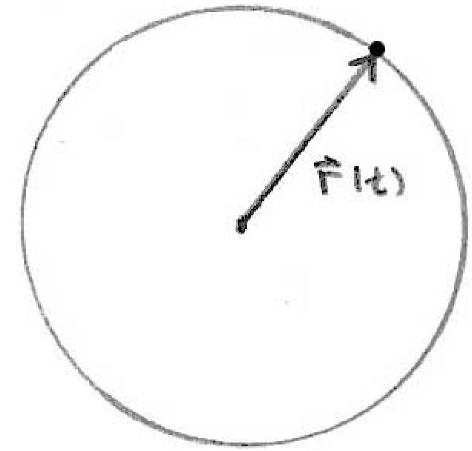
Circular Motion

8.01T

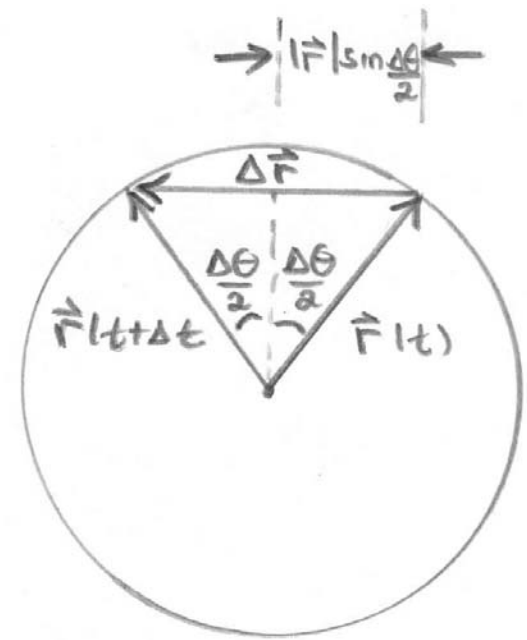
Sept 27, 2004

Position and Displacement

Position vector of an object moving in a circular orbit of radius R



Change in position $\Delta\vec{r}$ between time t and time t+Δt .

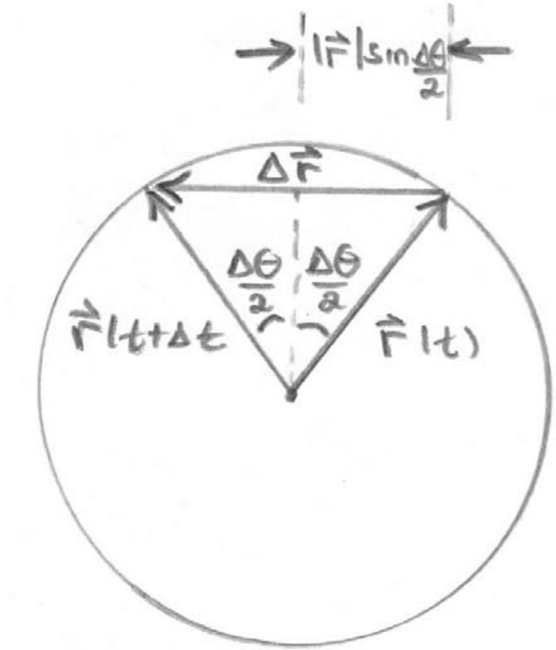


Position vector is changing in direction not in magnitude.

Magnitude of Displacement

- The magnitude of the displacement , is the length of the chord of the circle

$$\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta / 2)$$



Small Angle Approximation

- When the angle is small, approximate

$$\sin\Delta\theta \cong \Delta\theta$$

- infinite power series expansion

$$\sin\Delta\theta = \Delta\theta - \frac{1}{3!}(\Delta\theta)^3 + \frac{1}{5!}(\Delta\theta)^5 - ...$$

- Using the small angle approximation, the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| \cong R\Delta\theta$$

