Cross Product, Torque, and Static Equilibrium

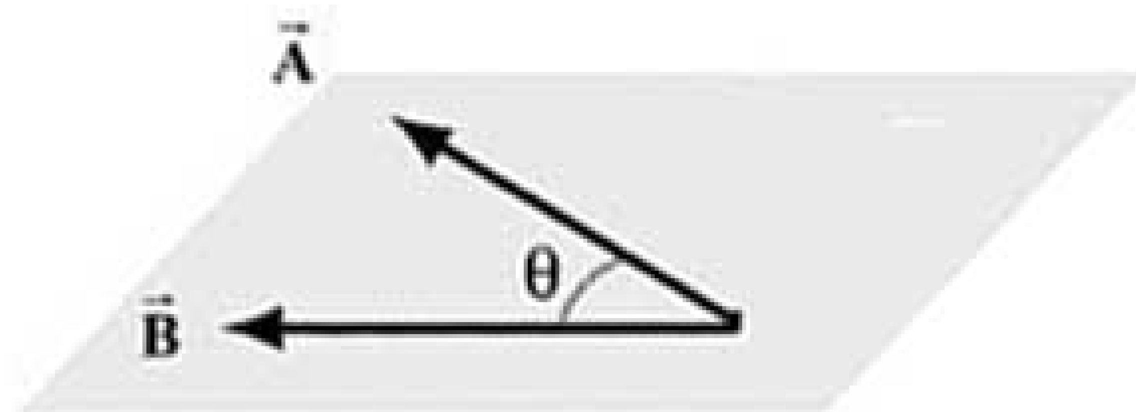
8.01t

Oct 6, 2004

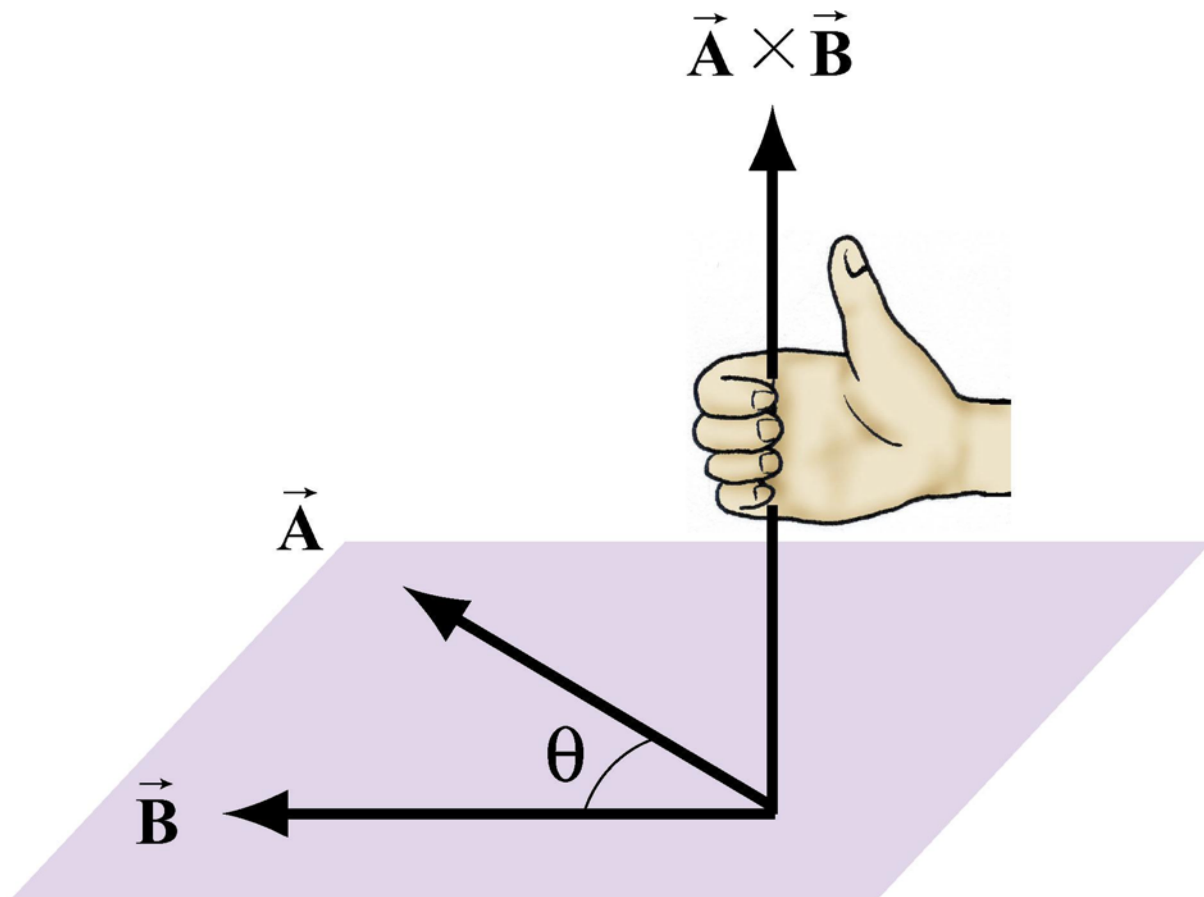
Cross Product

- The magnitude of the cross product

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta \qquad 0 \le \theta \le \pi$$



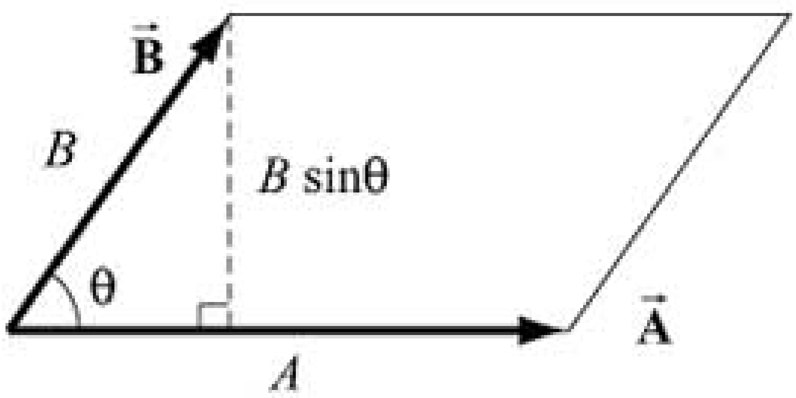
Direction of Cross Product



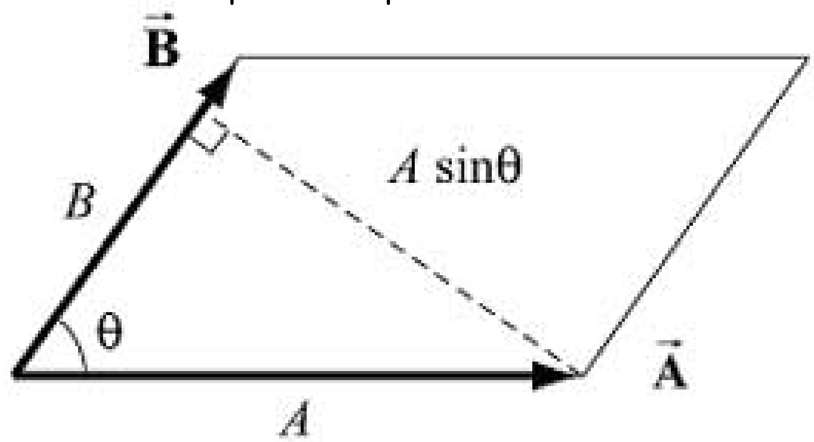
Area and the Cross Product

- The area of the parallelogram equals the height times the base, which is the magnitude of the cross product.

