

Experiment 08: Physical Pendulum



Purpose of the Experiment:

In this experiment you investigate rigid body rotational dynamics by studying a pendulum made of a rigid body. This is called a “physical pendulum” to distinguish it from the “ideal pendulum” which consists of a point mass on a string. You will use the *DataStudio* motion sensor to measure the period of the pendulum. This will give you practice in

- calculating moments of inertia,
- calculating the period of a physical pendulum, and
- possibly determining how good the approximation $\sin \theta = \theta$ (made in deriving the equation of motion for the pendulum) is.

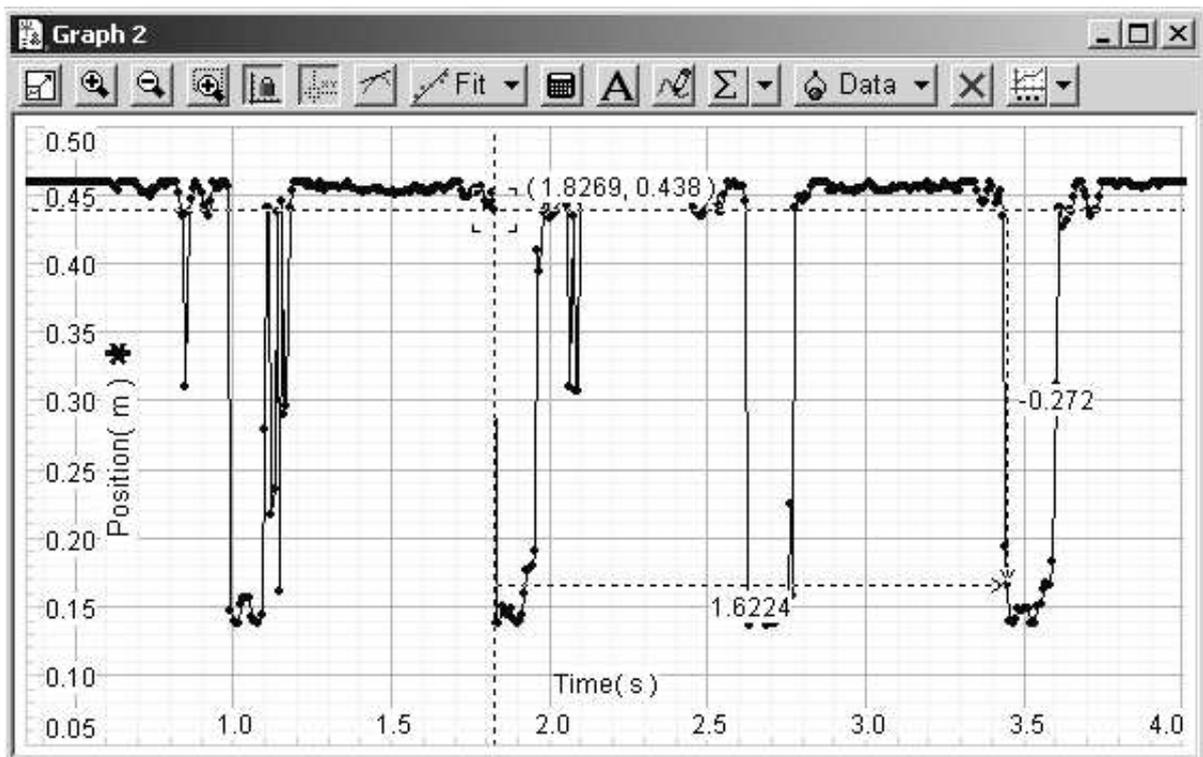
Setting Up the Experiment:

The pendulum is made by hanging a metal ruler over a $\frac{1}{4}$ in. dia. rod clamped to the vertical rod of a support stand. As the ruler is 1.0 m long (longer than the support rod for the stand), place the rod in the hole near the apex of the triangular base and locate the stand so the ruler can hang over the edge of the table—as shown in the photo to the left. The motion sensor should be placed over the rod so that the ruler is four to six inches front of it when hanging at rest. Set the motion sensor on its narrow beam. When the ruler is in front of the motion sensor, it will report a distance 0.10 to 0.15 m; when it is not, a larger distance will be reported. The motion sensor should be set to a sample rate of 120; that should let you measure the pendulum period to an accuracy of ± 0.01 s. **Note:** when the motion sensor tries to detect a distant target (e.g., with the ruler out of the way) it will sample at a much lower rate. To time the pendulum period accurately, you should place a target (such as the back of a chair) about 0.3 to 0.4 m away—for the motion sensor to detect when the ruler has moved aside.

