REVIEW #2B

Circular Dynamics

Statics and Torque

Work and Energy

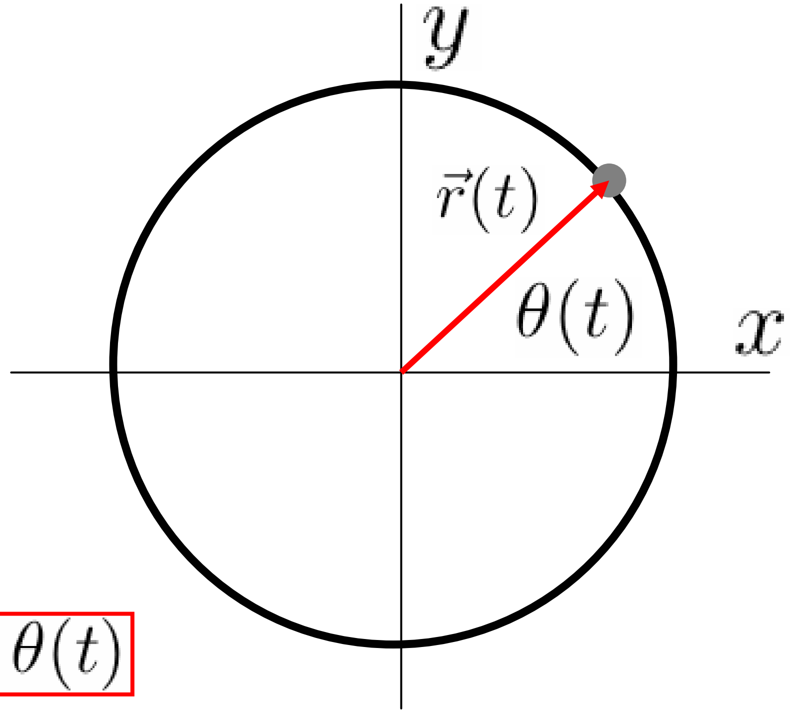
Harmonic Motion

Circular Dynamics

Circular Motion: Position

- The position vector of an object moving in a circular orbit of radius R

$$\vec{r}(t) = R\hat{r}(t)$$



- Components:

$$\vec{r}(t) = R\cos\theta(t)\hat{i} + R\sin\theta(t)\hat{j}$$

$$x(t) = R\cos\theta(t) \qquad y(t) = R\sin\theta(t)$$

- The angle θ(t) is a function of time!
- Uniform Circular motion:

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \text{const.}$$

Angular velocity and velocity

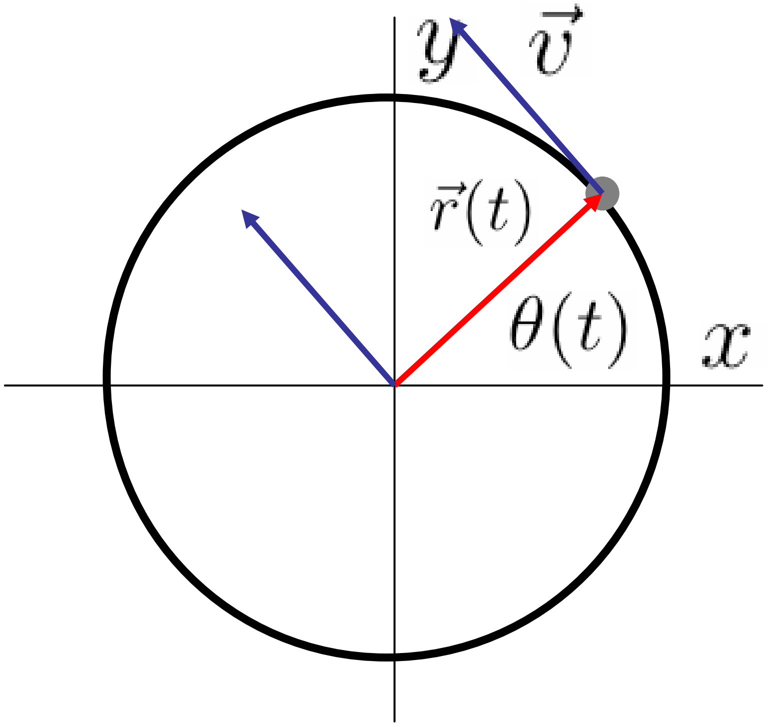
- Angular velocity

$$\omega \equiv \frac{d\theta}{dt}$$

- Units

  Not degrees/sec!

$$\left[\mathrm{rad} \cdot \mathrm{sec}^{-1}\right]$$



- Magnitude and direction of velocity

$$\vec{r}(t) = R\cos\theta(t)\hat{i} + R\sin\theta(t)\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -R\omega(t)\sin\theta(t)\hat{i} + R\omega(t)\cos\theta(t)\hat{j}$$

$$|\vec{v}| = R\omega$$

- Period (T) and frequency (f):