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = A\left(B\sin\theta\right)$$



$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = (A\sin\theta)B$$



Properties

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c\vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

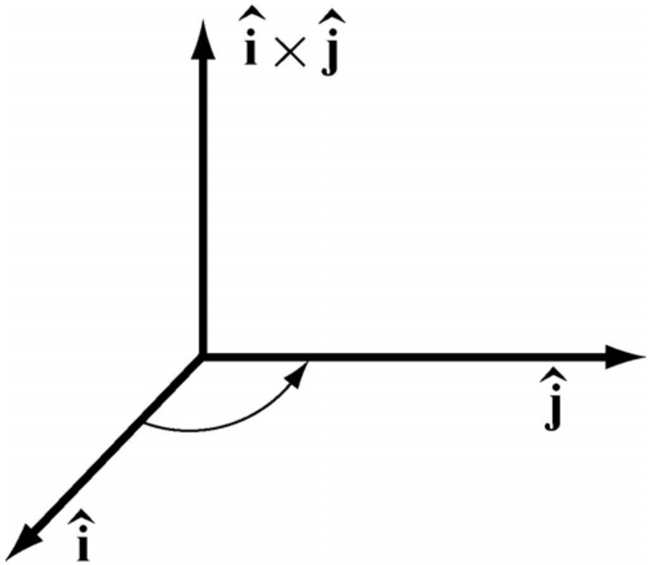
$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}}$$

Unit Vectors and the Cross Product

- Unit vectors

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right| = \left|\hat{\mathbf{i}}\right|\left|\hat{\mathbf{j}}\right| \sin\left(\pi/2\right) = 1$$

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{i}}\right| = \left|\hat{\mathbf{i}}\right|\left|\hat{\mathbf{i}}\right| \sin(0) = 0$$



$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Vector Components of Cross Product

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}$$

PRS Question 1

Consider two vectors $\vec{\mathbf{r}}_{P,F} = x\hat{\mathbf{i}}$ with x >0 and

$\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$

The cross product $\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}}$

points in the

1) + x-direction
2) -x-direction
3) +y-direction
4) -y-direction
5) +z-direction
6) -z-direction
7) None of the above directions

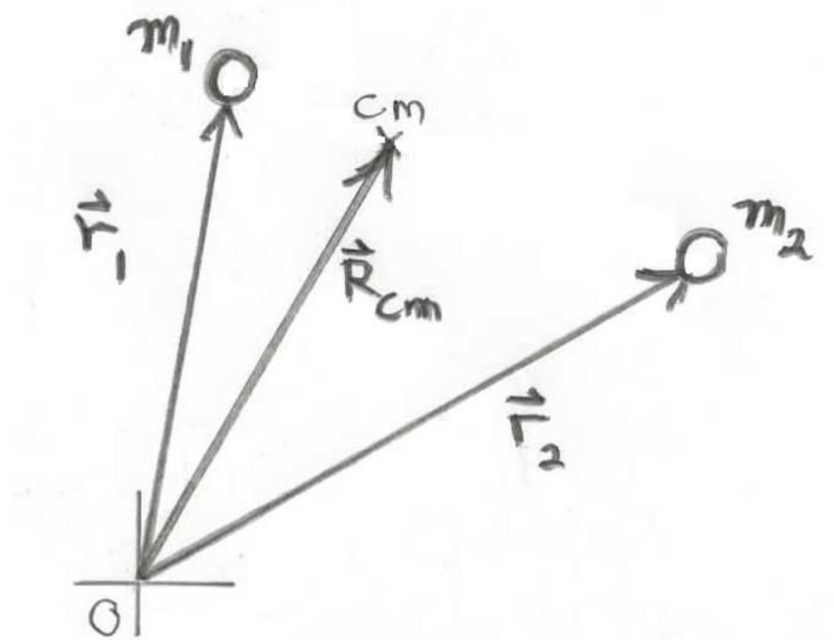
Rigid Bodies

- external forces make the center of the mass translate

- external `torques’ make the body rotate about the center of mass

Center of Mass

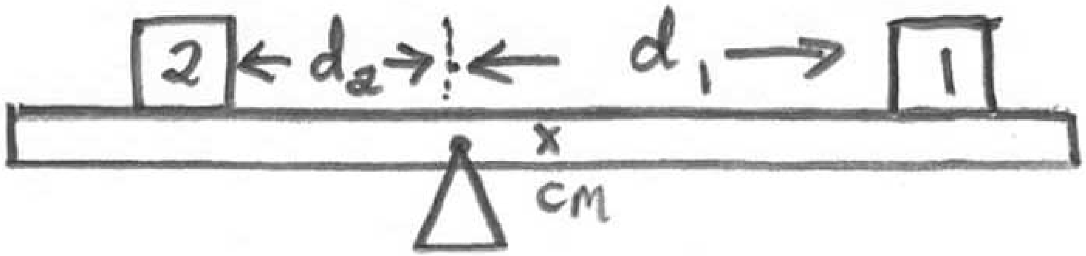
A rigid body can be balanced by pivoting the body about a special point known as the center of mass

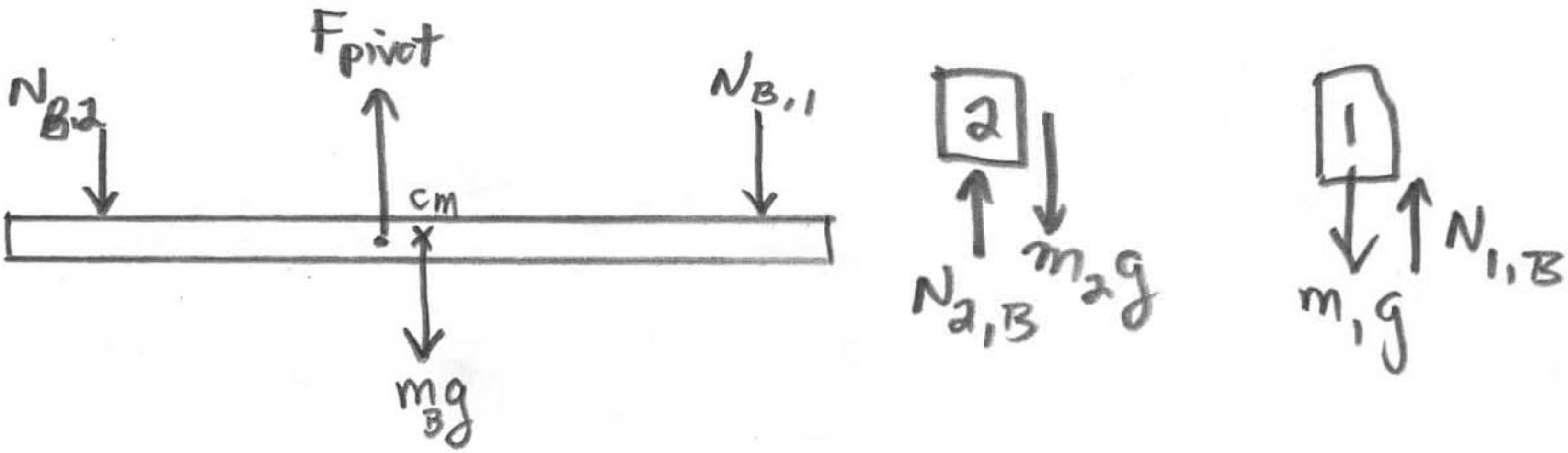


$$\vec{\mathbf{R}}_{cm} = \frac{\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i}{\sum_{i=1}^{i=N} m_i}$$

