Rotational Dynamics

8.01t

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Fixed Axis Rotation: Angular Velocity and Angular Acceleration

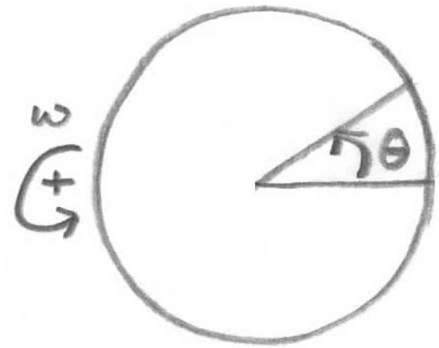
Angle variable θ

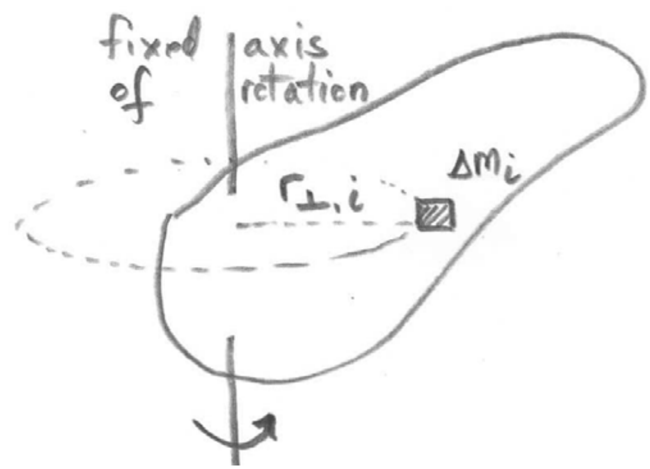
Angular velocity $\omega \equiv \dfrac{d\theta}{dt}$

Angular acceleration $\alpha \equiv \dfrac{d^2\theta}{dt^2}$

Mass element Δm_i

Radius of orbit $r_{\perp,i}$





Fixed Axis Rotation: Tangential Velocity and Tangential Acceleration

- Individual mass elements Δm_i

- Tangential velocity $v_{\tan,i} = r_{\perp,i}\omega$

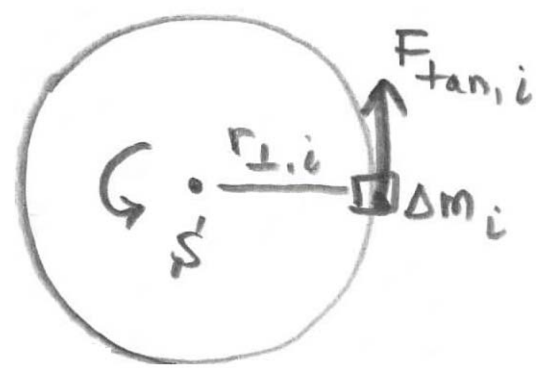
- Tangential acceleration $a_{\tan,i} = r_{\perp,i}\alpha$

- Radial Acceleration

$$a_{rad,i} = \frac{v_{\tan,i}^2}{r_{\perp,i}} = r_{\perp,i}\omega^2$$

Newton's Second Law

- Tangential force on mass element produces torque



- Newton's Second Law

$$F_{\tan,i} = \Delta m_i a_{\tan,i}$$

$$F_{\tan,i} = \Delta m_i r_{\perp,i} \alpha$$

- Torque

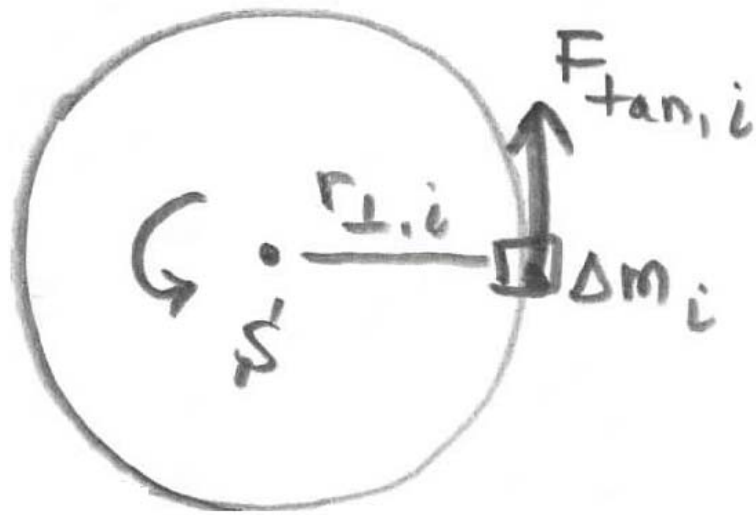
$$\vec{\boldsymbol{\tau}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i$$