You should set up *DataStudio* with Automatic Start “None” and Delayed Stop after 10 s. (The period of the pendulum is about 1.6 s.) Set the motion sensor to measure only position and make a graph to plot position vs. time.

For each measurement pull the bottom end of ruler aside by the desired distance (between

10 and 50 cm), then release the ruler and click the *DataStudio* start button at about the same time. When the ruler is in the ultrasound beam from the motion sensor, it will measure the distance to the ruler, about 10 to 15 cm. When it is to the side of the beam, the motion sensor will measure the distance to the chair you placed in the beam, about 40 cm. You do not care about the distance to the ruler and only want to detect the times when the ruler passes in front of the sensor. You should be aware that the motion sensor can give weird results for the distance to objects closer than 15 cm. If you seem to be getting strange behavior, try sliding the ruler out on the rod to get it farther away from the motion sensor.



The graph above has my results with the graph scaled to show only the first four seconds. As adjacent pulses correspond to the ruler swinging in opposite directions, to determine the period of the pendulum, you must find the time between two pulses that have one pulse in between them. Choose a point that is easy to identify (on my graph the point when the motion sensor first detects the ruler seems to be sharply defined) and use the Smart Tool on the graph—make sure it “locks on” to the data points—to find the time when this occurs. If you position the cursor close to the point the Smart Tool is locked on to, the cursor becomes a hand with a small triangle near the index finger  .

Drag this cursor to the corresponding point on the graph one period later and release it. *DataStudio* will calculate and display the time difference between the two points, which is the period of the pendulum. As you can see on my graph, it is 1.62 s.

You should measure the period with the ruler started 10, 25, and 50 cm to the side of its vertical equilibrium position; these displacements do not need to be very accurate, so you may guess them. They correspond to angular amplitudes θ_0 of about 0.10, 0.25, and 0.52 radians. Complete a table like this one:

Displacement	θ_0	Period
0.10 m	0.10	
0.25 m	0.25	
0.50 m	0.52	

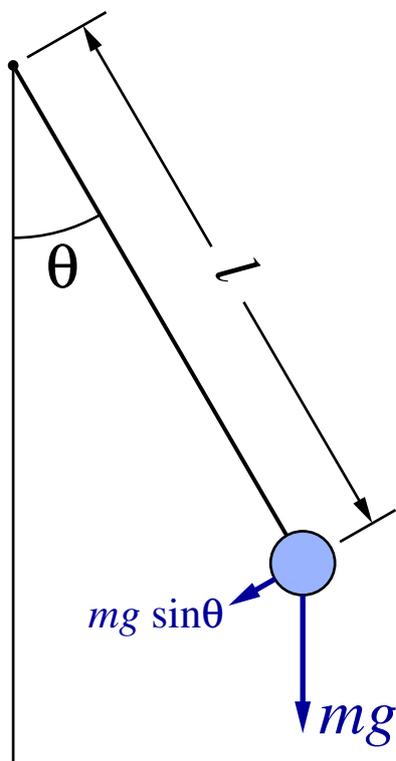
Next, try modifying the moment of inertia of the pendulum and observing how its period changes. Do that by using a $1\frac{1}{4}$ in. binder clip to attach a 50 gm brass weight to the ruler. The mass of the clip is 8.6 gm. The small hole in the center of the weight can be used to identify fairly accurately where the center of mass of the weight is located.

You should measure the period and report the results for the three positions in the table below. For convenience, you may carry out these measurements with an initial ruler displacement of 20 cm; it does not matter much what you choose. The “position” in the table is the distance from the top end of the ruler where the center of mass of the extra weight and clip is located. The position 0.50 m is where the center of mass of the ruler is; the mass of the ruler is 0.159 kg.