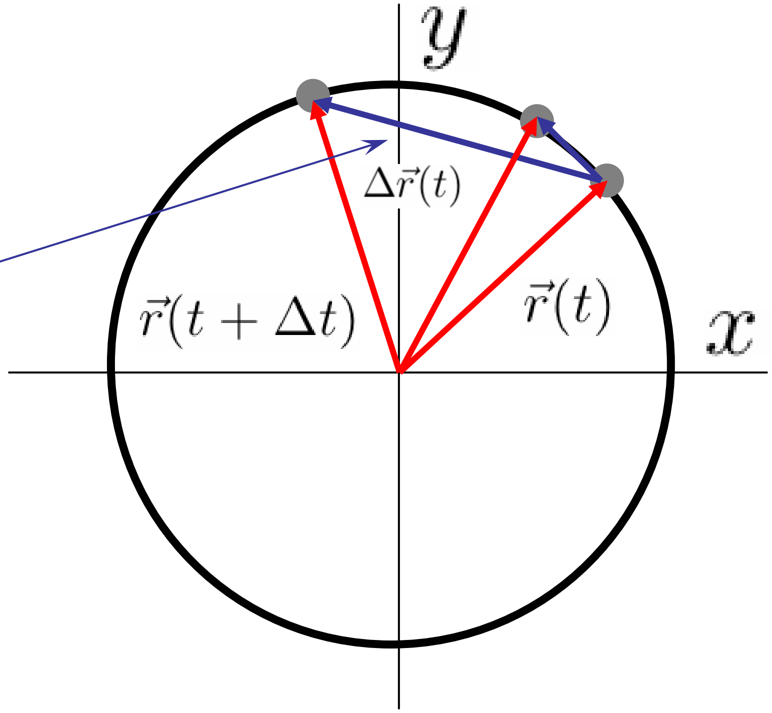
$$T = \frac{2\pi}{\omega}$$

$$f = \frac{1}{2\pi\omega}$$

Direction of the velocity

- Sequence of chord directions as Δt approaches zero.

Δt $\Delta\vec{\mathbf{r}}$



$$\Delta\vec{r}(t) = \vec{r}(t+\Delta t) - \vec{r}(t)$$

- The direction of the velocity at time t is perpendicular to the position vector i.e. tangent to the circular orbit!

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} =$$

$$= R\omega(t)(-\sin\theta(t)\hat{i} + \cos\theta(t)\hat{j})$$

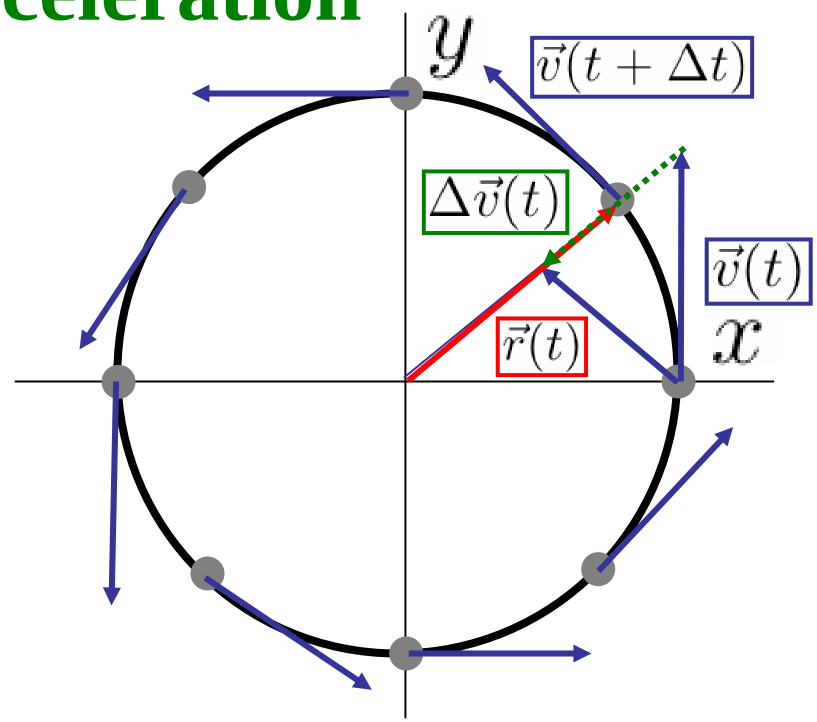
$$= R\omega(t)\hat{\theta}$$

Tangential Velocity Gives Radial Acceleration

- Direction of velocity is constantly changing

$$\Delta\vec{v}(t) = \vec{v}(t+\Delta t) - \vec{v}(t)$$

- An object traveling with uniform circular motion (ω(t)=ω) is always accelerating towards the center!



$$\Delta\vec{v}(t) = \vec{v}(t+\Delta t) - \vec{v}(t)$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} =$$

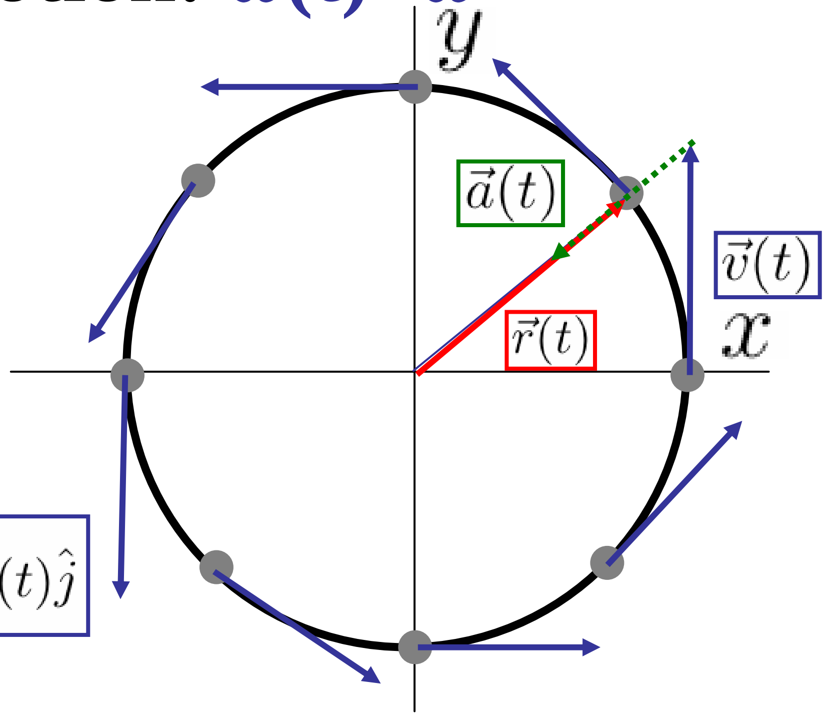
$$= -\omega^2 R(\cos\theta(t)\hat{i} + \sin\theta(t)\hat{j}) =$$

$$= -\omega^2 R\hat{r} = -\frac{v^2}{R}\hat{r}$$

Uniform Circular Motion: ω(t)=ω

- Object is constrained to move in a circle at uniform angular velocity.

$$\vec{r}(t) = R\cos\theta(t)\hat{i} + R\sin\theta(t)\hat{j}$$



- Magnitude of the velocity (speed) remains constant.