Torque

Torque about is S:

$$\vec{\boldsymbol{\tau}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i$$

- Counterclockwise
- perpendicular to the plane



$$\tau_{S,i} = r_{\perp,i} F_{\tan,i} = \Delta m_i (r_{\perp,i})^2 \alpha$$

Moment of Inertia

- Total torque is the sum over all mass elements

$$\tau_S^{total} = \tau_{S,1} + \tau_{S,2} + \ldots = \sum_{i=1}^{i=N} \tau_{S,i} = \sum_{i=1}^{i=N} r_{\perp,i} F_{\tan,i} = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \alpha$$

- Moment of Inertia about S:

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2$$

- Unit: $[kg - m^2]$

- Summary: $\tau_S^{total} = I_S \alpha$

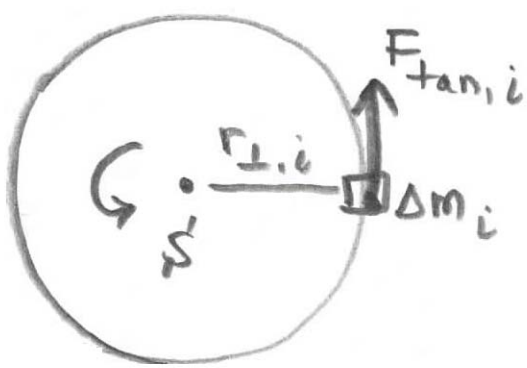
Rotational Work

- tangential force $\vec{\mathbf{F}}_{\tan,i} = F_{\tan,i}\hat{\boldsymbol{\theta}}$

- displacement vector $\Delta\vec{\mathbf{r}}_{S,i} = \left(r_{S,\perp}\right)_i \Delta\theta\hat{\boldsymbol{\theta}}$

- infinitesimal work

$$\Delta W_i = \vec{\mathbf{F}}_{\tan,i} \cdot \Delta\vec{\mathbf{r}}_{S,i} = F_{\tan,i}\hat{\boldsymbol{\theta}} \cdot \left(r_{S,\perp}\right)_i \Delta\theta\hat{\boldsymbol{\theta}} = F_{\tan,i}\left(r_{S,\perp}\right)_i \Delta\theta$$



Rotational Work

- Newton's Second Law $$F_{\tan,i} = \Delta m_i a_{\tan,i}$$
- tangential acceleration $$a_{\tan,i} = \left(r_{S,\perp}\right)_i \alpha$$
- infinitesimal work $$\Delta W_i = \Delta m_i \left(r_{S,\perp}\right)_i^2 \alpha \Delta\theta$$
- summation

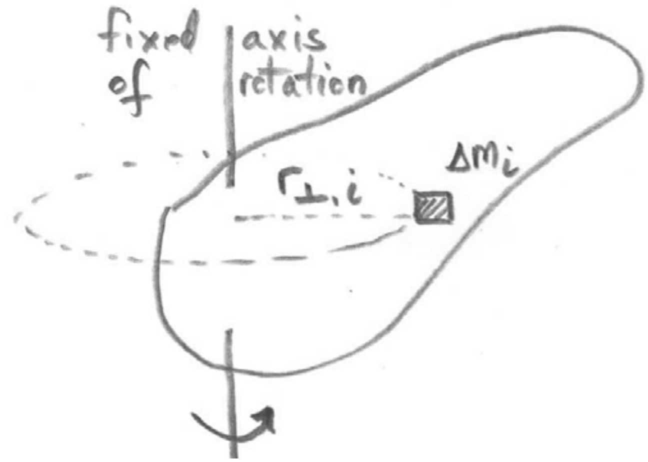
$$\Delta W = \left(\sum_i \Delta m_i \left(r_{S,\perp}\right)_i^2 \right) \alpha \Delta\theta = \left(\int_{body} dm \left(r_{S,\perp}\right)^2 \right) \alpha \Delta\theta = I_S \alpha \Delta\theta$$

