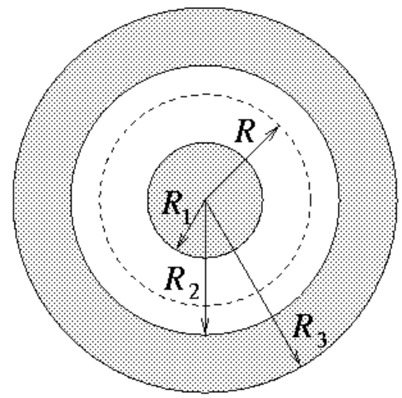
$$\begin{aligned} F_{+40} &= (2 \times 10^{-7})(0.02) \frac{(2 \times 10^8)(25)}{(56)(8)} = 0.045\,\mathrm{N} \ , \\ F_{-40} &= (2 \times 10^{-7})(0.02) \frac{(2 \times 10^8)(25)}{(38)(8)} = 0.066\,\mathrm{N} \ . \end{aligned}$$

Since the currents in the two wires run in opposite directions, this force between the wires is repulsive.

Problem 5.2

Ampere's law in action. (Giancoli 28-27.)

Consider the diagram at right, showing a cross-section through the cable. Assume the current I_0 flows "out of the page" in the core and "into the page" in the shell. The cylindrical symmetry of the system tells us that the magnetic field lines will be circles centered on the cable axis, and the magnetic field strength will depend only on distance R from the axis. So we may construct an "Amperian loop" of radius R centered on the cable axis as shown, knowing that $\mathbf{B}$ will always be tangent to the loop and will have a constant magnitude around the loop.



Applying Ampere's law to our loop with counterclockwise circulation, we find

$$\oint \mathbf{B}\cdot d\boldsymbol{l} = \oint B\,dl = B\oint dl = B(2\pi R) = \mu_o I_{\text{encl}}$$
$$\Rightarrow B = \frac{\mu_0 I_{\text{encl}}}{2\pi R} \quad ,$$

where B is the component of $\mathbf{B}$ in the counterclockwise azimuthal direction (the only non-zero component of $\mathbf{B}$). Our task now is to find I_{encl} for the various regions of interest.

(a) The current I_0 in the core is distributed uniformly throughout its cross-sectional area of πR_1^2. An Amperian loop of radius $R < R_1$ encloses an area of πR^2, and will therefore enclose a fraction of this total current given by the ratio of these two areas: $I_{\text{encl}} = I_0(R^2/R_1^2)$. Plugging this into our expression for B, we have

$$B = \frac{\mu_0 I_0 R}{2\pi R_1^2} \quad (R < R_1).$$

(Note that we have regarded the enclosed current as *positive*, since it flows in the same direction as the normal to the surface bounded by our Amperian loop, given our choice of counterclockwise circulation.)

(b) An Amperian loop of radius R in the region $R_1 < R < R_2$ will enclose *all* of the current I_0 in the core and *none* of the current in the shell, so we simply have

$$B = \frac{\mu_0 I_0}{2\pi R} \quad (R_1 < R < R_2).$$

(c) The outer shell carries a current $-I_0$ (negative because it is directed opposite to the normal to the surface bounded by our counterclockwise Amperian loop) which is distributed uniformly throughout its cross-sectional area of $\pi(R_3^2 - R_2^2)$. A loop in the region $R_2 < R < R_3$ will enclose $\pi(R^2 - R_2^2)$ worth of shell cross-section, and thus will enclose a fraction of the shell current given by $-I_0(R^2 - R_2^2)/(R_3^2 - R_2^2)$. The loop also encloses the full I_0 of the core, so in this region $I_{\text{encl}} = I_0[1 - (R^2 - R_2^2)/(R_3^2 - R_2^2)]$. Plugging in to get B:

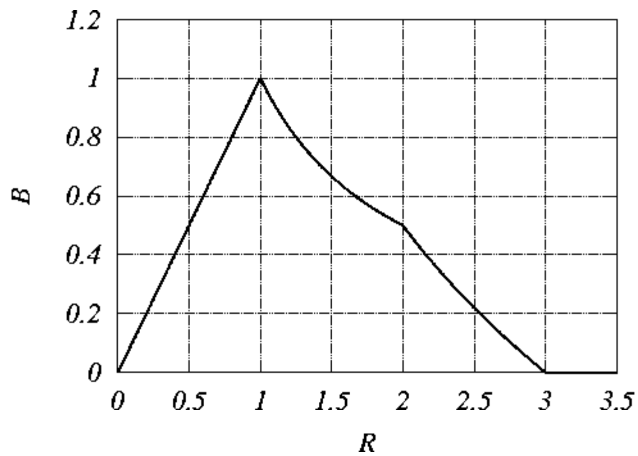
$$B = \frac{\mu_0 I_0}{2\pi R}\left[1 - \frac{(R^2 - R_2^2)}{(R_3^2 - R_2^2)}\right] \qquad (R_2 < R < R_3).$$

(d) Finally, in the region $R > R_3$, our loop encloses the full I_0 of the core and the full $-I_0$ of the shell. $I_{\text{encl}} = 0$, giving

$$B = 0 \qquad (R > R_3).$$

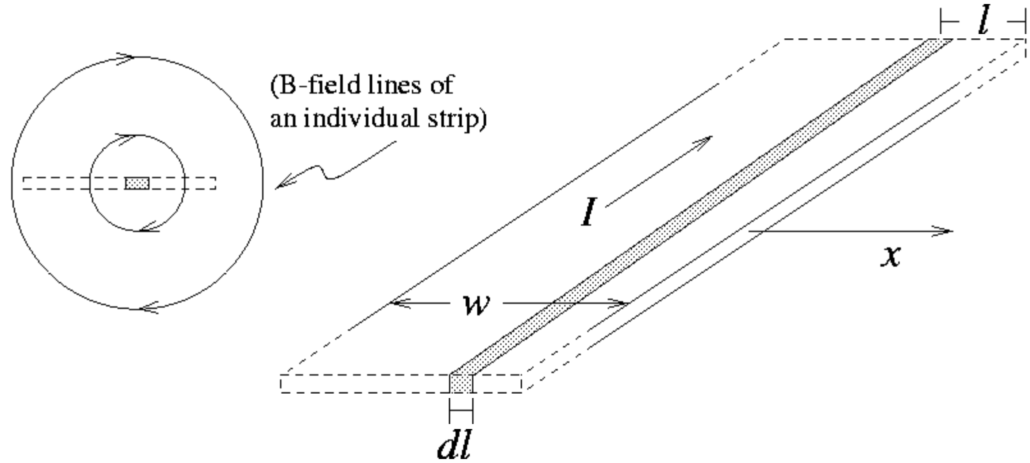
(Note: when we say "loop X encloses current Y", strictly speaking we mean "current Y passes through an open surface bounded by loop X".)

<u>Beyond the call of homework-duty:</u>
Just to get a better feel for the dependence of B on R, we can take $R_2 = 2R_1$, $R_3 = 3R_1$ and plot $B(R)$, as shown at right. R is plotted in units of R_1 and B in units of $\mu_0 I_0/2\pi R_1$. Note that $B(R)$ is continuous everywhere. Discontinuities in B only come from sheets of current, just as discontinuities in E only come from sheets of charge.



Problem 5.3

Magnetic field of a current-carrying ribbon.



To solve for the magnetic field of a current-carrying ribbon, we break it up conceptually (and mathematically) into many infinitesimal strips which may be regarded as wires. The principle of linear superposition tells us that the magnetic field of the ribbon will be the vector sum of the magnetic field contributions of the individual strips.