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -R\omega(t)\sin\theta(t)\hat{i} + R\omega(t)\cos\theta(t)\hat{j}$$

$$a_\theta = 0$$

- The tangential acceleration is zero so by Newton's Second Law, total tangential force acting on the object is zero.

- Object is always accelerated towards center!

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} =$$

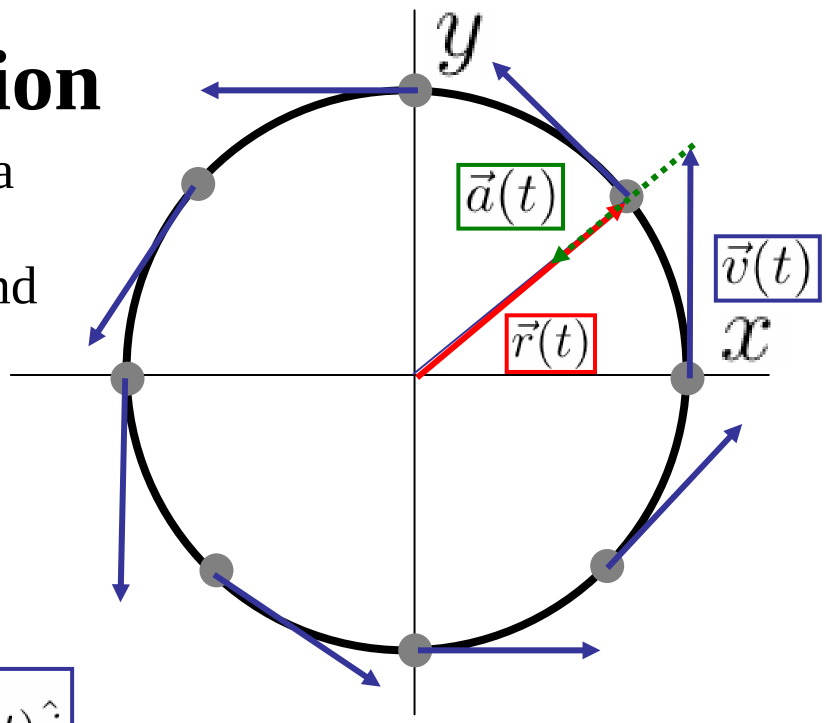
Note: ω(t)=ω

$$= -\omega^2 R(\cos\theta(t)\hat{i} + \sin\theta(t)\hat{j}) =$$

$$= -\omega^2 R\hat{r} = -\frac{v^2}{R}\hat{r}$$

Non-uniform Circular Motion: Acceleration

- When an object moves non-uniformly in a circular orbit, the acceleration has two components, the tangential component, and the radial component.



$$\vec{r}(t) = R\cos\theta(t)\hat{i} + R\sin\theta(t)\hat{j}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -R\omega(t)\sin\theta(t)\hat{i} + R\omega(t)\cos\theta(t)\hat{j}$$

$$\vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} =$$

$$= -\omega(t)^2 R(\cos\theta(t)\hat{i} + \sin\theta(t)\hat{j})$$

$$+\frac{d\omega(t)}{dt}R(-\sin\theta(t)\hat{i} + \cos\theta(t)\hat{j})$$

$$a_r = -\omega(t)^2 R$$ Radial component

$$a_\theta = R\frac{d\omega}{dt} = R\alpha$$ Tangential component

Comparison of linear and circular motion at constant acceleration

- Linear motion

$$a = \text{const.}$$

$$v(t) = v_0 + at$$

$$s(t) = s_0 + v_0 t + \frac{1}{2} a t^2$$

- Circular motion

$$\alpha = \text{const.}$$

$$\omega(t) = \omega_0 + \alpha t$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$$

Problem solving strategy involving circular motion

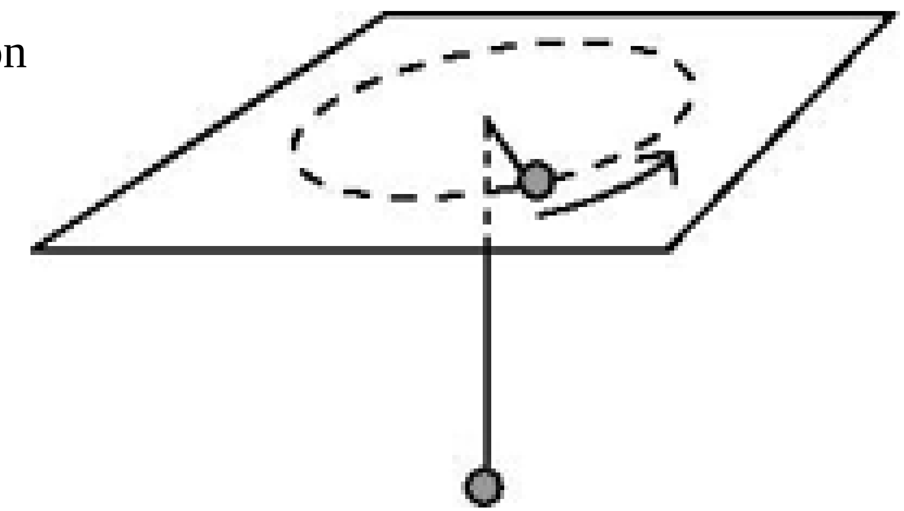
- Always has a component of acceleration pointing radially inward.
- May or may not have tangential component of acceleration.
- Draw a free body diagram for all forces.
- Note: mv^2/R is not a force but mass times acceleration and does not appear on force diagram.
- Choose one unit vectors to point in the radial direction.

$$\vec{\mathbf{F}} \doteq m\vec{\mathbf{a}}$$

	physics		mathematics
$\hat{\mathbf{r}}:$	force decom-	$\doteq$	$-mv^2/r$
$\hat{\boldsymbol{\theta}}:$	position from	$\doteq$	$mrd^2\theta/dt^2$
$\hat{\mathbf{k}}:$	force diagram	$=$	0

PRS Question

A puck of inertia M is moving in a circle at uniform speed on a frictionless table as shown above. It is held by a string which holds a suspended bob, also of inertia M, at rest below the table. Half of the length of the string is above the tabletop and half below. What is the centripetal acceleration of the moving puck?