Rotational Work

- infinitesimal rotational work $\Delta W = I_S \alpha \Delta\theta$
- torque $\tau_S = I_S \alpha$
- infinitesimal rotational work $\Delta W = \tau_S \Delta\theta$
- Integrate total work $W = \int_{\theta=\theta_0}^{\theta=\theta_f} dW = \int_{\theta=\theta_0}^{\theta=\theta_f} \tau_S d\theta$

Rotational Kinetic Energy

Rotational kinetic energy



$$K_{cm,i} = \frac{1}{2}\Delta m_i v_{cm,i}{}^2 = \frac{1}{2}\Delta m_i \left(r_{cm,\perp}\right)_i^2 \omega_{cm}{}^2$$

$$K_{cm} = \sum_i K_{cm,i} = \left(\sum_i \frac{1}{2}\Delta m_i \left(r_{cm,\perp}\right)_i^2\right)\omega_{cm}{}^2 = \left(\frac{1}{2}\int_{body} dm\left(r_{cm,\perp}\right)^2\right)\omega_{cm}{}^2 = \frac{1}{2}I_{cm}\omega_{cm}{}^2$$

- Total kinetic energy

$$K_{total} = K_{trans} + K_{rot} = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega_{cm}^2$$

PRS - kinetic energy

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. Its total kinetic energy is:

1. $1/4\ M\ R^2\ \omega^2$
2. $1/2M\ R^2\ \omega^2$
3. $3/4\ M\ R^2\ \omega^2$
4. $1/4M\ R\ \omega^2$
5. $1/4\ M\ R\ \omega^2$
6. $1/2\ M\ R\ \omega$

Rotational Work-Kinetic Energy Theorem

- angular velocity $\omega \equiv d\theta / dt$

- angular acceleration $\alpha \equiv d\omega / dt$

- infinitesimal rotational work

$$dW_{rot} = I_S \alpha d\theta = I_S \frac{d\omega}{dt} d\theta = I_S d\omega \frac{d\theta}{dt} = I_S d\omega \omega$$

- integrate rotational work

$$W_{rot} = \int_{\omega=\omega_0}^{\omega=\omega_f} dW_{rot} = \int_{\omega=\omega_0}^{\omega=\omega_f} I_S d\omega \omega = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_0^2$$

Rotational Work-Kinetic Energy Theorem

- Fixed axis passing through a point S in the body

$$W_{rot} = \frac{1}{2} I_{cm} \omega_{cm,f}^2 - \frac{1}{2} I_{cm} \omega_{cm,0}^2 = K_{rot,f} - K_{rot,0} \equiv \Delta K_{rot}$$

- Rotation and translation

$$W_{total} = \Delta K_{trans} + \Delta K_{rot}$$

$$W_{total} = W_{trans} + W_{rot} = \left(\frac{1}{2} m v_{cm,f}^2 - \frac{1}{2} m v_{cm,0}^2 \right) + \left(\frac{1}{2} I_{cm} \omega_f^2 - \frac{1}{2} I_{cm} \omega_0^2 \right)$$

Concept Question

- Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder *A* has most of its mass concentrated at the rim, while cylinder *B* has most of its mass concentrated at the center. Which is moving faster at the bottom?

1) A

2) B

3) Both have the same

Concept Question

- Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder *A* has most of its mass concentrated at the rim, while cylinder *B* has most of its mass concentrated at the center. Which has more total kinetic energy at the bottom?

1) A

2) B

3) Both have the same

Rotational Power

- rotational power is the time rate of doing rotational work

$$P_{rot} \equiv \frac{dW_{rot}}{dt}$$

- product of the applied torque with the angular velocity

$$P_{rot} \equiv \frac{dW_{rot}}{dt} = \tau_{S,\perp} \frac{d\theta}{dt} = \tau_{S,\perp} \omega$$

Class Problem

A turntable is a uniform disc of mass *m* and a radius R. The turntable is spinning initially at a constant frequency f . The motor is turned off and the turntable slows to a stop in t seconds. Assume that the angular acceleration is constant. The moment of inertia of the disc is I.

a) What is the initial angular velocity of the turntable?

b) What is the angular acceleration of the turntable?

c) What is the magnitude of the frictional torque acting on the disc?

d) How much work is done by the frictional torque?