Displacement	Weight	Position	Period
0.20 m	58.6 gm	0.25 m	
0.20 m	58.6 gm	0.50 m	
0.20 m	58.6 gm	0.90 m	

Appendix: Theoretical Discussions

1. The Ideal Pendulum



For the ideal pendulum the tangential restoring force is $mg \sin \theta$ and the displacement is $l\theta$, giving rise to the equation of motion ($F = ma$)

$$ml \frac{d^2\theta}{dt^2} + mg \sin \theta = 0$$

with the approximation $\sin \theta \approx \theta$ this becomes

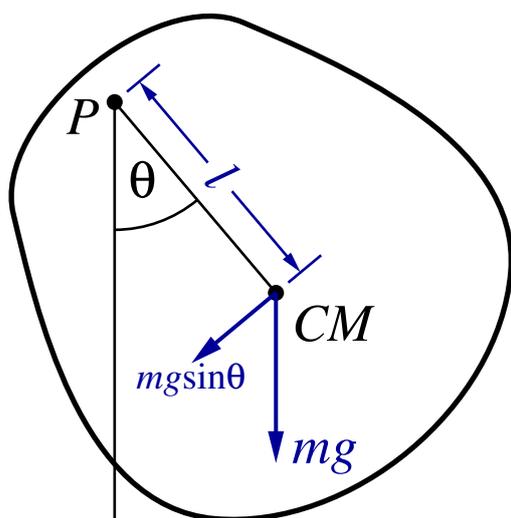
$$ml \frac{d^2\theta}{dt^2} + mg\theta = 0$$

This is the equation of motion for a harmonic oscillator whose angular frequency ω_0 is $\sqrt{g/l}$. Thus the period of oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{l/g}.$$

Note: ω_0 is the angular frequency of oscillation of the pendulum; it is **not** $d\theta/dt$. It can be confusing that the same greek letter is commonly used for both quantities.

2. The Physical Pendulum



The physical pendulum is a rigid body that rotates about an axis, shown as P in the figure to the left.

Its equation of motion will be $\tau = I\alpha$, where I is the moment of inertia about the axis of rotation. Let us suppose that the distance from the axis P to the center of mass CM is l . Then the restoring torque will be $mgl \sin \theta$, which leads to the equation of motion

$$I \frac{d^2\theta}{dt^2} + mgl \sin \theta = 0$$

with the approximation $\sin \theta \approx \theta$ this becomes

$$I \frac{d^2\theta}{dt^2} + mgl\theta = 0$$

This is the equation of motion for a harmonic oscillator whose angular frequency ω_0 is $\sqrt{mgl/I}$. Thus the period of oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{I/(mgl)} = 2\pi\sqrt{l/g}\sqrt{\frac{I}{ml^2}}.$$

3. The Ruler Pendulum

You need to know the moment of inertia of the ruler about an axis through the hole near one end. Suppose the ruler has a length b and a width a . The textbook (Table 9.2) gives an expression for its moment of inertia about an axis that passes through the *CM* and is perpendicular to the flat of the ruler as

$$I_{CM} = \frac{m}{12}(a^2 + b^2).$$

If the distance from the *CM* to the pivot point P is l , we can use the parallel axis theorem (section 9.5 in the text) to find the moment of inertia about P as

$$I_P = \frac{m}{12}(a^2 + b^2) + ml^2$$

which gives the period of the ruler pendulum

$$T = 2\pi\sqrt{\frac{a^2 + b^2 + 12l^2}{12gl}} = 2\pi\sqrt{l/g}\sqrt{1 + \frac{a^2 + b^2}{12l^2}} = 1.622 \text{ s}$$

using the dimensions of the ruler: $a = 0.028$ m, $b = 1.000$ m, and $l = 0.479$ m. (The mass of the ruler is 0.159 kg.)

4. Changing the Moment of Inertia

To calculate the period of the pendulum when a weight is clipped to it, you must know the moment of inertia of the ruler with the weight attached to it. Suppose the weight and the clip are point masses and their centers of mass are both located at position p on the ruler. As the hole is 2.1 cm from the top end of the ruler, the distance from the pivot to the *CM* of the weight will be $d = p - 0.021$ m. The moment of inertia (about P) of the ruler with the weight attached will then be