1. less than g
2. g
3. greater than g
4. zero

Cross Product, Torque, and Static Equilibrium

Point Masses → **Rigid Bodies**

Generalization of Newton's Laws:

- External forces accelerate the **center of mass**
- External **torques** cause angular acceleration

Torque

Quantitative measure of the tendency of a force to cause or change the rotational motion of a body!

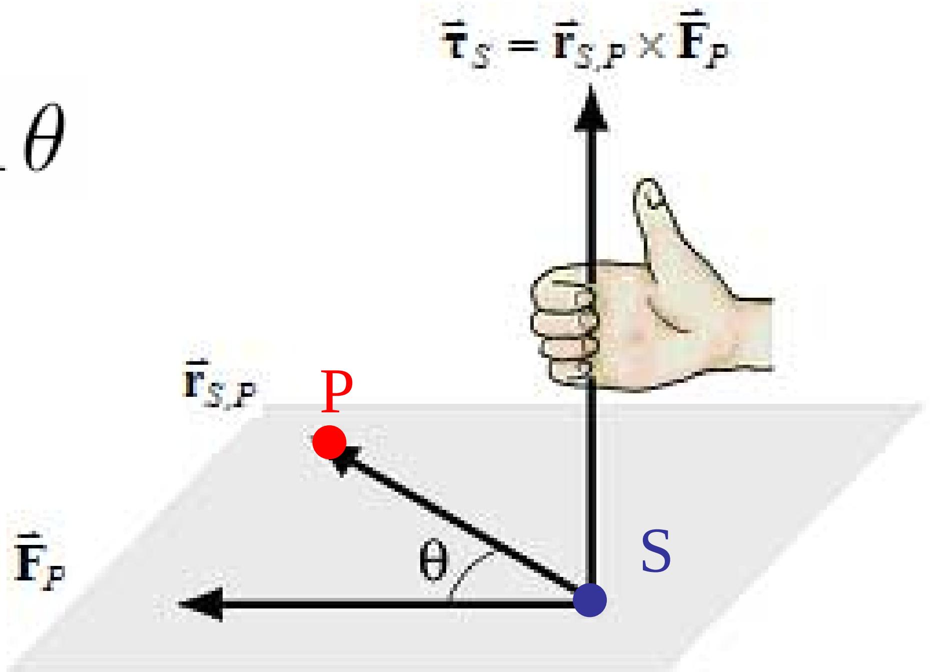
- (1) Magnitude of the torque about S

$$|\vec{\tau}_s| = |\vec{r}_{S,P}||\vec{F}_P|\sin\theta$$

- (2) Direction

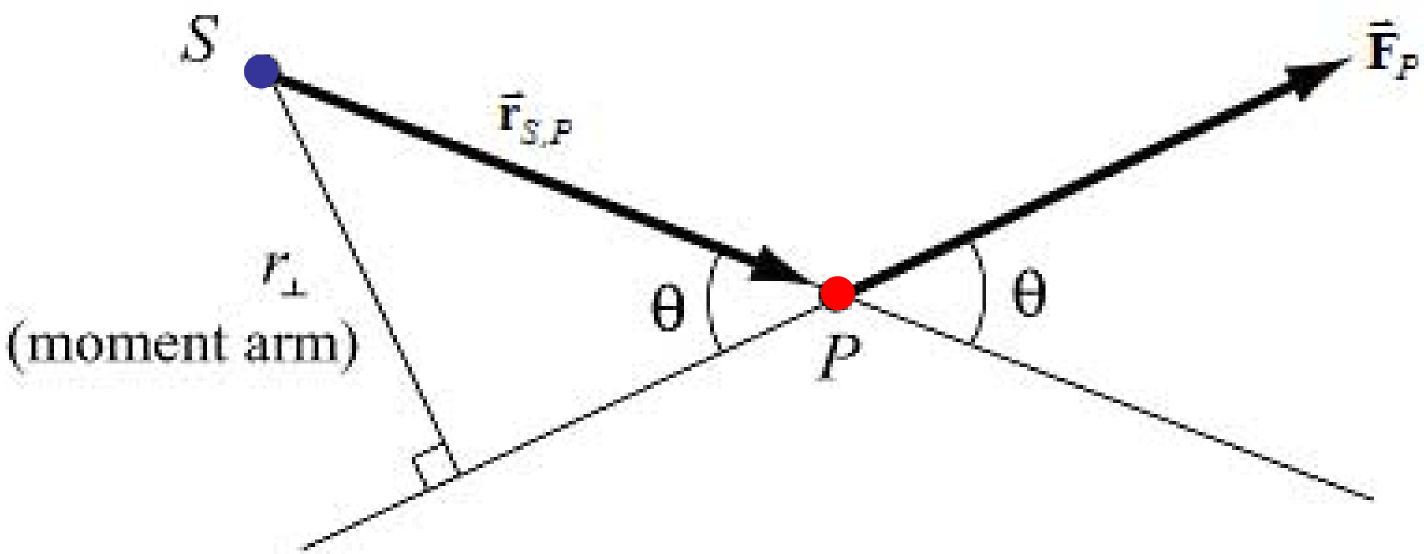
$$\vec{\tau}_s = \vec{r}_{S,P} \times \vec{F}_P$$

Sign convention: Counterclockwise always positive!