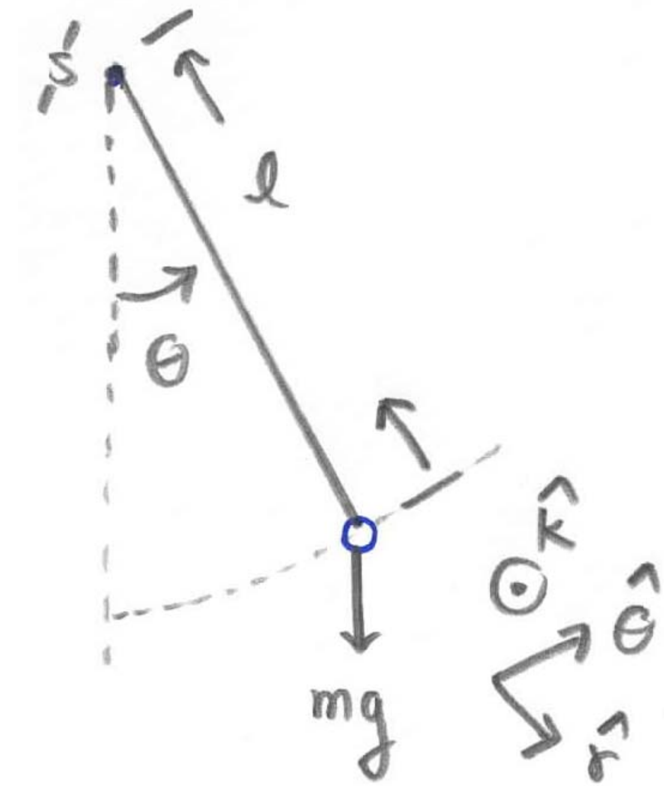
e) What is the change in kinetic energy of the turntable?

f) Graph the rotational power as a function of time.

Simple Pendulum

Pendulum: bob hanging from end of string

- Pivot point
- bob of negligible size
- massless string



Simple Pendulum: Torque Diagram

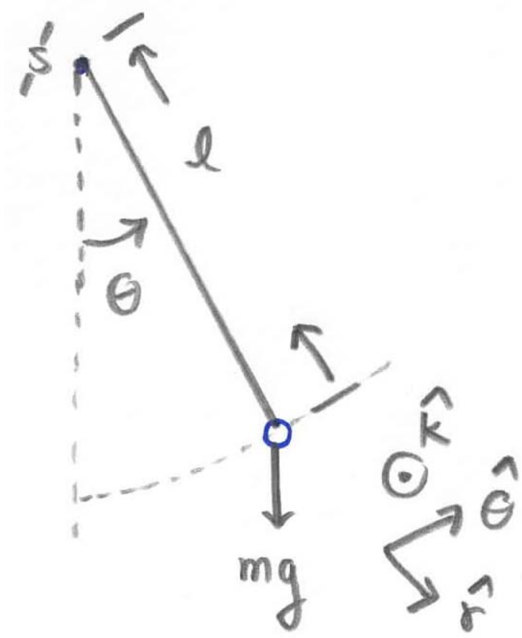
torque about the pivot point

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{s,m} \times m\vec{\mathbf{g}} = l\hat{\mathbf{r}} \times mg(-\sin\theta\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -lmg\sin\theta\hat{\mathbf{k}}$$

angular acceleration

(vector quantity)

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}$$



- Points along axis
- Positive or negative

Simple Pendulum: Rotational Equation of Motion

- moment of inertial of a point mass about the pivot point

$$I_S = ml^2$$

- Rotational Law of Motion $\vec{\boldsymbol{\tau}}_S = I_S \vec{\boldsymbol{\alpha}}$