$$\vec{\mathbf{R}}_{cm} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2}$$

Pivoted Lever

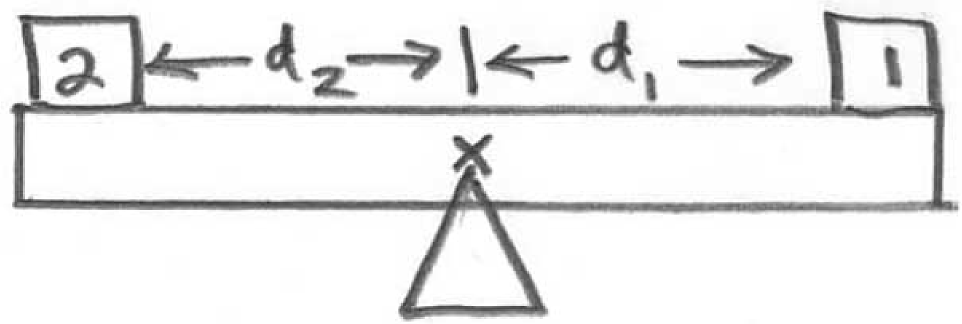




$$F_{pivot} - m_{beam} g - N_1 - N_2 = 0$$

Lever Law

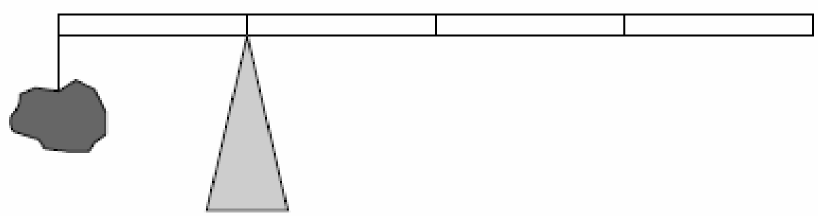
- *Pivoted Lever at Center of Mass*



$$d_1 N_1 = d_2 N_2$$

PRS Question 2

A 1-kg rock is suspended by a massless string from one end of a 1-m measuring stick. What is the weight of the measuring stick if it is balanced by a support force at the 0.25-m mark?



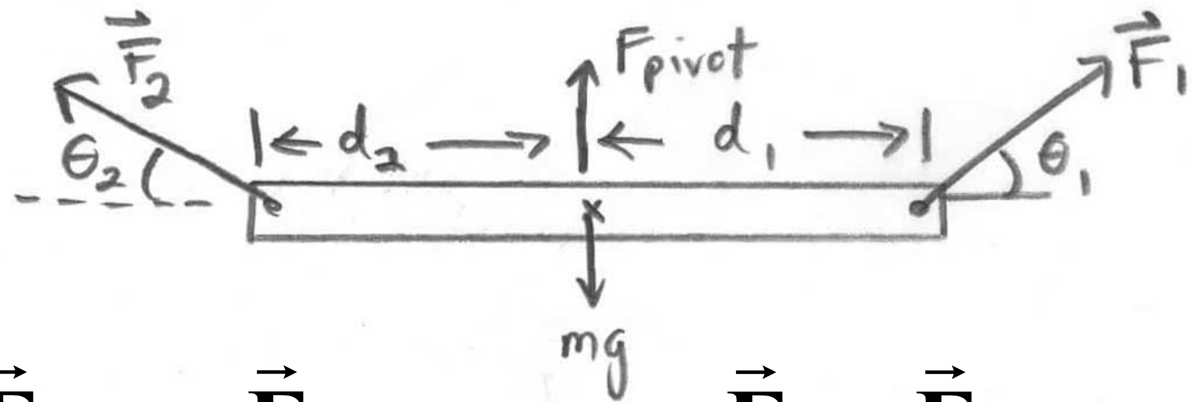
1. 0.25 kg
2. 0.5 kg
3. 1 kg
4. 2 kg
5. 4 kg
6. impossible to determine

Class Problem 1

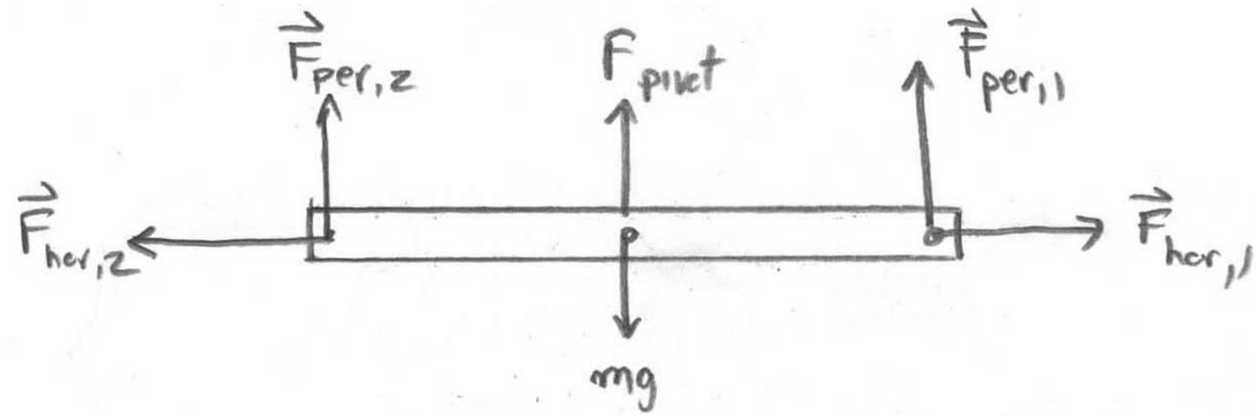
Suppose a beam of length s = 1.0 m and mass m = 2.0 kg is balanced on a pivot point that is placed directly beneath the center of the beam. Suppose a mass m_1 = 0.3 kg is placed a distance d_1 = 0.4 m to the right of the pivot point. A second mass m_2 = 0.6 kg is placed a distance d_2 to the left of the pivot point to keep the beam static.

1. What is the force that the pivot exerts on the beam?
2. What is the distance d_2 that maintains static equilibrium?

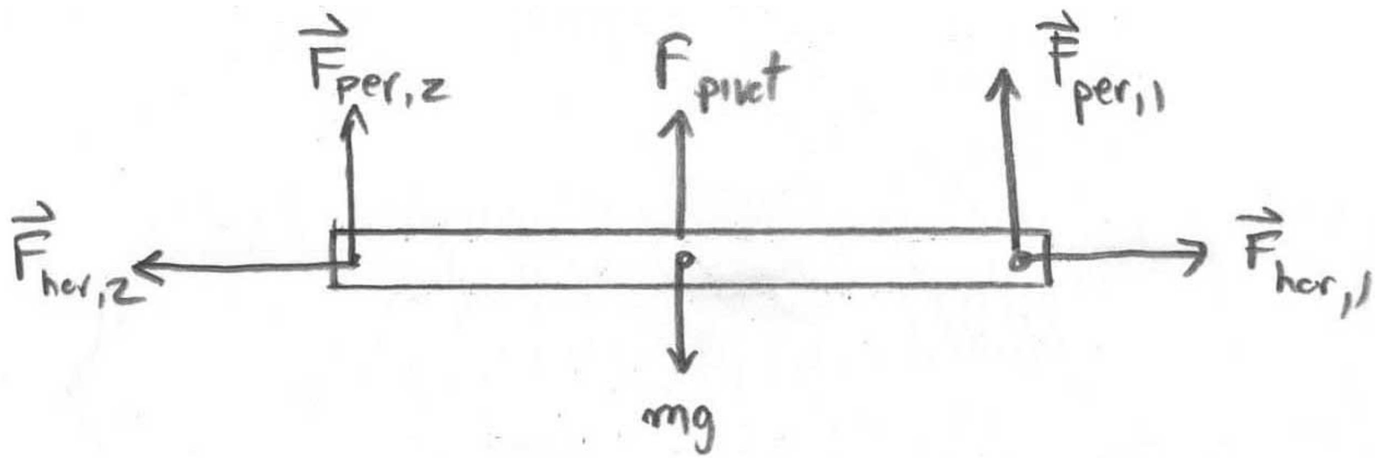
Generalized Lever Law



$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{hor,1} + \vec{\mathbf{F}}_{per,1} \qquad \vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{hor,2} + \vec{\mathbf{F}}_{per,2}$$