Moment Arm of the Force

- Moment Arm:

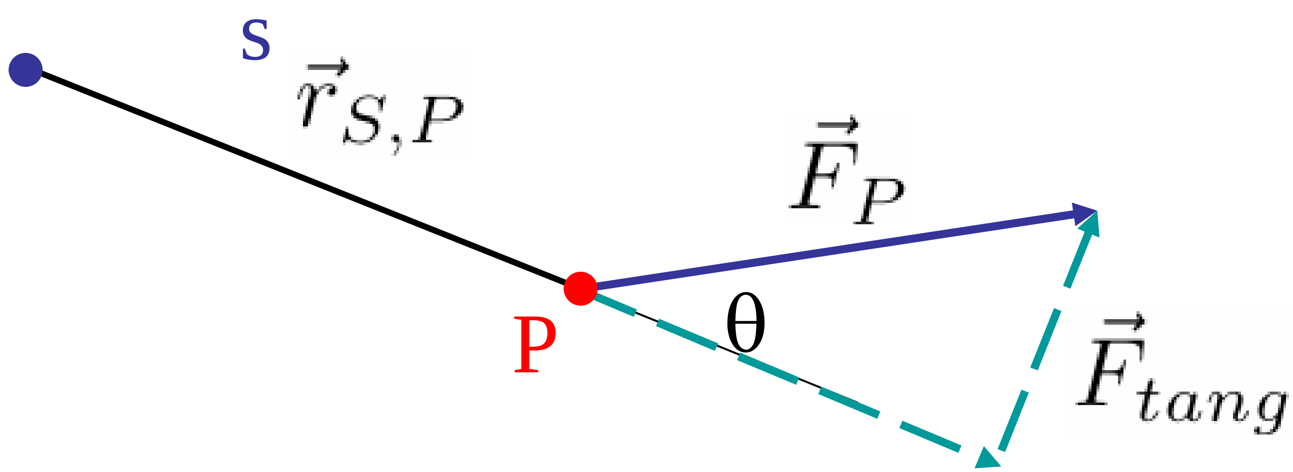


- Torque:

$$|\vec{\tau}_s| = +r_\perp |\vec{F}_P| = |\vec{r}_{S,P}| \sin\theta |\vec{F}_P|$$

Torque from Tangential Force

Break the force into components parallel and perpendicular to the displacement of the force from the axis of the torque:



$$|\vec{\tau}_s| = |\vec{r}_{S,P}||\vec{F}_{tang}| = |\vec{r}_{S,P}||\vec{F}_P|\sin\theta$$

Static Equilibrium

(1) The sum of the forces acting on the rigid body is zero:

$$\Sigma \vec{F} = 0$$

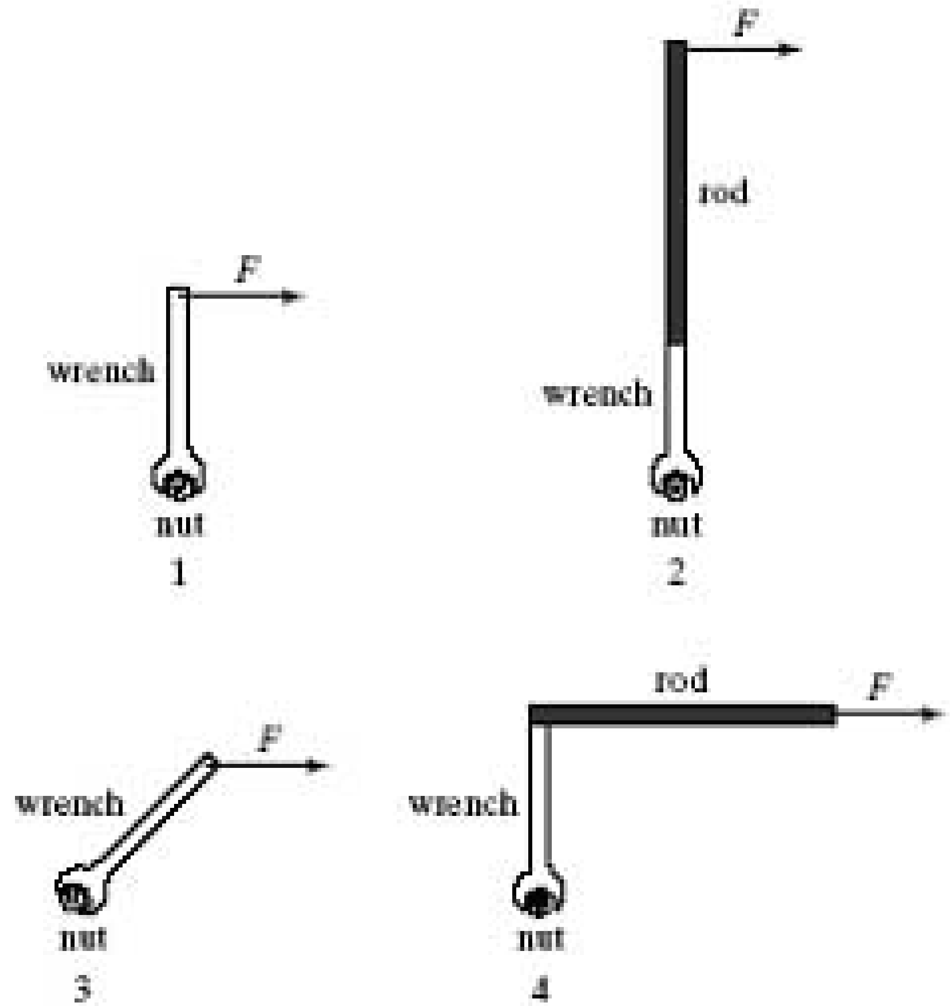
(2) The vector sum of the torques about any point S in a rigid body is zero:

S

$$\Sigma \vec{\tau}_{ip} = 0$$

PRS Question

You are using a wrench to loosen a rusty nut. Which of the arrangements shown is most effective in loosening the nut?