- Simple pendulum oscillator equation

$$-lmg\sin\theta = ml^2 \frac{d^2\theta}{dt^2}$$

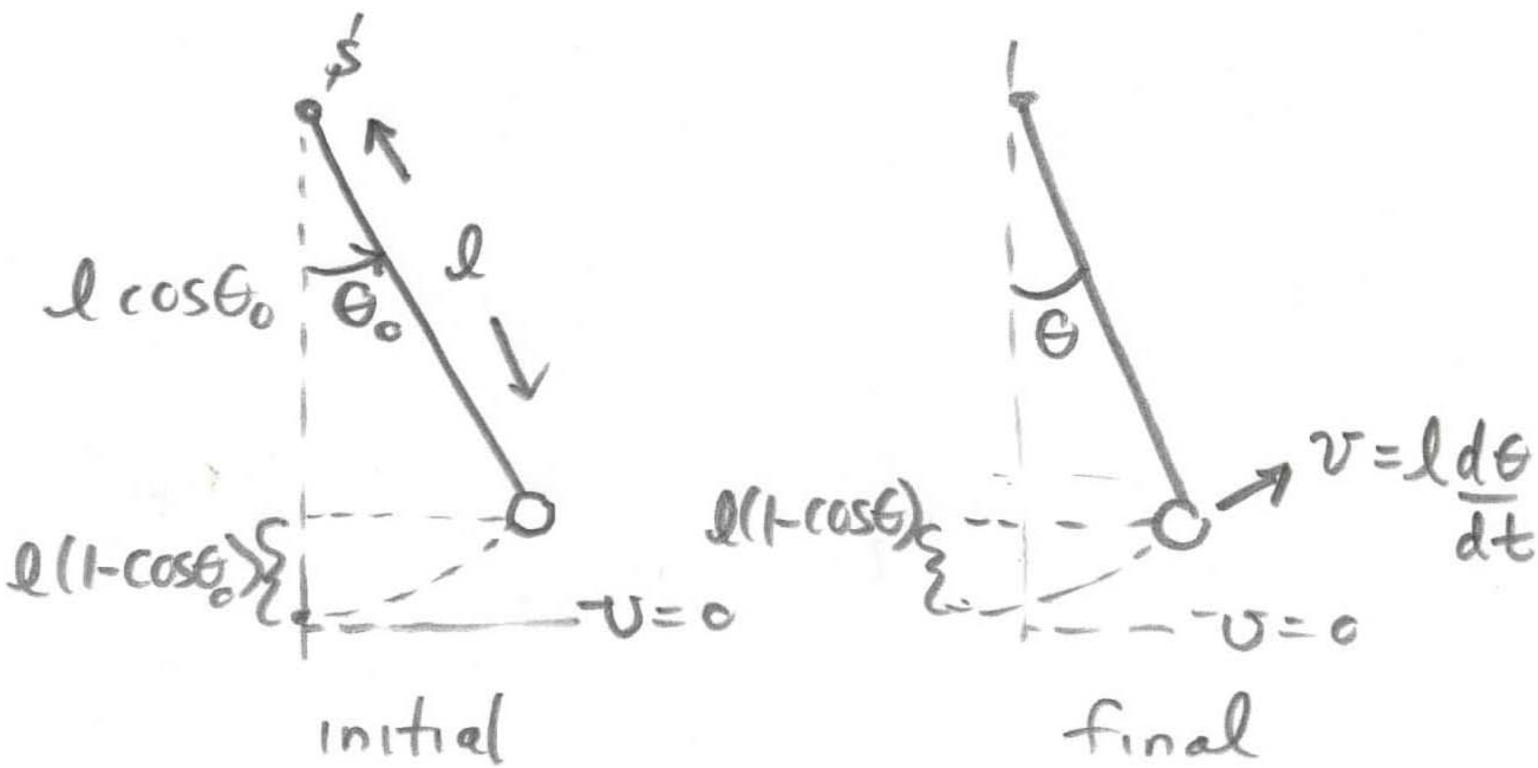
Simple Pendulum: Small Angle Approximation

Angle of oscillation is small $\sin\theta \cong \theta$

- Simple harmonic oscillator $\dfrac{d^2\theta}{dt^2} \cong -\dfrac{g}{l}\theta$
- Analogy to spring equation $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$
- Angular frequency of oscillation $\omega_{pendulum} \cong \sqrt{\dfrac{g}{l}}$
- Period $T_0 = \dfrac{2\pi}{\omega_p} \cong 2\pi\sqrt{\dfrac{l}{g}}$