Consider a little strip of width dl, located a distance l from the right edge of the ribbon as shown in the diagram above. The strip carries a current $I\,dl/w$ and is located a distance $l+x$ from the point in the plane of the ribbon at which we wish to determine the field. The differential contribution to the field at the point of interest due to this strip is just the field due to a long straight wire:

$$dB = \frac{\mu_0 I\,dl/w}{2\pi(x+l)} \quad ,$$

directed downward by the right-hand rule. To find the total field we simply integrate over the ribbon from $l = 0$ to $l = w$ (making the substitution $u = x + l$, $du = dl$):

$$B = \int dB = \frac{\mu_0 I}{2\pi w}\int_0^w \frac{dl}{(x+l)} = \frac{\mu_0 I}{2\pi w}\int_x^{x+w}\frac{du}{u} = \frac{\mu_0 I}{2\pi w}\ln\left(1+\frac{w}{x}\right) \quad .$$

(Again, the field is directed downward, perpendicular to both the plane of the ribbon and the direction of current flow.)

To examine the limit $w \to 0$, we make use of the first-order Taylor expansion of the natural logarithm: $\ln(1+\delta) \simeq \delta$ for $\delta \ll 1$ (check out Giancoli Appendix A-3). As $w \to 0$, $w/x \ll 1$, so

$$B \to \frac{\mu_0 I}{2\pi w}\left(\frac{w}{x}\right) = \frac{\mu_0 I}{2\pi x} \quad .$$

This is just what we would expect a distance x away from a wire carrying current I.

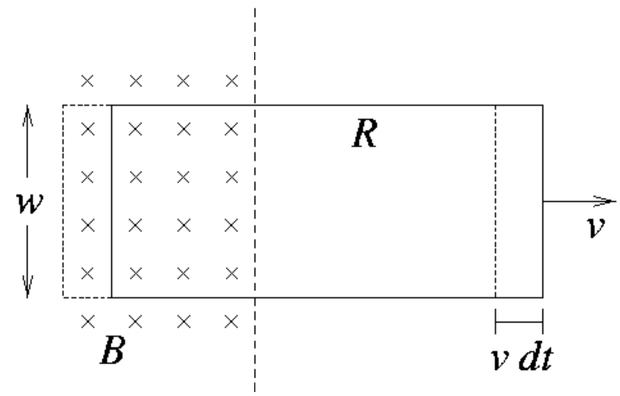
We have taken the limit $w \to 0$ at fixed x (the "skinny ribbon limit"), but we could equally well view this as the limit $x \to \infty$ at fixed w (the "far away limit"): far enough away, *any* ribbon will look like a wire. What is important is that $w/x \ll 1$.

Problem 5.4

Force on loop. (Giancoli 29-14.)

For convenience, we'll define the following variables to work with:

$$\begin{aligned} w &= 0.350\,\text{m} \\ B &= 0.450\,\text{T} \\ R &= 0.230\,\Omega \\ v &= 3.40\,\text{m/s} \end{aligned}$$



In a small time interval dt, the loop moves a distance $v\,dt$, and thus the area of the loop within the B-field region decreases by an amount $dA = wv\,dt$. The field magnitude remains constant, so the magnetic flux through the plane surface bounded by the loop decreases by an amount $d\Phi_B = B\,dA = Bwv\,dt$. By Faraday's law (Giancoli Equation (29-2a), p. 736), the magnitude of the induced emf will be

$$|\mathcal{E}| = \frac{d\Phi_B}{dt} = Bwv \quad .$$

This will induce a clockwise current in the loop (clockwise to oppose the change in flux: that's Lenz's law) of magnitude

$$I = |\mathcal{E}|/R = Bwv/R \quad .$$

As this current flows upwards through the left edge of the loop in the B-field region, that edge will experience a force directed to the left as given by the Lorentz force law (in the form of Giancoli Equation (27-3), p. 691):

$$F = IwB = B^2w^2v/R = (0.450)^2(0.350)^2(3.40)/(0.230) = 0.367\,\text{N} \quad .$$

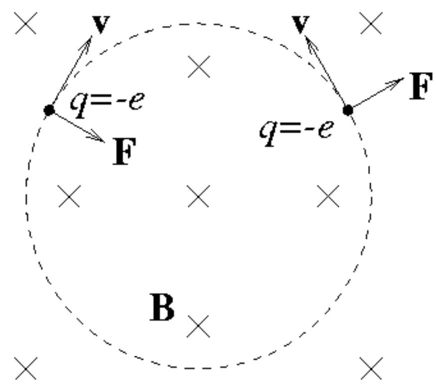
So to keep the loop moving with constant speed, someone or something must pull the loop to the right with an equal 0.367 N of force.