Work, Kinetic Energy, and Power

Work-energy Definitions

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_i^2 + F_x(x_f - x_i)$$

Now we give each term a descriptive name:

$\frac{1}{2}mv^2$ Kinetic Energy

$F_x(x_f - x_i)$ Work

Restating the Work-Kinetic Energy relationship conventionally:

$$K_f = K_i + W_{f,i}$$

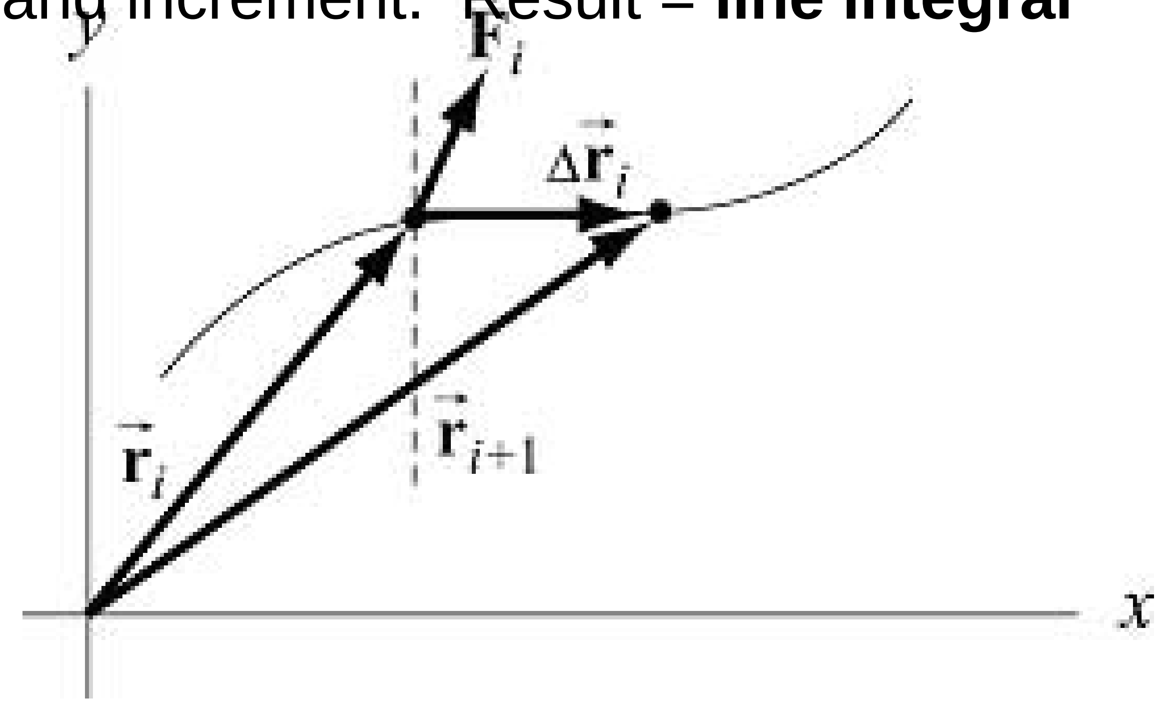
Kinetic energy is a physical quantity that changes due to work

The work $W_{f,i}$ depends on the path to f from i and the force, F_x *and* the relative direction of the force and the path

The expression for work is only valid for constant force

Work Done Along an Arbitrary Path

Key Idea: Add work along incremental path, proportional to scalar product of force and increment. Result = **line integral**



For straight path:

$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i$$

For Curved Path:

Add straight increments:

$$W_{f,o} = \lim_{\substack{N\to\infty \\ |\Delta\vec{\mathbf{r}}_i|\to 0}} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i = \int_{\text{path}\ \vec{r}_0}^{\vec{r}_f} \vec{\mathbf{F}}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}$$

PRS Question on Work

Consider the work associated with three actions that a person might take:

A. Lift a 4 kg mass from the floor to h=1m

B. Hold a 5 kg mass 1 m above the floor

C. Lower a 3 kg mass from h=2m to the floor

Order these actions from greatest to least by the work done:

1. A,B,C
2. C,B,A
3. C,A,B
4. A,C,B
5. None of above

PRS: Work Due to Friction

A force F_{xa} is applied to a block and causes it to slide from x = 0m to x = 5m and back to x = 3m.
The block experiences a friction force, F_f = 7N.
At the start and finish the block is not moving.
The total work done by the friction force on the block is:

1. 21j
2. 49j
3. 56j
4. -21j
5. -49j
6. -56j
7. Need more information about motion

Instantaneous Power

- limit of the average power

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = F_{applied,x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = F_{applied,x} v_x$$

Potential Energy
Total Mechanical Energy

Mechanical Energy - Model

$$E_f = E_i + W_{f,i}^{noncons}$$

Similar to Work-Kinetic Energy

Only non-conservative Work changes Mechanical Energy

$$E_f \equiv K_f + U(x_f)$$

Definition of total Mechanical Energy

$$W_{f,o}^{noncons} = \int_{r_0,\ \mathrm{path}}^{r_f} \vec{\mathbf{F}}^{noncons}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}$$

Usual Definition but only
For non-conservatives

W^{cons} Replaced by Potential Energy (difference)

$$W_{f,i}^{cons} \equiv -U(x_f) + U(x_i)$$

Model: Potential Energy and Force

The potential energy must represent the work:

$$W_{f,0}^{cons} \equiv -U(x_f) + U(x_0)$$

i.e. Work depends on endpoints

This is possible ONLY if **(DEFINITION of CONSERVATIVE)**

$$W_{f,o}^{cons} = \int_{\vec{r}_0,\ \text{path}}^{\vec{r}_f} \vec{\mathbf{F}}^{cons}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}$$

is **independent** of the **path**

Define $$U(\vec{\mathbf{r}}_f) = -\int_{\vec{r}_0,\ \text{any convenient path}}^{\vec{r}_f} \vec{\mathbf{F}}^{cons}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}}$$