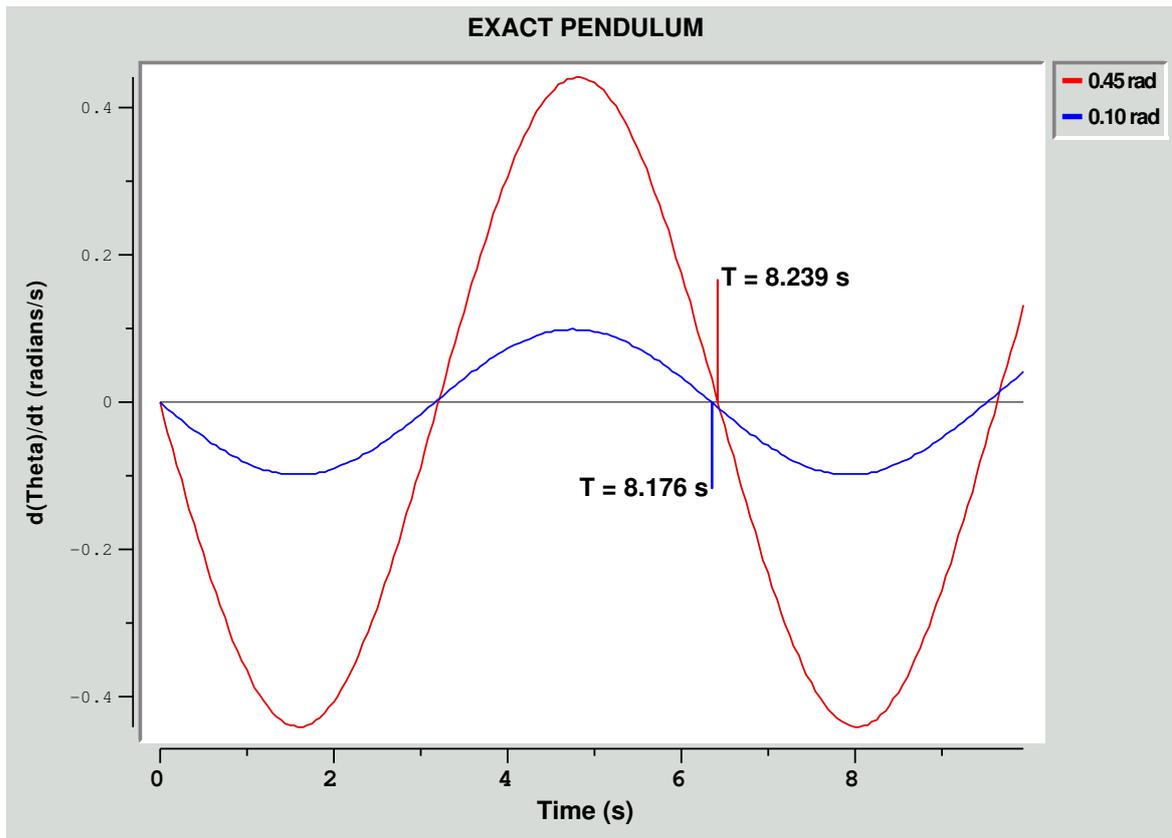
$$I_P = \frac{m_r}{12}(a^2 + b^2) + m_rl^2 + m_cd^2 + m_wd^2$$

where m_r , m_c and m_w are the masses of the ruler, the clip and the weight, respectively. The restoring torque that tries to return the pendulum to a vertical position will be

$$\tau = (m_rl + m_cd + m_wd)g \sin \theta \approx (m_rl + m_cd + m_wd)g\theta.$$

You can use these expressions to calculate the period of oscillation. As part of the homework assignment due November 16, you should derive an expression for the period of the pendulum with the weight attached, evaluate it, and compare with the measured results you reported in the table at the top of page 3. (The mass of the ruler is 0.159 kg.)

A Note on the $\sin \theta = \theta$ Approximation.



The figure above plots the numerical solutions to the exact pendulum equation of motion for initial amplitudes of $\theta_0 = 0.10$ and $\theta_0 = 0.45$ radians. The $\sin \theta = \theta$ approximation is very good when $\theta_0 = 0.10$, but the numerical solution shows the actual period is about 0.8% longer than the approximation would predict when $\theta_0 = 0.45$. This difference (about 0.013 s for your experiment) is about the same as the experimental error, but you may detect a slightly longer period for your largest amplitude oscillations.

The first order analytic correction to the $\sin \theta = \theta$ approximation gives a period

$$T(\theta_0) = T(0)[1 + \theta_0^2/16]$$

where θ_0 is the angular amplitude (in radians) of the pendulum motion and $T(0)$ is the period assuming the approximation is exact. (This is discussed in the Course Notes on Rotational Dynamics.)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.01T

Fall Term 2004

Part of Problem Set 10

Section and Group: _____

Your Name: _____

Part One: Ruler Pendulum

The ruler has a mass $m_r = 0.159$ kg, a width $a = 0.028$ m, a length $b = 1.00$ m, and the distance from the pivot point to the center of mass is $l = 0.479$ m. Enter your measured period into the T_{meas} column in the table below and calculate the other entries using the formulas

$$T_{\text{ideal}} = 2\pi\sqrt{l/g}$$

and

$$T_{\text{theory}} = 2\pi\sqrt{l/g}\sqrt{\frac{I}{ml^2}}\left(1 + \frac{\theta_0^2}{16}\right)$$

with $g = 9.805$ ms⁻².