Generalized Lever Law



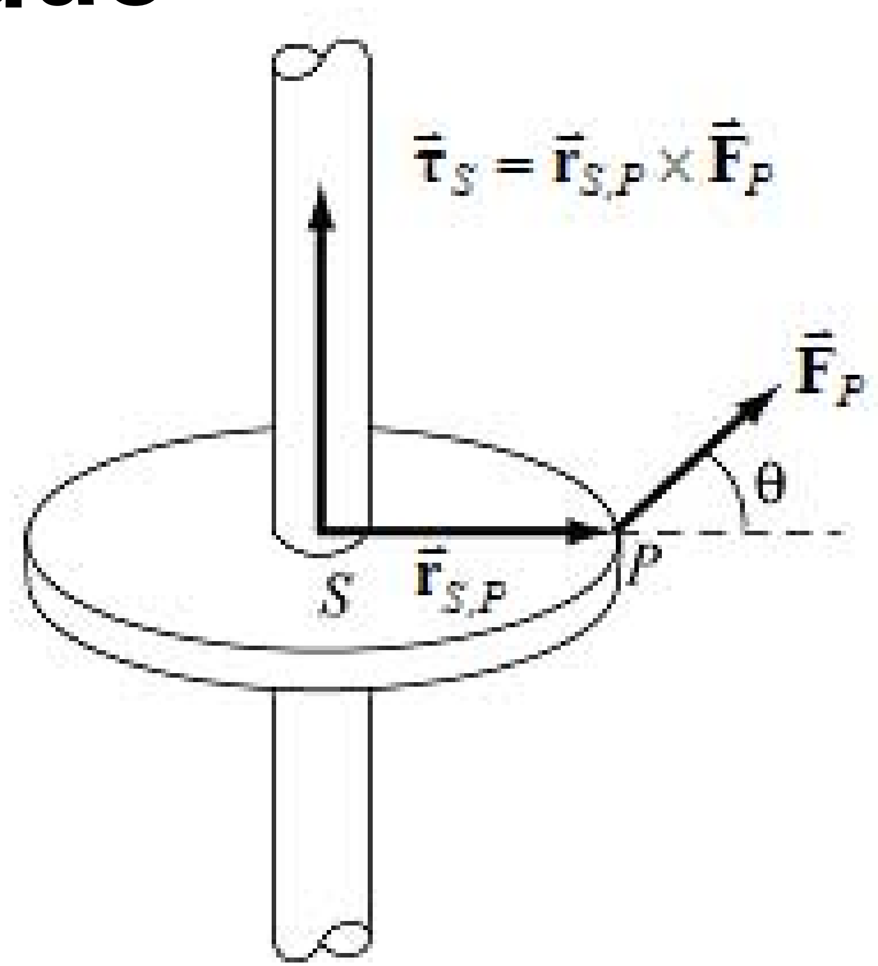
$$F_{1,\perp} \equiv F_{per,1} = F_1 \sin(\theta_1)$$

$$F_{2,\perp} \equiv F_{per,2} = F_2 \sin(\theta_2)$$

$$d_1 F_{1,\perp} = d_2 F_{2,\perp}$$

Torque

- Let a force $\vec{\mathbf{F}}_P$ act at a point P
- Let $\vec{\mathbf{r}}_{S,P}$ be the vector from the point *S* to a point *P*

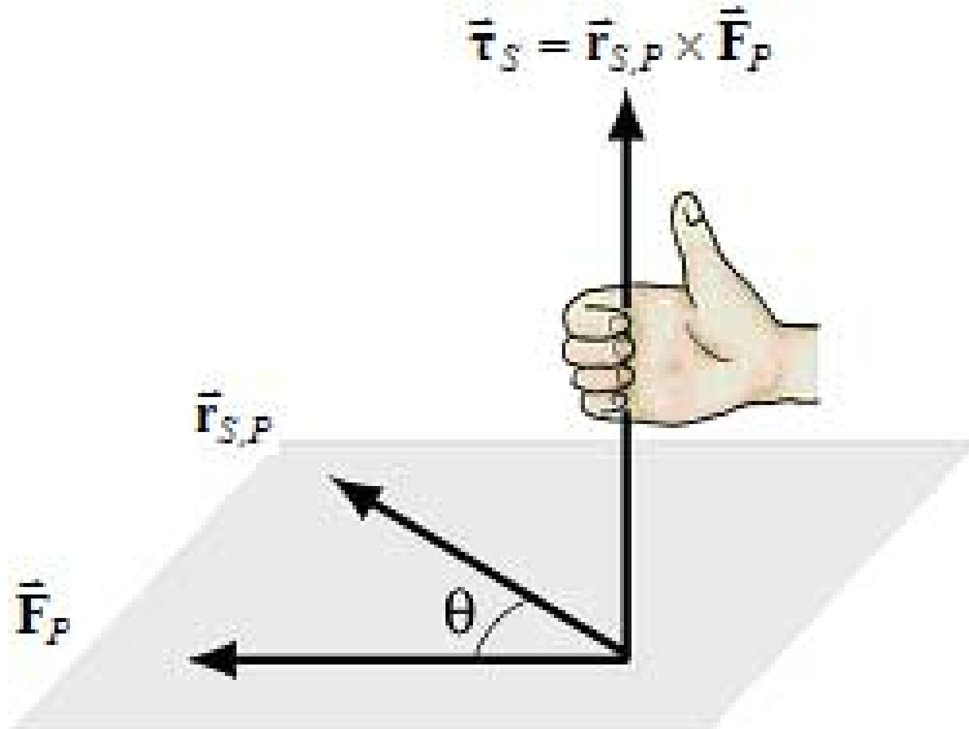


$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P$$

Torque

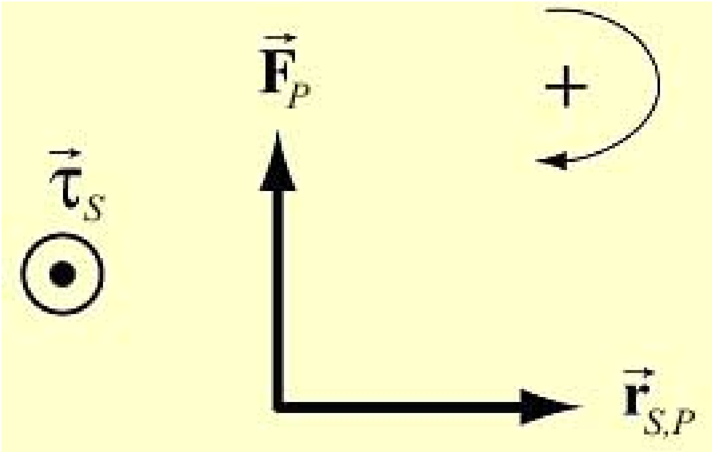
- (1) Magnitude of the torque about S $\tau_S = rF_\perp = rF\sin\theta$