Path independence allows this

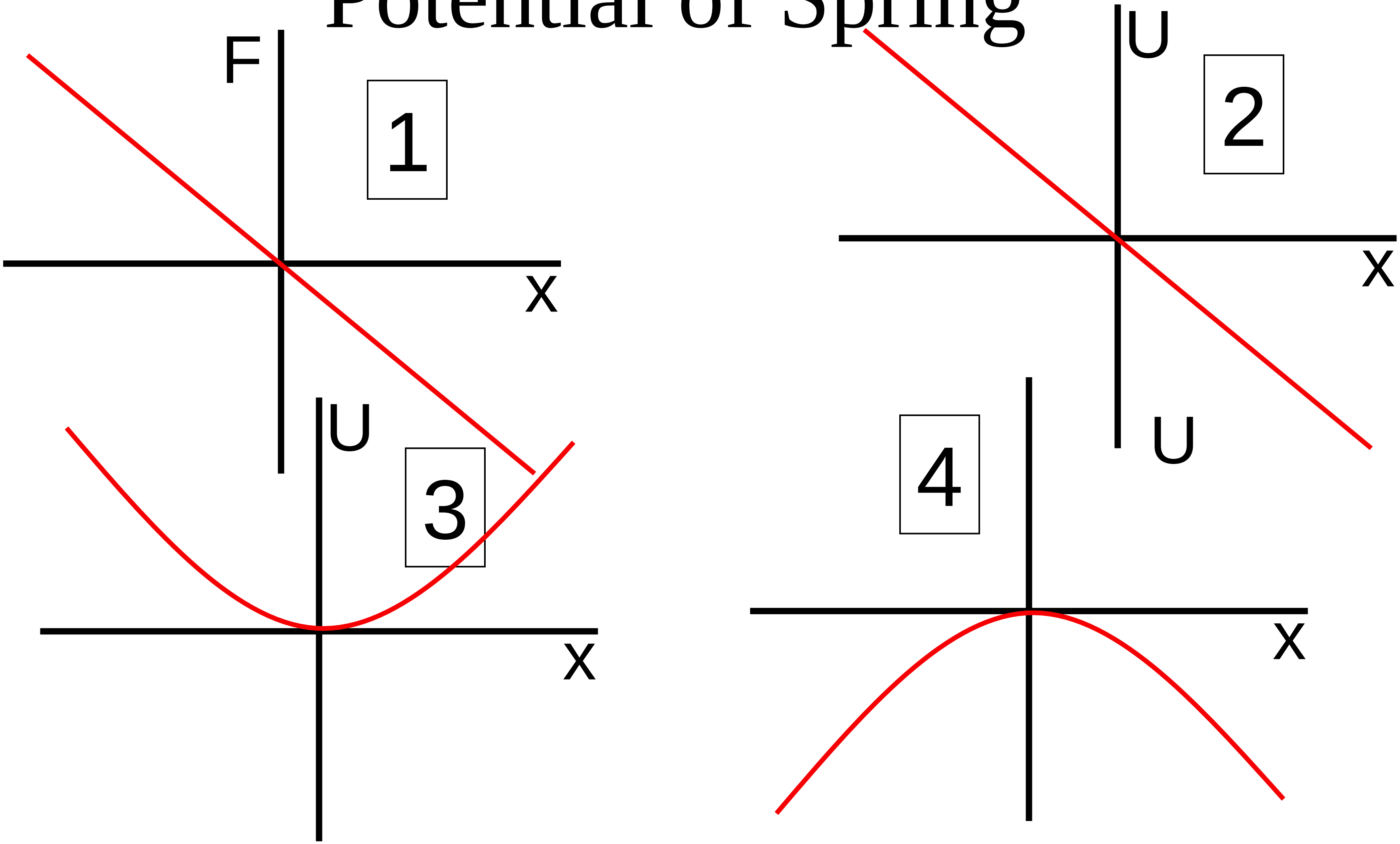
Usually $$U(x) = -\int_{x_0}^{x} \mathbf{F}^{cons}(x) \cdot dx$$

X_0 defines zero of U

Like an indefinite integral

Get Force from U: $$F_x(x) = -\frac{dU(x)}{dx}$$

Potential of Spring

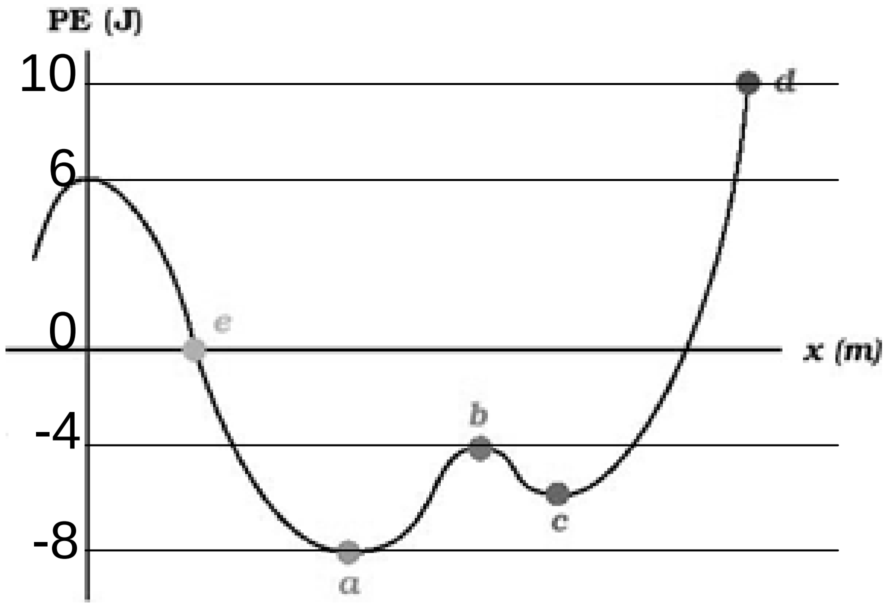


Which graph is potential for spring?