Magnitude of Velocity and Angular Velocity

Magnitude of the velocity is proportional to the rate of change of the magnitude of the angle with respect to time

$$v \equiv \left|\vec{\mathbf{v}}\right| = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t \to 0} \frac{R\left|\Delta \theta\right|}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\left|\Delta \theta\right|}{\Delta t} = R\left|\frac{d\theta}{dt}\right|$$

angular velocity $\omega \equiv \frac{d\theta}{dt}$

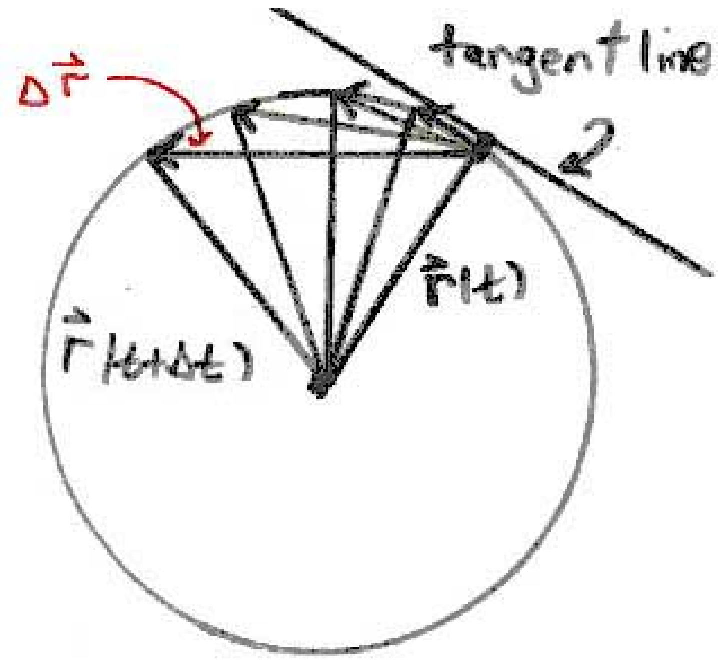
units: [rad-sec^{-1}]

magnitude of velocity $v = R\omega$

Direction of Velocity

- sequence of chord $\Delta\vec{\mathbf{r}}$ directions as Δt approaches zero

- the direction of the velocity at time t is perpendicular to position vector and tangent to the circular orbit



Acceleration

When an object moves in a circular orbit, the acceleration has two components, tangential and radial.

Tangential Acceleration

the *tangential acceleration* is just the rate of change of the magnitude of the velocity

$$a_\theta = \lim_{\Delta t \to 0} \frac{\Delta v_\theta}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = R \frac{d\omega}{dt} = R \frac{d^2\theta}{dt^2}$$

Angular acceleration: rate of change of angular velocity with time

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

units are $\left[\mathrm{rad} \cdot \mathrm{sec}^{-2}\right]$

tangential acceleration $a_{\tan} = R\alpha$

Uniform Circular Motion

- object is constrained to move in a circle and total tangential force acting on the object is zero, then by Newton's Second Law, the tangential acceleration is zero

$$a_{\tan} = 0$$

- magnitude of the velocity (speed) remains constant

Period and Frequency

- The amount of time to complete one circular orbit of radius R is called the period.
- In one period the object travels a distance equal to the circumference,

$$s = 2\pi R = vT$$

- Period:

$$T = 2\pi R / v = 2\pi R / R\omega = 2\pi / \omega$$

- Frequency is the inverse of the period

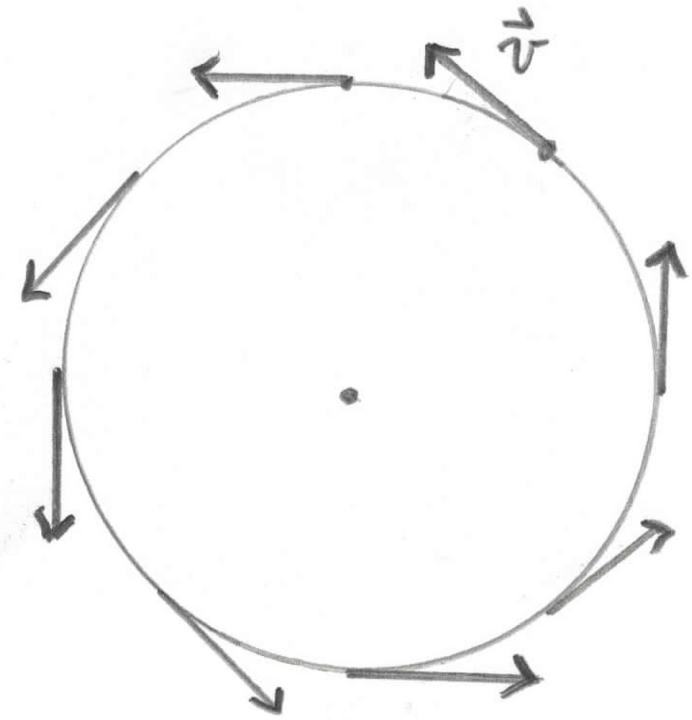
$$f = 1/T = \omega / 2\pi$$

- Units [sec^{-1}] ≡ [Hz]