- (2) Direction

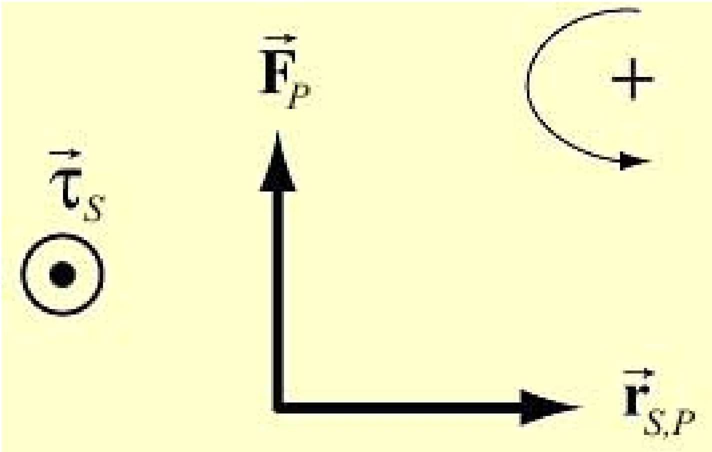


Sign Convention

- Clockwise positive



- Counterclockwise
- positive



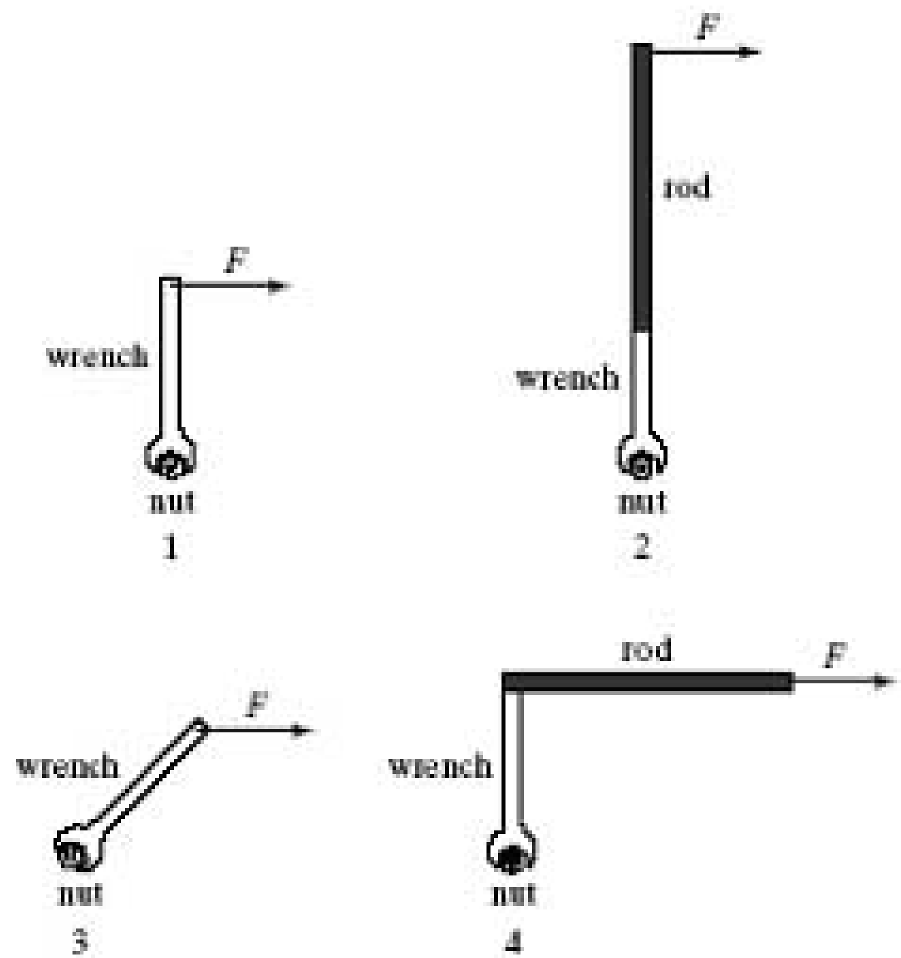
PRS Question 3

You are trying to open a door that is stuck by pulling on the doorknob in a direction perpendicular to the door. If you instead tie a rope to the doorknob and then pull with the same force, is the torque you exert increased?

1. yes
2. no

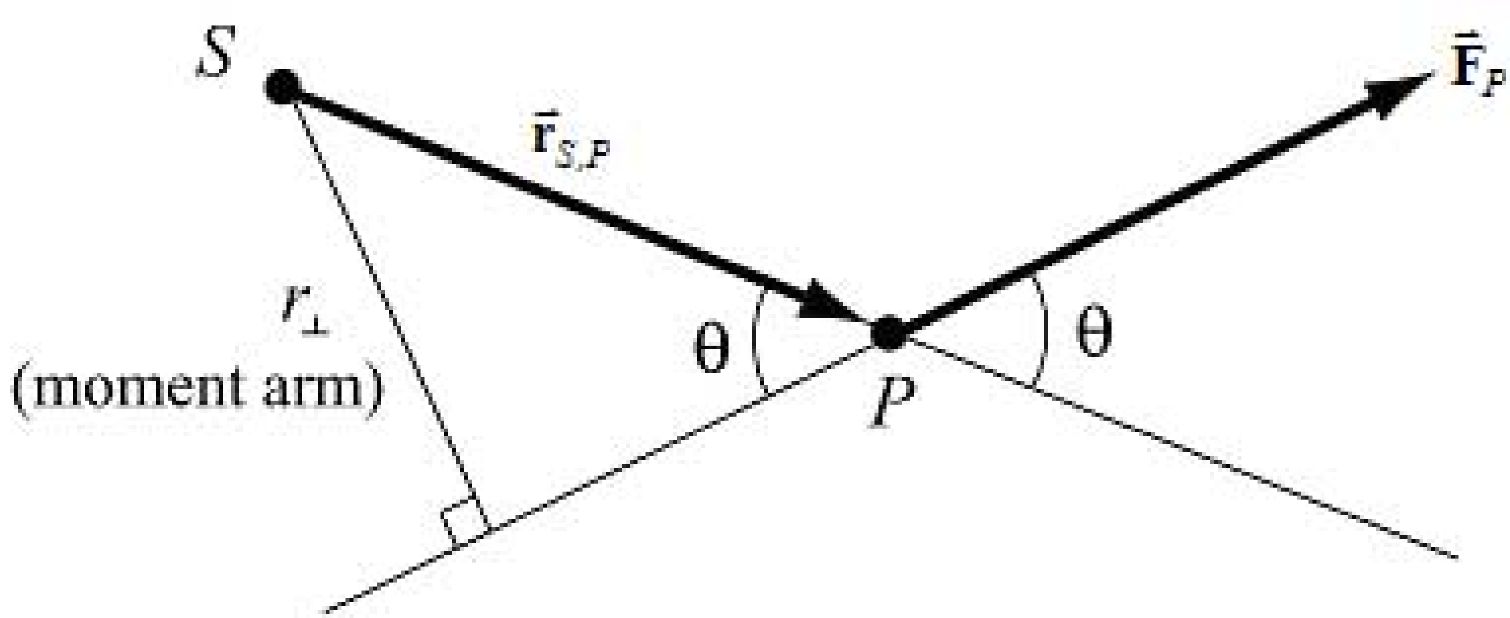
PRS Question 4

You are using a wrench to loosen a rusty nut. Which of the arrangements shown is most effective in loosening the nut?



Line of Action of the Force

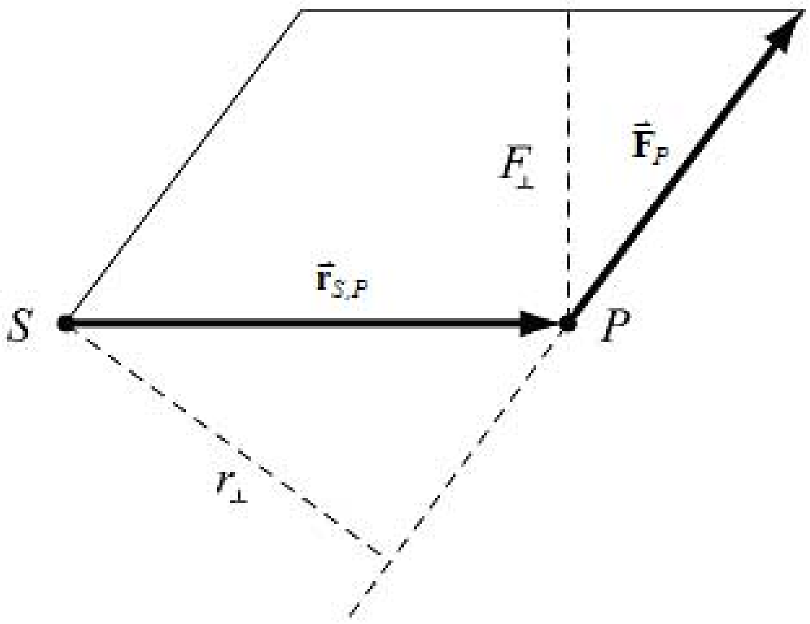
- Moment Arm:



- Torque: $$\tau_S = rF_\perp = rF\sin\theta = r_\perp F$$

Two Geometric Interpretations of Torque

- Area of the torque parallelogram.