Simple Pendulum: Mechanical Energy

- released from rest at an angle θ_0



Extra Topic: Simple Pendulum: Mechanical Energy

- Velocity $$v_{\tan} = l\frac{d\theta}{dt}$$
- Kinetic energy $$K_f = \frac{1}{2}mv_{\tan}^2 = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2$$
- Initial energy $$E_0 = K_0 + U_0 = mgl(1-\cos\theta_0)$$
- Final energy $$E_f = K_f + U_f = \frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta\ )$$
- Conservation of energy

$$\frac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

Simple Pendulum: Angular Velocity Equation of Motion

- Angular velocity

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{(\cos\theta - \cos\theta_0)}$$

- Integral form

$$\int \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}} = \int \sqrt{\frac{2g}{l}}dt$$

Simple Pendulum: Integral Form

- Change of variables

$$b\sin a = \sin\left(\theta/2\right)$$

$$b = \sin\left(\theta_0/2\right)$$

- Integral form

$$\int \frac{da}{\left(1-b^2\sin^2 a\right)^{1/2}} = \int \sqrt{\frac{g}{l}}dt$$

- Power series approximation

$$\left(1-b^2\sin^2 a\right)^{-1/2} = 1+\frac{1}{2}b^2\sin^2 a+\frac{3}{8}b^4\sin^4 a+\ldots$$

- Solution

$$2\pi+\frac{1}{2}\pi\sin^2\left(\theta_0/2\right)+\ldots=\sqrt{\frac{g}{l}}T$$

Simple Pendulum: First Order Correction

- period $$T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\left(\theta_0/2\right)+\ldots\right) \qquad T_0 = 2\pi\sqrt{\frac{l}{g}}$$

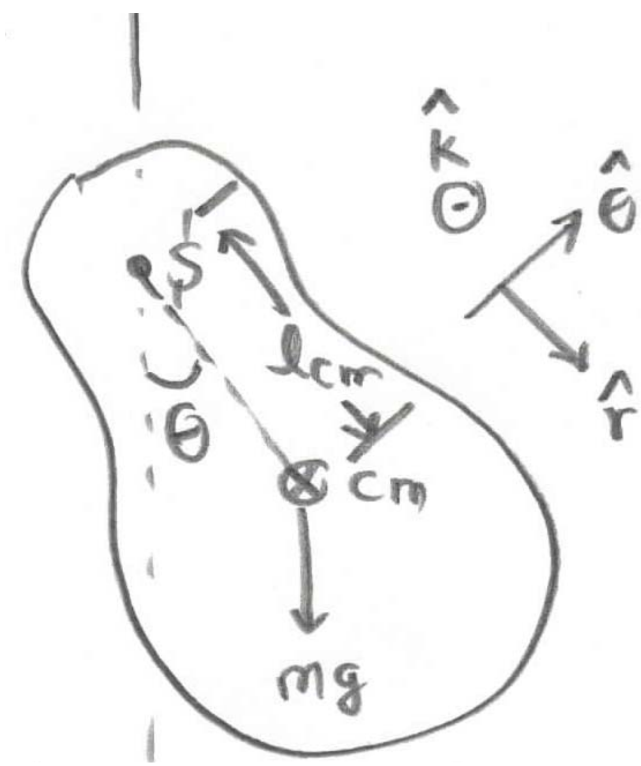
- initial angle is small $$\sin^2\left(\theta_0/2\right) \cong \theta_0^{\ 2}/4$$

- Approximation $$T \cong 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_0^{\ 2}\right) = T_0\left(1+\frac{1}{16}\theta_0^{\ 2}\right)$$

- First order correction $$\Delta T_1 \cong \frac{1}{16}\theta_0^{\ 2}T_0$$

Physical Pendulum

- pendulum pivoted about point S
- gravitational force acts center of mass
- center of mass distance l_{cm} from the pivot point



Physical Pendulum

- torque about pivot point

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{s,cm} \times m\vec{\mathbf{g}} = l_{cm}\hat{\mathbf{r}} \times mg(-\sin\theta\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -l_{cm}mg\sin\theta\hat{\mathbf{k}}$$

- moment of inertial about pivot point I_S

- Example: body is a uniform rod of mass m and length l .