Radial Acceleration

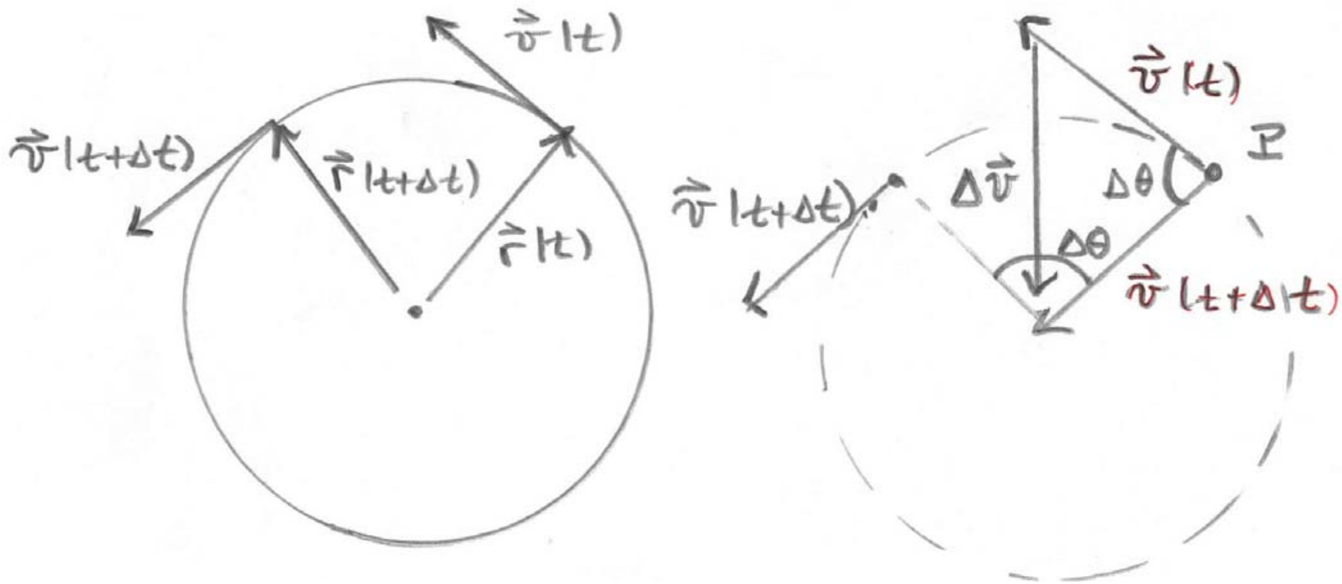
- Any object traveling in a circular orbit with a constant speed is always accelerating towards the center

- Direction of velocity is constantly changing



Change in Magnitude of Velocity

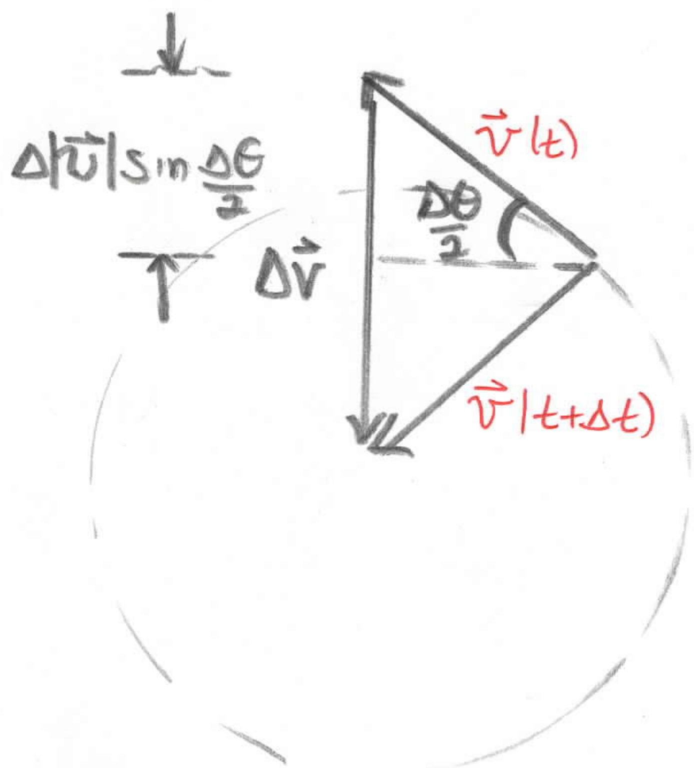
- The velocity vector $\vec{\mathbf{v}}(t+\Delta t)$ has been transported to the point P ,
- change in velocity $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$