$$A = \tau_S = r_\perp F = r F_\perp$$

Static Equilibrium

(1) The sum of the forces acting on the rigid body is zero

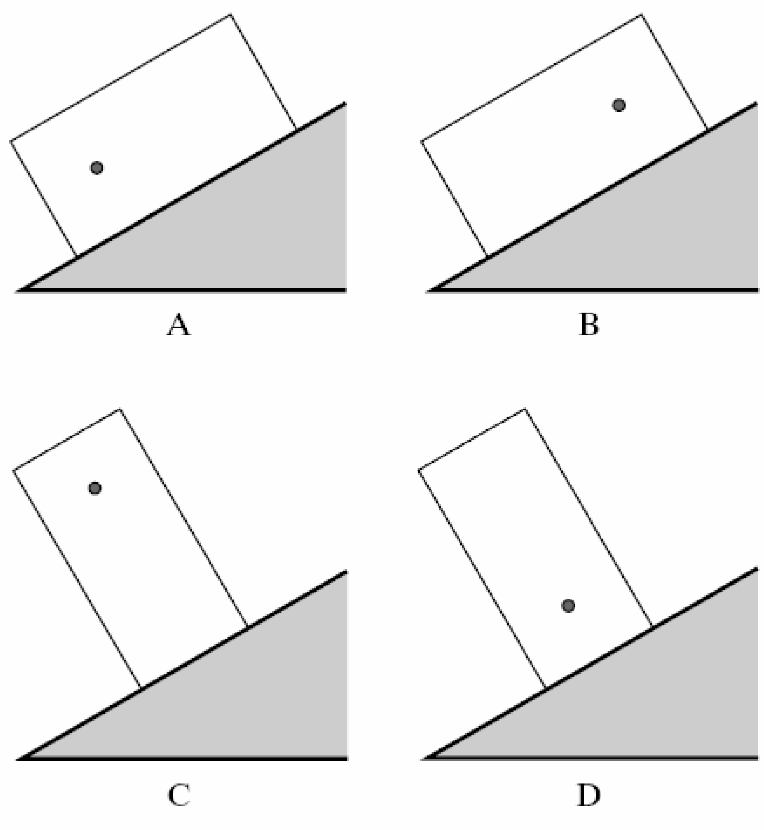
$$\vec{\mathbf{F}}_{total} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + ... = \vec{\mathbf{0}}$$

(2) The vector sum of the torques about any point S in a rigid body is zero

$$\vec{\boldsymbol{\tau}}_S^{total} = \vec{\boldsymbol{\tau}}_{S,1} + \vec{\boldsymbol{\tau}}_{S,2} + ... = \vec{\mathbf{0}}$$

PRS Question 5

A box, with its center-of-mass off-center as indicated by the dot, is placed on an inclined plane. In which of the four orientations shown, if any, does the box tip over?

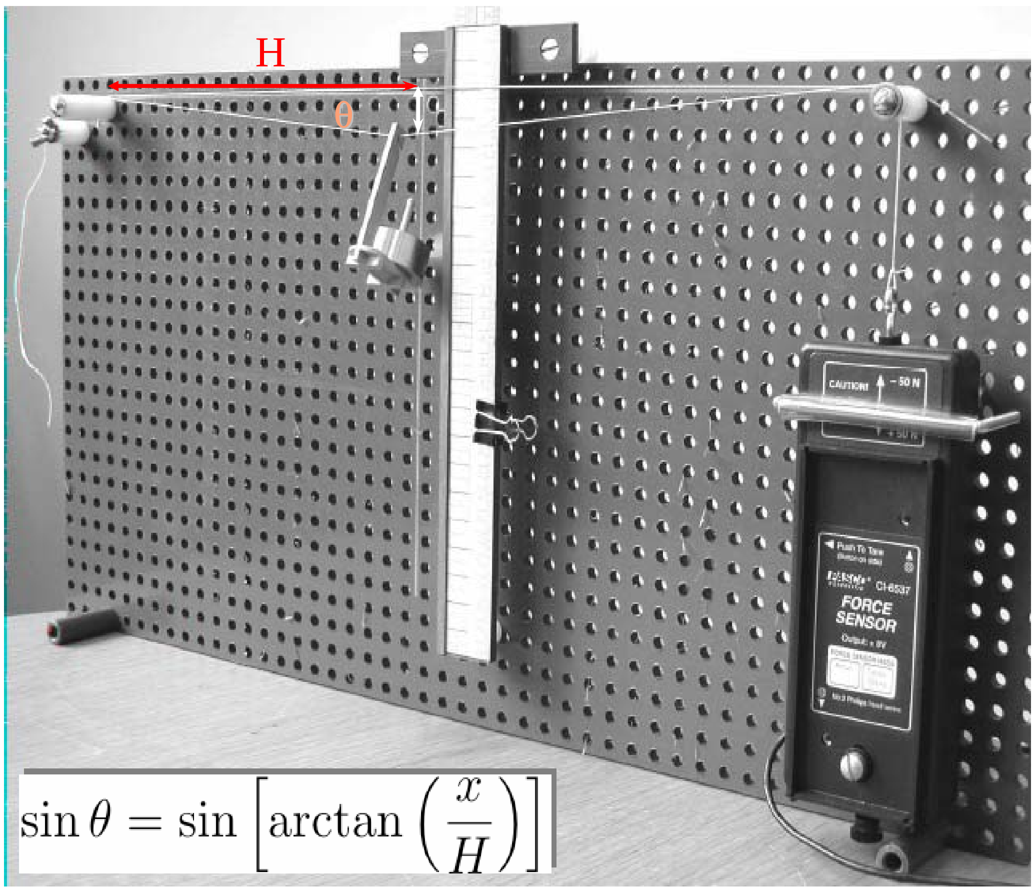


Experiment 05A: Static equilibrium

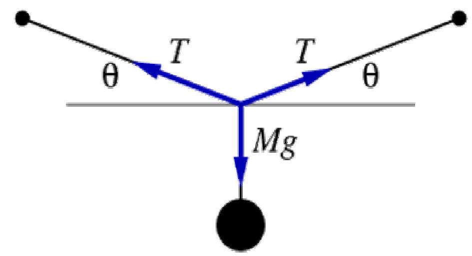
Goal

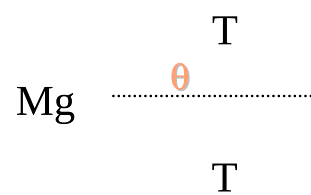
When a weight is suspended by two strings in the center as shown in the photograph below, the tension is given as follows:

$$T = \frac{Mg}{2\sin\theta}$$



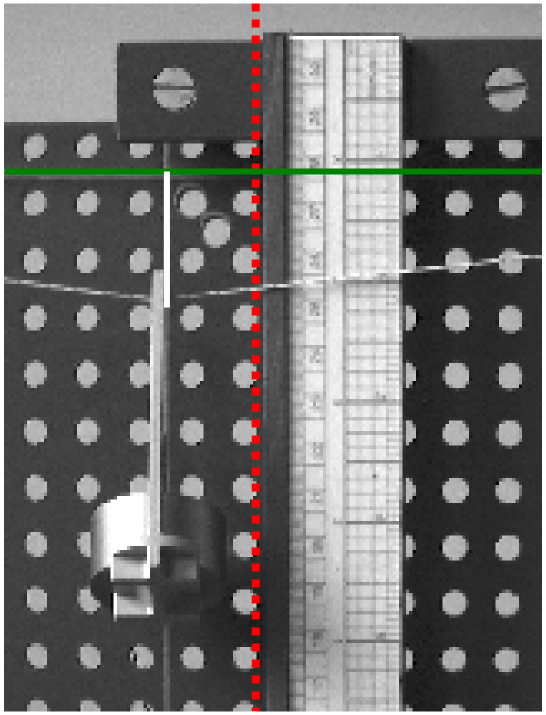
$$\sin\theta = \sin\left[\arctan\left(\frac{x}{H}\right)\right]$$





Goal: Measure T for several values of θ using measurements of , H (fixed), to verify the equation above!