Problem 5.5

Betatron. (Giancoli 29-49.)

(a) The changing magnetic field changes the magnetic flux through an open surface bounded by the electron orbital path. By Faraday's law this will induce an emf around the vacuum tube: a non-zero $\oint \mathbf{E} \cdot d\boldsymbol{l}$, the $\mathbf{E}$ of which will serve to accelerate the electrons.

(b) The diagram at right corresponds to Giancoli Figure 29-37 (p. 754) as viewed looking down from above. Electrons moving in the vacuum tube are held in their circular orbits by a magnetic force given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. If the electrons were moving counterclockwise, this force would be directed *outwards*, and would not serve to hold the electrons in orbit (see diagram, and remember: the charge of the electron is *negative*). However, clockwise-moving electrons will feel an *inward* magnetic force. So the electrons must be moving clockwise.



(c) To accelerate our clockwise-moving electrons to higher speeds, we need a *counter*clockwise emf, since negatively charged particles accelerate in the direction opposite to the E-field. Faraday-Lenz tells us that *increasing* the magnitude of the B-field "into the paper" in our diagram above will induce such a counterclockwise emf, for the magnetic field due to a counterclockwise electric current would tend to offset the change in flux due to such an increase.

(d) If the electromagnet is AC, then the magnetic field will vary sinusoidally in time. Say it starts from zero and begins increasing into the page. During the first quarter-cycle, the B-field will have the right direction for confining clockwise-orbiting electrons *and* it will be changing in such a way as to induce a counterclockwise emf that will accelerate these electrons to higher speeds. But starting with the second quarter-cycle, the B-field magnitude will be decreasing. The field direction will still serve to confine clockwise electrons, but the induced emf will now be *clockwise*, and will tend to slow these electrons down. So useful acceleration is only possible during one quarter-cycle.

Problem 5.6

Intuition breaks down.
("Test 1" of March 15 lecture supplement.)

We may analyze this situation in exacly the same way as is done for the example circuit in the Lecture Supplement. There is one important change: *if we attach to the left closed loop an open surface, there now is a magnetic flux change through that surface.* So equations (3)–(5) of the supplement become

$$\begin{array}{ll} \text{Left loop:} & I_1 R_i + I_1 R_1 - I R_1 = \mathcal{E} \quad , \\ \text{Middle loop:} & I R_1 + I R_2 - I_1 R_1 - I_2 R_2 = \mathcal{E} \quad , \\ \text{Right loop:} & I_2 R_2 - I R_2 + I_2 R_i = 0 \quad . \end{array}$$

Only the left-loop equation has changed, to reflect the use of Faraday's law instead of Kirchhoff's 2nd. The same approximations as described in the supplement hold, so the equations become

$$I_1 R_i - I R_1 \approx \mathcal{E} \quad , \tag{1}$$
$$I(R_1 + R_2) \approx \mathcal{E} \quad , \tag{2}$$
$$I_2 R_i - I R_2 \approx 0 \quad . \tag{3}$$

If we connect the two voltmeter terminals as described in the supplement ("+" to A-side and "−" to D-side), then positive I_1 will give positive V_1 and positive I_2 will give negative V_2. From (1) and (3) above, we have

$$\begin{aligned} |V_1| &= I_1 R_i \approx I R_1 + \mathcal{E} \quad , \\ |V_2| &= I_2 R_i \approx I R_2 \quad . \end{aligned}$$

(2) tells us that $\mathcal{E}$ and I will always have the same sign, and therefore it follows from (1) and (3) that I_1 and I_2 have the same sign as I. Therefore V_1 and V_2 will always have opposite signs. The relative voltage magnitude is

$$|V_1/V_2| \approx \frac{I R_1 + \mathcal{E}}{I R_2} = \frac{R_1}{R_2} + \frac{\mathcal{E}}{I R_2} \quad .$$

Eliminating $\mathcal{E}/I$ using (2), we get

$$|V_1/V_2| \approx 1 + 2R_1/R_2 \quad .$$

If we take the numerical values given in the supplement example, with $R_1/R_2 = 1/9$, we have $|V_1/V_2| \approx 11/9$.