Magnitude of Change in Velocity

- The magnitude of the change in velocity is

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2)$$



- small angle approximation

$$\sin\Delta\theta \cong \Delta\theta$$

- Conclusion $\left|\Delta\vec{\mathbf{v}}\right| \cong v\left|\Delta\theta\right|$

Magnitude of Radial Acceleration

- Magnitude

$$a_r = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{\mathbf{v}}\right|}{\Delta t} = \lim_{\Delta t \to 0} \frac{v\left|\Delta \theta\right|}{\Delta t} = v \lim_{\Delta t \to 0} \frac{\left|\Delta \theta\right|}{\Delta t} = v \frac{d\theta}{dt} = v\omega$$

- Recall the magnitude of velocity $v = R\omega$

- Conclusion: $\left|a_r\right| = R\,\omega^2$

Alternative Forms for Magnitude of Radial Acceleration

- Radius and speed $$\left|a_r\right| = \frac{v^2}{R}$$

- Radius and frequency $$\left|a_r\right| = 4\pi^2 R\, f^2$$

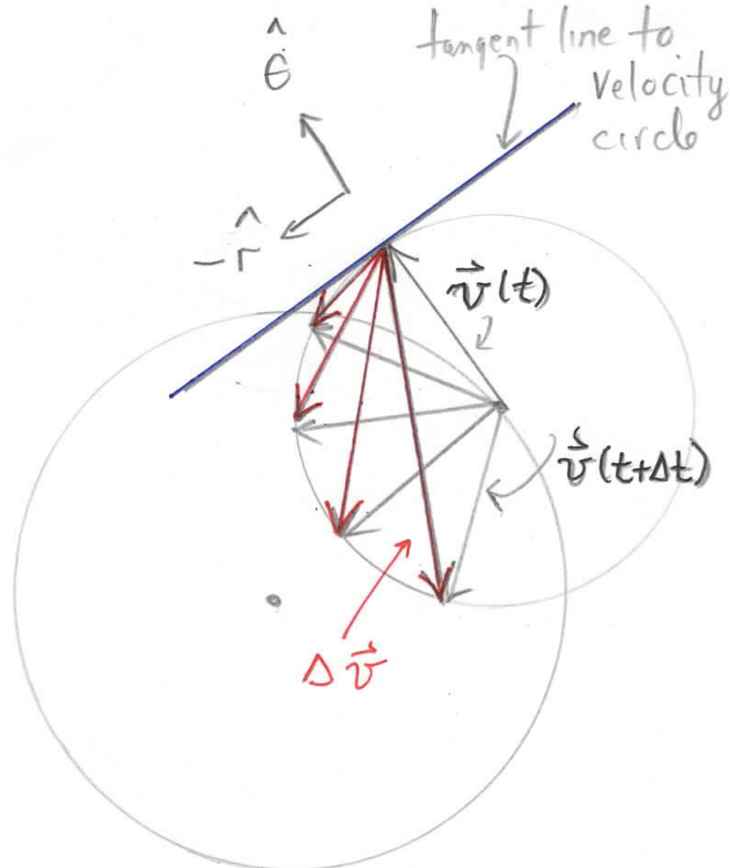
- Radius and period $$\left|a_r\right| = \frac{4\pi^2 R}{T^2}$$