$$I_S = \frac{1}{3}ml^2$$

Physical Pendulum

- rotational dynamical equation $\vec{\boldsymbol{\tau}}_S = I_S \vec{\boldsymbol{\alpha}}$
- small angle approximation $\sin\theta \cong \theta$
- Equation of motion $\dfrac{d^2\theta}{dt^2} \cong -\dfrac{l_{cm} mg}{I_S}\theta$
- Angular frequency $\omega_{pendulum} \cong \sqrt{\dfrac{l_{cm} mg}{I_S}}$
- Period $T = \dfrac{2\pi}{\omega_p} \cong 2\pi\sqrt{\dfrac{I_S}{l_{cm} mg}}$

PRS - linear momentum

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. The magnitude of its linear momentum is:

1. $1/2\ M\ R^2\ \omega$

2. $M\ R^2\ \omega$

3. $1/2\ M\ R\ \omega^2$

4. $M\ R\ \omega^2$

5. $1/2\ M\ R\ \omega$

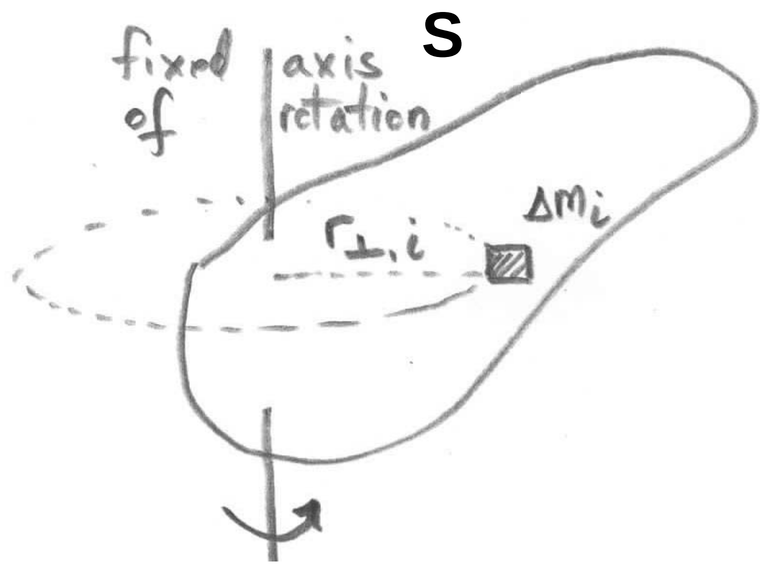
6. $M\ R\ \omega$

PRS: Rigid Body Rotation

A rigid body rotates about an axis, S.

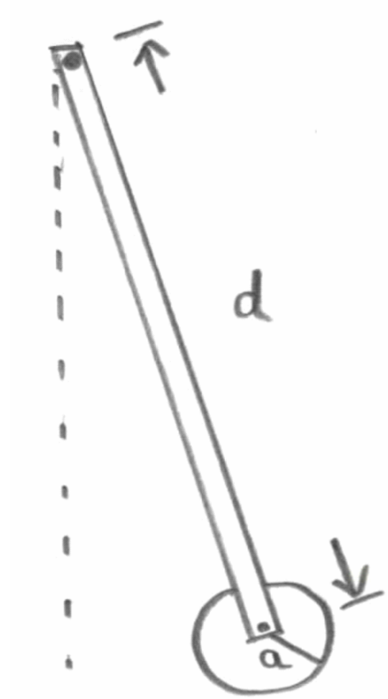
What is the relationship between the rotation rate about S vs about the center of mass? Also, what is the relationship between the rotational accelerations about these points?

1. $\alpha_{cm} = \alpha_S$ and $\omega_{cm} = \omega_S$
2. $\alpha_{cm} = \alpha_S$ but $\omega_{cm} \neq \omega_S$
3. $\omega_{cm} = \omega_S$ but $\alpha_{cm} \neq \alpha_S$
4. $\alpha_{cm} \neq \alpha_S$ and $\omega_{cm} \neq \omega_S$
5. None of above is consistently true



Concept Question: Physical Pendulum

A physical pendulum consists of a uniform rod of length l and mass m pivoted at one end. A disk of mass m_1 and radius a is fixed to the other end. Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin. Does the period of the pendulum



a) increase?
b) stay the same?
c) decrease?

Class Problem: Physical Pendulum

A physical pendulum consists of a uniform rod of length l and mass m pivoted at one end. A disk of mass m_1 and radius a is fixed to the other end.

a) Find the period of the pendulum.

Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin.

b) Find the new period of the pendulum.