θ_0	T_{meas}	T_{theory}	T_{ideal}
0.10			
0.52			

Part Two: Added Mass

Consider the effect of a brass weight clipped to the ruler. The weight is shaped like a washer with an outer radius $r_o = 0.016$ m and an inner radius $r_i = 0.002$ m; it has a mass $m_w = 0.050$ kg. It is clipped to the ruler so that the center of the inner hole is over the 0.500 m mark on the ruler, or $l = 0.479$ m from the pivot point. The clip has a mass of $m_c = 0.0086$ kg and you may assume its center of mass is also over the 0.500 m mark on the ruler. If you treat the washer and the clip as point masses, then, as was discussed in the notes for Experiment 08, the combined unit has a moment of inertia about the pivot point

$$I_P = \frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2$$

where $d = l$ for this situation. The restoring torque that tries to return the pendulum to a vertical position will be

$$\tau = (m_r l + m_c d + m_w d)g \sin \theta \approx (m_r l + m_c d + m_w d)g \theta .$$

1. Use these two expressions to derive an equation of motion for the pendulum and calculate its period T in the small amplitude ($\sin \theta = \theta$) approximation. Express your answer algebraically in terms of the variables a , b , d , l , m_r , m_w , m_c and g .

2. Evaluate the result numerically and compare with the value you measured in your experiment.

2. If you treat the brass washer as a point mass, its moment of inertia about the pivot point is $I_{w,P} = m_w l^2$. However the washer also has a moment of inertia about its center of mass that is given by $I_w = \frac{1}{2} m_w (r_o^2 - r_i^2)$. When this is taken into account, what is the new (and more accurate) expression for $I_{w,P}$? How many percent does this differ from the simpler expression $I_{w,P} = m_w l^2$?

Solution to the Exact Equation

Solve the equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta$$

with an amplitude θ_0 , which gives a boundary condition $\omega = d\theta/dt = 0$ when $\theta = \theta_0$. Make the substitution $\omega = d\theta/dt$; then

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta}$$

and the equation becomes

$$\omega \frac{d\omega}{d\theta} = -\frac{g}{l} \sin \theta \quad \text{or} \quad \omega d\omega = -(g/l) \sin \theta d\theta.$$

Integrating gives

$$\omega^2 = (2g/l) \cos \theta + C_0.$$

The boundary condition requires $C_0 = -(2g/l) \cos \theta_0$. Replacing ω by $d\theta/dt$ gives the equation

$$\frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \sqrt{\frac{2g}{l}} dt$$

Using the identity

$$\cos \theta = 1 - 2 \sin^2(\theta/2)$$

the equation may be written as

$$\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} = 2\sqrt{\frac{g}{l}} dt = \frac{2 d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

where $k = \sin(\theta_0/2)$ and $\sin \alpha = (1/k) \sin(\theta/2)$. This equation may be integrated to give

$$\int [1 - k^2 \sin^2 \alpha]^{-\frac{1}{2}} d\alpha = t\sqrt{\frac{g}{l}} + C_1.$$

The integral on the left is an elliptic integral of the first kind, and cannot be evaluated analytically. However if k is small, we may expand the square root using the binomial theorem

$$\int \left[1 + \frac{k^2}{2} \sin^2 \alpha + \dots \right] d\alpha = \alpha + \frac{k^2}{4} (\alpha - \sin \alpha \cos \alpha) = t\sqrt{\frac{g}{l}} + C_1.$$

Suppose the boundary condition when $t = 0$ is $\theta = \alpha = 0$; this requires $C_1 = 0$. Then

$$t\sqrt{\frac{g}{l}} = \alpha + \frac{k^2}{4}(\alpha - \sin \alpha \cos \alpha) .$$

After $\frac{1}{4}$ period ($t = T/4$) one will have $\theta = \theta_0$ or $\alpha = \pi/2$. Thus the period is

$$T = 2\pi\sqrt{\frac{l}{g}} \left(1 + \frac{k^2}{4}\right) = 2\pi\sqrt{\frac{l}{g}} \left(1 + \frac{\theta_0^2}{16} + \dots\right)$$