Setup

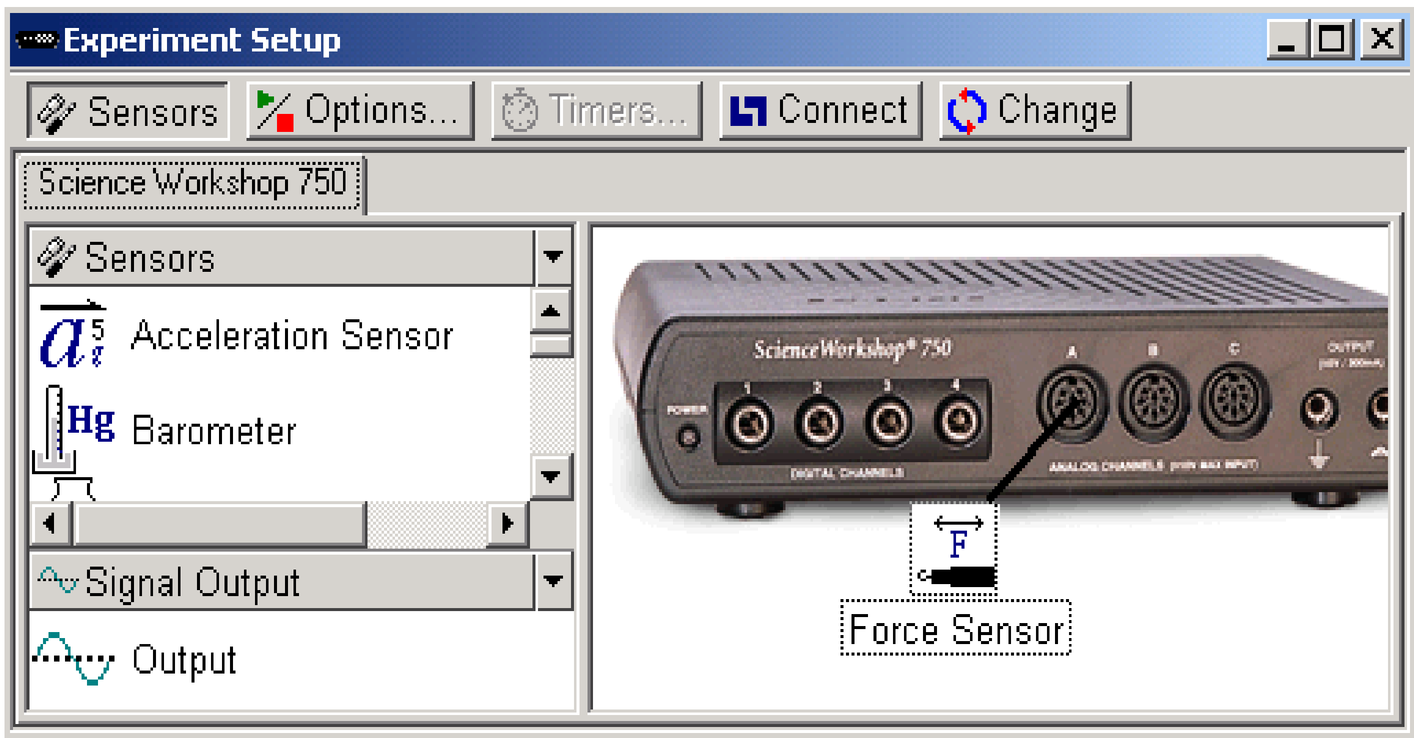


- Align the right edge of the ruler with the center of a column of holes.
- Maintain the same horizontal distance for all measurements.
- A second string along the top marks the horizontal line between the two string support lines.
- The vertical drop ( ) from this line is what you have to measure to determine the angle θ.

- Ensure string passes over pulley before all measurements.
- Keep line of sight perpendicular to board to minimize parallax.

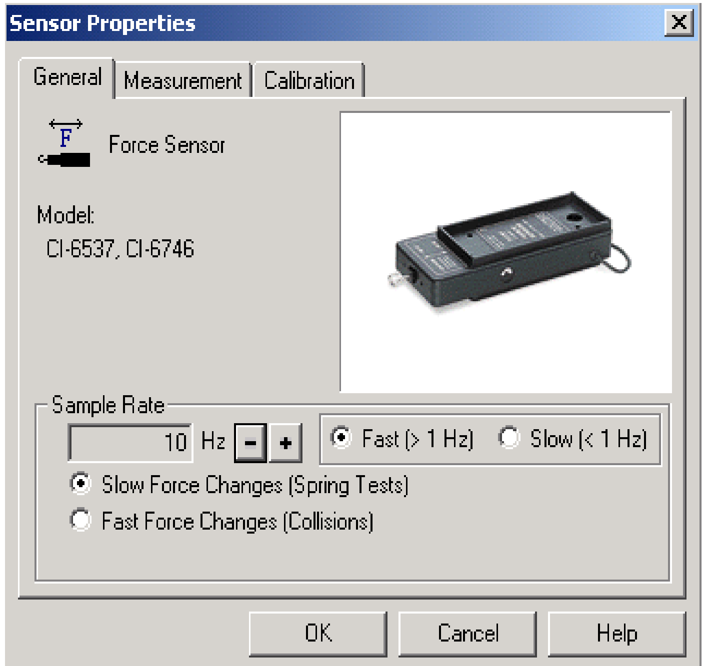
Setting DataStudio

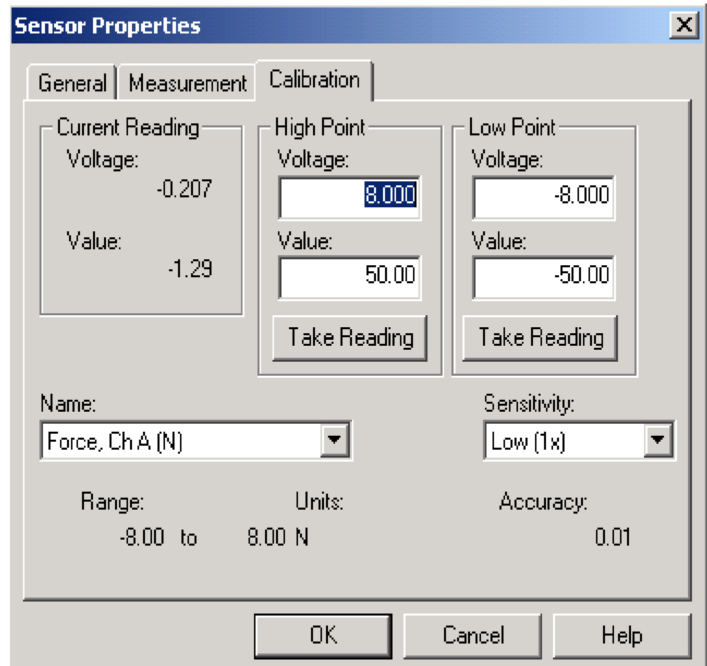
- Create a new experiment. Drag the force sensor to the interface in the Experiment Setup window.



- Double-click the force sensor icon to open a window to set the Sensor Properties.