Consider now the possibility of not 1 but 100 windings of the left-hand loop about the entire circuit (this is the "Test 2" scenario): $\mathcal{E} \to 100\mathcal{E}$ in equation (1) above ($\mathcal{E}$ here is taken to be the value of $-d\Phi_B/dt$ through the surface of ONE LOOP!), leading to

$$\begin{aligned} |V_1| &= I_1 R_i \approx I R_1 + 100\mathcal{E} \quad , \\ |V_2| &= I_2 R_i \approx I R_2 \quad , \end{aligned}$$

and then

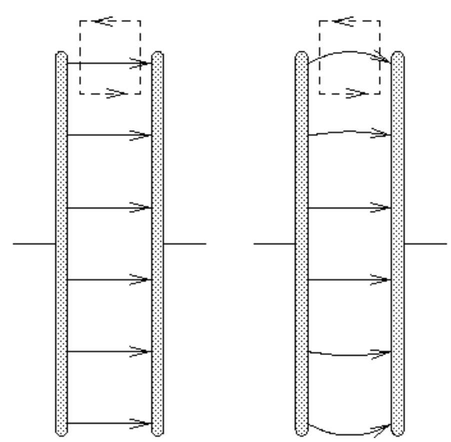
$$|V_1/V_2| \approx \frac{IR_1 + 100\mathcal{E}}{IR_2} = \frac{R_1}{R_2} + 100\frac{\mathcal{E}}{IR_2} = 100 + 101\frac{R_1}{R_2} \quad .$$

For our numbers, this gives $|V_1/V_2| \approx 111$. V_2 has the same value as it had before ($\frac{9}{10}$ Volt), but now V_1 reads $\frac{9}{10}(111) \approx 100$ Volts, which is one hundred times the EMF $\mathcal{E}$! This is the basic idea behind transformers.

Problem 5.7

Fringe fields. (Giancoli 29-69.)

Consider first the possibility labeled (a) in the figure at right: the electric field between the capacitor plates is uniform and directed exactly to the right, and drops abruptly to zero outside. If we integrate $\oint \mathbf{E} \cdot d\boldsymbol{l}$ counterclockwise around the dashed path, we will get a positive contribution from the bottom segment, no contribution from the top segment ($\mathbf{E} = 0$ there), and no contributions from either of the side segments ($\mathbf{E} \perp d\boldsymbol{l}$ there). So we will definitely get $\oint \mathbf{E} \cdot d\boldsymbol{l} \neq 0$. At the same time, this is supposedly an electrostatic configuration with *no* magnetic fields (time-varying or otherwise) so that $d\Phi_B/dt = 0$ through any open surface bounded by the loop. But Faraday's law in the form of Giancoli Equation (29-8) (p. 747) tells us that $\oint \mathbf{E} \cdot d\boldsymbol{l} = -d\Phi_B/dt$! So this is not a realizable static electric field.



(a) (b)

Now consider the possibility illustrated by (b): the E-field "fringes" a bit at the edge of the capacitor plates. Integrating $\oint \mathbf{E} \cdot d\boldsymbol{l}$ around the same closed path as before, we will still get a significant positive contribution from the bottom segment. But now, as we integrate up the right-side segment, we will get some negative contribution due to the slight downward component of $\mathbf{E}$. Integrating to the left along the top segment, we will get another negative contribution, since $\mathbf{E}$ will not have dropped exactly to zero. Integrating down along the left-side edge, we get yet another negative contribution, due to the slight upward component of $\mathbf{E}$ there. Again we conclude that $d\Phi_B/dt = 0$ due to the absence of magnetic fields. However, our results are no longer inconsistent with Faraday's law, for the positive and negative contributions to our integral $\oint \mathbf{E} \cdot d\boldsymbol{l}$ may (and in fact must!) cancel one another out to give zero. Therefore (b) depicts the actual static electric field configuration of a charged parallel-plate capacitor.

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