Hint: $$U(x) = -\int_{x_0}^{x} \mathbf{F}^{cons}(x)\cdot dx$$

PRS: Energy Curve - E_{min}

Consider the following sketch of potential energy for a particle as a function of position. There are no dissipative forces or internal sources of energy.



What is the minimum total mechanical energy that the particle can have if you know that it has travelled over the entire region of X shown?

1. -8
2. 6
3. 10
4. It depends on direction of travel
5. Can't say - Potential Energy uncertain by a constant

Potential Energy of Spring

- Force:

$$\vec{\mathbf{F}} = F_x \hat{\mathbf{i}} = -kx\hat{\mathbf{i}}$$

- Work done:

$$W_{f,0}^{spring} = \int_{x=x_0}^{x=x_f} (-kx)\,dx = -\frac{1}{2}k\left(x_f^{\,2} - x_0^{\,2}\right)$$

- Potential Energy Change:

$$\Delta U_{f,0}^{spring} \equiv -W_{f,0}^{spring} = \frac{1}{2}k\left(x_f^{\,2} - x_0^{\,2}\right)$$

- Pick Potential Zero Point:

$$U_{spring}(x = 0) \equiv 0$$

- Potential Energy Function

$$\boxed{U_{spring}(x) = \frac{1}{2}kx^2}$$

Change in Potential Energy: Constant Gravity

Force:

$$\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}} = F_{grav,y}\hat{\mathbf{j}} = -mg\hat{\mathbf{j}}$$

Work:

$$W_{gravity} = F_{gravity,y}\Delta y = -mg\Delta y$$

Potential Energy:

$$\Delta U^{grav}_{f,0} = -W^{grav}_{f,0} = mg\Delta y = mgy_f - mgy_0$$

Choice of Zero Point: Whatever "origin" is convenient

Potential for uniform gravity:

$$U^{grav}(y) = mg(y - y_{origin})$$

Change in Potential Energy: Inverse Square Gravity

- Force: $$\vec{\mathbf{F}}_{m_1,m_2} = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}}$$

- Work done: $$W = \int_{r_0}^{r_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{r_0}^{r_f} \left(-\frac{Gm_1 m_2}{r^2}\right) dr = \frac{Gm_1 m_2}{r}\Bigg|_{r_0}^{r_f} = Gm_1 m_2\left(\frac{1}{r_f} - \frac{1}{r_0}\right)$$

- Potential Energy Change: $$\Delta U_{gravity} \equiv -W_{gravity} = -Gm_1 m_2\left(\frac{1}{r_f} - \frac{1}{r_0}\right)$$

- Potential Energy Function for Universal Gravity $$U_{gravity}(\mathrm{r}) = -\frac{Gm_1 m_2}{r}$$

- Associated Zero Point: $U_{gravity}(\mathrm{r}) \equiv 0 \text{ as } \mathrm{r} \to \infty$

Energy Strategies

Hallmarks of Energy:

Kinetic energy depends on speed squared
Potential Energy from position-dependent Forces
Work by non-conservative forces

- Mechanical Energy changed by $W^{non\text{-}cons}$
- M.E. switches between Kinetic and Potential

$$K_f + U_f = K_i + U_i + W_{f,i}^{noncons}$$

1. Identify forces as F^{cons} or $F^{non\text{-}cons}$
2. Represent F^{cons} by Potential Energy
3. Pick the System
4. Select Initial and Final States to simplify algebra
5. Evaluate $W^{non\text{-}cons}$

Class Problem: Block-Spring Oscillator

Example 1: A block of mass m is attached to a spring and is free to slide along a horizontal frictionless surface. At $t = 0$ the block-spring system is stretched an amount A from the equilibrium position ($x=0$) and is released from rest.

What is the speed of the block when it is at the position x ($|x|<A$)?

Energy Initial and Final

Initial state: at rest with maximum stretch A (take > 0)
and initial velocity $v_0 = 0$

- Kinetic energy — $K_0 = 0$ since $v_0 = 0$ — $K_0 = 0$
- Potential energy — $U_0 = \frac{1}{2} kA^2$
- Mechanical energy — $E_0 = K_0 + U_0 = \frac{1}{2} kA^2$

Final state: position x and speed v(x) (can be > or < 0)

Kinetic energy $K = \frac{1}{2} mv^2$

Potential energy $U = \frac{1}{2} kx^2$

Mechanical energy $E = \frac{1}{2} mv^2$

Mechanical Energy is Constant

$$E(x) = E_0 \Rightarrow \frac{1}{2}mv(x)^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Solve for velocity at equilibrium position

$$v^2 = \frac{k}{m}[A^2 - x^2]$$

$$\Rightarrow v(x) = \sqrt{\frac{k}{m}[A^2 - x^2]}$$

Problem 2: Position of Oscillator

You must solve the central problem of mechanics:
given the force as a function of position,
find the position as a function of time.

REMEMBER that $\mathbf{x(t) = A\ \ cos[\omega t+\phi]}$
Is **a valid solution** for the position of the oscillator.
Adjust A and ϕ to match initial conditions

PRS: Derivative of oscillator position

Motivation: The kinematics of harmonic oscillators involves the derivatives of trigonometric functions and pervades physics.

Find the velocity associated with $\mathbf{x(t) = A \ \cos[2\pi(t/T)]}$

1. $\mathbf{v(t) = A \ \sin[2\pi(t/T)]}$
2. $\mathbf{v(t) = -A \ \sin[2\pi(t/T)]}$
3. $\mathbf{v(t) = -(2\pi A/T) \sin[2\pi(t/T)]}$
4. $\mathbf{v(t) = -(A/T) \ \sin[2\pi(t/T)]}$
5. $\mathbf{v(t) = -(2\pi A/T) \ \cos[2\pi(t/T)]}$
6. $\mathbf{v(t) = -(A/T) \ \cos[2\pi(t/T)]}$
7. **None of above**