Force sensor

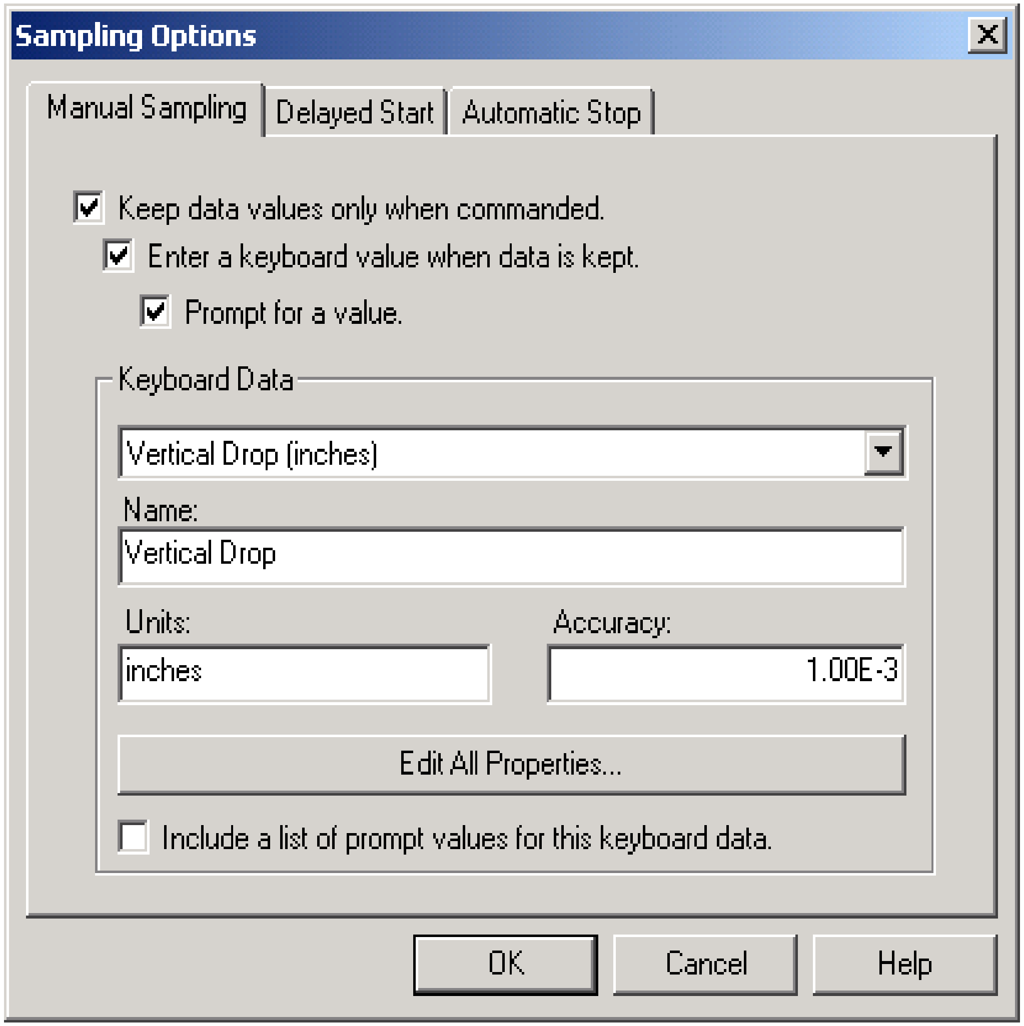




- Under General set Sample Rate to 10Hz and select Slow Force Changes.
- Under Calibration choose Sensitivity Low (1x)

Next: Click Options...

Options for force sensor

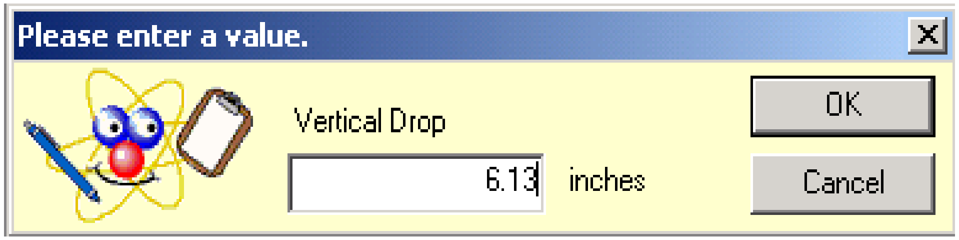


- Check all three boxes.
- Choose New Keyboard Data from the pull-down list in the Keyboard Data area.
- Click Edit all Properties tab which will open another window which allows to name variables and assign units (e.g. Vertical drop and units in mm)
- Click OK on Manual Sampling window. A new variable should appear in the Data window.

Ready to go…!

Data taking

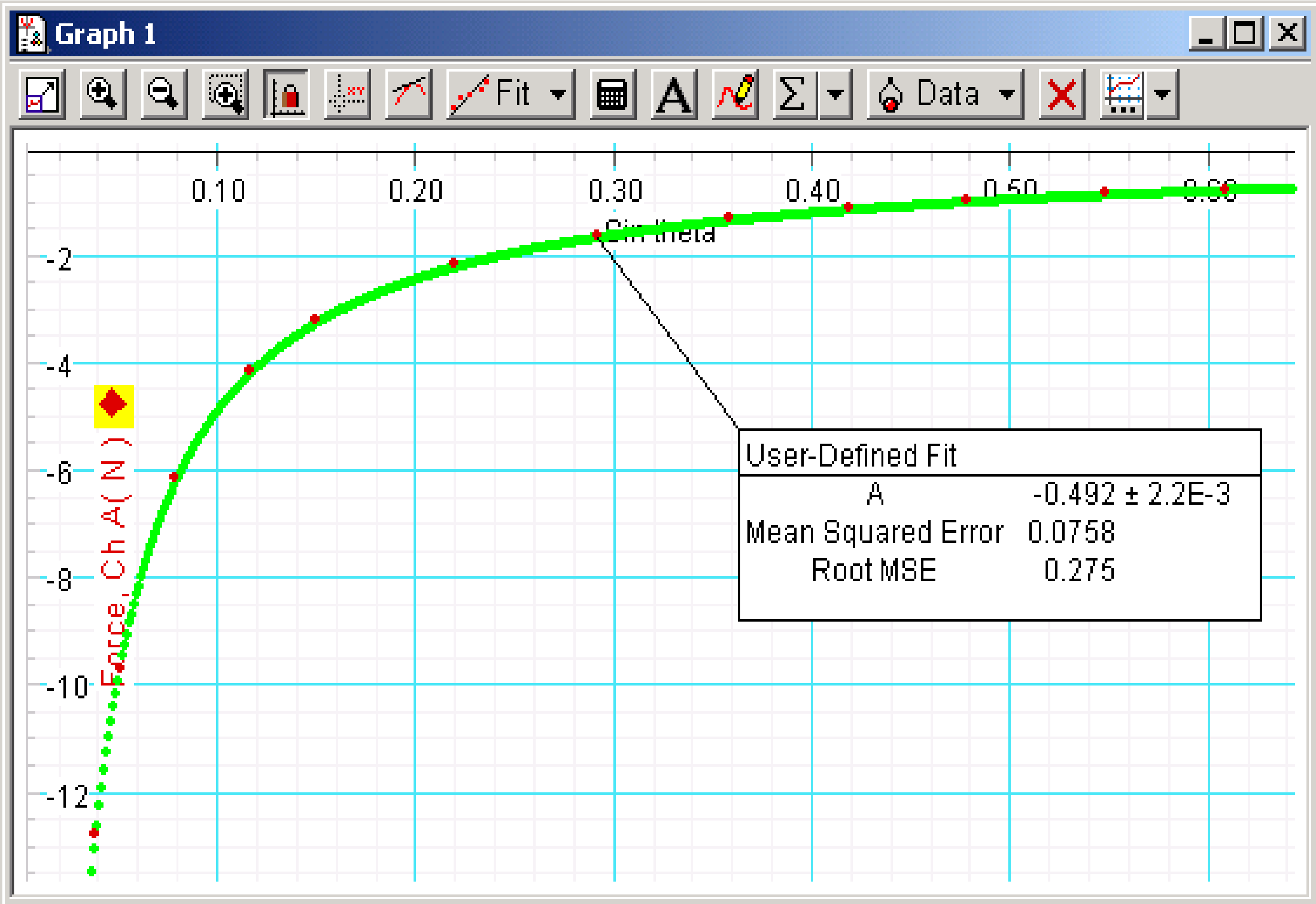
- Click Start! Button turns to Keep.
- Measure vertical drop, click Keep.
- Enter vertical drop into window.



- Shorten string, repeat for 10 to 12 measurements.
- Ensure string passes over pulley.
- Make 2-3 measurements with vertical drop 1.25" or less. (String will be tight even without the weight!)
- Click red stop button when finished.

Analyzing data

- Calculate sinθ from your vertical drop measurements (see write up).
- Plot force on y axis, sinθ on x axis.
- Fit y = A/x (User-defined fit) to your data.



Report

- Hand-in experiment report.
- There is a follow-up question as part of your PS!