Direction of Radial Acceleration

- sequence of chord
- directions $\Delta\vec{\mathbf{v}}$ as Δt approaches zero

- perpendicular to the velocity vector

- points radially inward

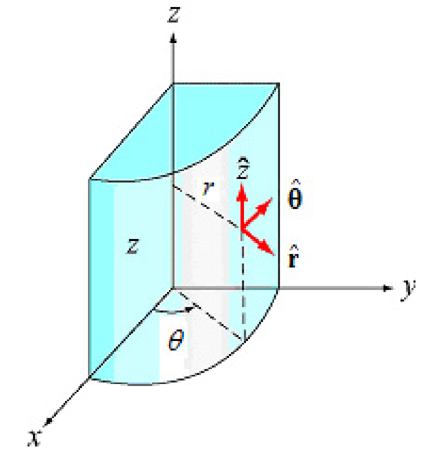


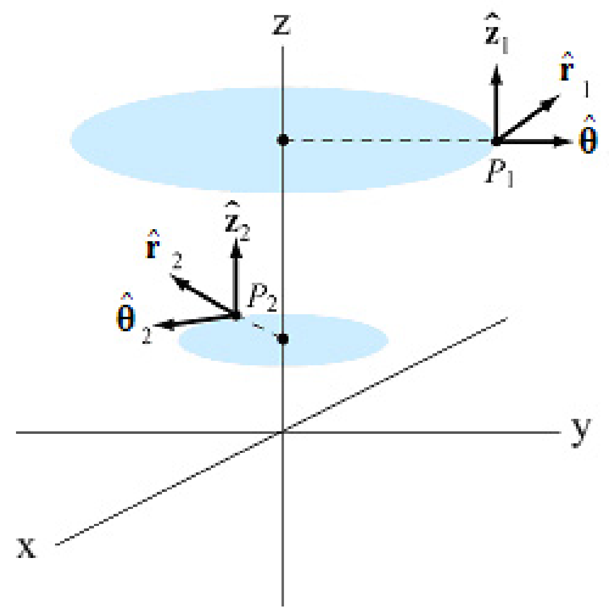
Summary: Circular Motion

- arc length $$s = R\theta$$
- tangential velocity $$v = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$
- tangential acceleration $$a_\theta = \frac{dv_\theta}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha$$
- centripetal acceleration $$\left|a_r\right| = v\omega = \frac{v^2}{R} = R\omega^2$$

Cylindrical Coordinate System

- coordinates (r, θ, z)
- unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}})$
- Unit vectors at different points





Circular Motion Vectorial Description

- Use plane polar coordinates
- Position

$$\vec{\mathbf{r}}(t) = R\hat{\mathbf{r}}(t)$$

- Velocity

$$\vec{\mathbf{v}}(t) = R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t) = R\omega\hat{\boldsymbol{\theta}}(t)$$

- Acceleration

$$\vec{\mathbf{a}} = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}$$

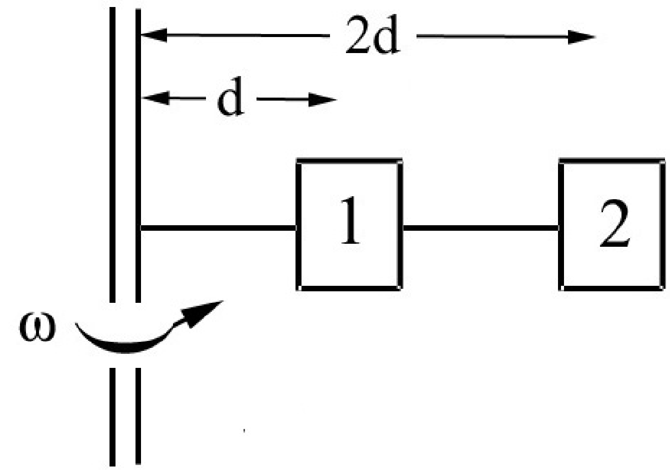
$$a_\theta = r\alpha$$

$$a_r = -r\omega^2 = -(v^2 / r)$$

Class Problem

Two objects of mass m are whirling around a shaft with a constant angular velocity ω. First object is a distance d from central axis, and second object is a distance 2d from the axis. You may ignore the effect of gravity.

a) Draw separate force diagrams for each object.



b) What is the tension in the string between the shaft and the first object?

c) What is the tension in the string between the first object and the second object?