Massachusetts Institute of Technology
Department of Physics

8.01T **Fall 2004**

Study Guide Final Exam Solutions

Part A: Kinetic Theory, First Law of Thermodynamics, Heat Engines

Problem 1 *Energy Transformation, Specific Heat and Temperature*

Suppose a person of mass $m = 6.5\times10^{2}\, kg$ is running at a speed $v = 3.8\, m/s$ and has a catabolic power output (rate of internal energy consumption) $9.45\times10^{2}\, W$ during a $1.0\times1\ ^{1}\, km$ workout. Suppose the runner converts 20% of the internal energy change into mechanical work. The rest of the energy goes into heat. If the specific heat of the runner is $c = 4.19\times10^{3}\, J/kg-K$, how much would the body temperature rise after running the $10\, km$?

The total energy generated by the runner is

$$\Delta U = P \Delta t$$

the runner runs for a time

$$\Delta t = \frac{d}{v} = \frac{1.0 \times 10^4 \, m}{3.8 \, m/s} = 2.6 \times 10^3 \, s$$

The change of internal energy

$$\Delta U = P \Delta t = \frac{Pd}{v} = (9.45 \times 10^2 \, W) \frac{(1.0 \times 10^4 \, m)}{3.8 \, m/s}$$

$$\Delta U = 2.49 \times 10^6 \, J$$

note if the runner is only 20% effecent then the mechanical power output

$$P_{mech} = \varepsilon P_{cat} = (.20)(9.45 \times 10^2 W)$$

$$= 1.89 \times 10^2 W$$

the change of internal energy

$$\Delta U = -W_{i \to f} + Q_{i \to f}$$

$$\frac{\Delta U}{\Delta t} = -\frac{W_{i,f}}{\Delta t} + \frac{Q_{i,f}}{\Delta t}$$

catabolic rate positive = mechanical power positive + power of heat generated

$$\frac{Q_{i,f}}{\Delta t} = \frac{\Delta U}{\Delta t} + \frac{W_{i,f}}{\Delta t}$$

$$= 9.45 \times 10^2 W - 1.89 \times 10^2 W$$

$$\frac{Q_{i,f}}{\Delta t} = m\,c\,\frac{\Delta T}{\Delta t}$$

$$\Delta T = \frac{Q_{i,f}}{mc} = \frac{Q_{i,f}\,\Delta t}{mc} = \frac{Q_{i,f}}{mc}\frac{d}{v}$$

$$= \frac{(7.56\times10^{2}\,W)(1.0\times10^{4}\,m)}{(6.5\times10^{1}\,kg)(3.8\,m/c)(4.19\times10^{3}\,J/kg\cdot K)}$$

$$= 7.3\ ^{\circ}K$$

the average force of friction of the atmosphere

$$\vec{f}_{air} = -\vec{f}_s$$

Since $P_{mech} = f_s\,v$, the friction force is $f_s = \frac{P_{mech}}{v} = \frac{1.89\times10^{2}\,W}{3.8\,m/s} = 4.97\times10^{1}\,N$

Problem 2 Kinetic Theory An ideal gas has a density of 1.78 kg/m^3 is contained in a volume of 44.8×10^{-3} m^3. The temperature of the gas is 273 K. The pressure of the gas is 1.01×10^5 Pa. The gas constant $R = 8.31\,\mathrm{J \cdot K^{-1} \cdot mole^{-1}}$.

a) What is the root mean square velocity of the air molecules?
b) How many moles of gas are present?
c) What is the gas?
d) What is the internal energy of the gas?

a) $$P = \frac{1}{3}\rho \langle v^2 \rangle \Rightarrow$$

$$v_{rms} = \left(\frac{3P}{\rho}\right)^{1/2} = \left(\frac{(3)(1.01 \times 10^5\ N/m^2)}{1.78\ kg/m^3)}\right)^{1/2}$$

$$= 4.13 \times 10^2\ m/s$$

b) $$\rho = \frac{m^{total}}{V} = \frac{n_m M_{molar}}{V}$$

$$PV = n_m RT \Rightarrow$$

$$n_m = \frac{PV}{RT} = \frac{(1.01 \times 10^5 N \cdot m^{-2})(44.8 \times 10^{-3} m^3)}{(8.31\ J \cdot K^{-1} \cdot mole^{-1})(273\ K)}$$

$$= 2.0\ moles$$

$$M_{molar} = \frac{\rho V}{n_m} = \frac{(1.78\ kg/m^3)(44.8 \times 10^{-3} m^3)}{(2.0\ moles)}$$

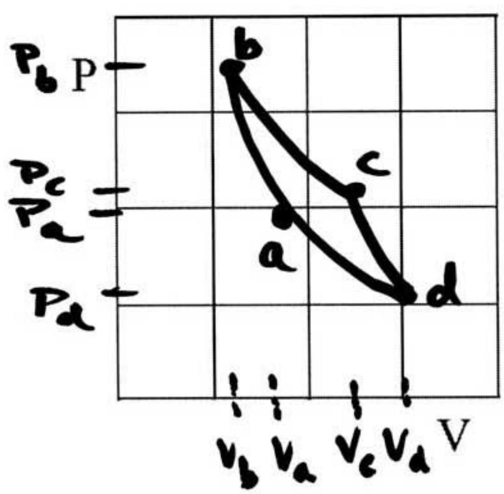
$$= 3\ 40.0 \times 10^{-3}\ kg/mole$$

The ideal gas is argon.

Problem 3: Carnot Cycle of an Ideal Gas

In this problem, the starting pressure P_a and volume V_a of an ideal gas in state a, are given. The ratio $R_V = V_c / V_a > 1$ of the volumes of the states c and a is given. Finally a constant $\gamma = 5/3$ is given. You do not know how many moles of the gas are present.

a) Read over steps (1)- (4) below and sketch the path of the cycle on a $P - V$ plot on the graph below. Label all appropriate points.



(1) In the first of four steps, a to b, an ideal gas is compressed from V_a to V_b while no heat is allowed to flow into or out of the system. The compression of the gas raises the temperature from an initial temperature T_1 and to a final temperature T_2. During this process the quantity $PV^\gamma = \text{constant}$, where $\gamma = 5/3$.

a) What is the pressure P_b and volume of the gas V_b after the compression is finished?

Answer: According to the ideal gas law, $P_b V_b = n_m R T_2$ and $P_a V_a = n_m R T_1$ so

$$P_b V_b = P_a V_a \frac{T_2}{T_1}.$$

So the pressure

$$P_b = P_a \frac{V_a}{V_b} \frac{T_2}{T_1}$$

The compression satisfies $P_b V_b^\gamma = P_a V_a^\gamma$ so using the above result for pressure P_b, we get

$$P_b V_b^{\gamma} = P_a \frac{V_a}{V_b} \frac{T_2}{T_1} V_b^{\gamma} = P_a V_a^{\gamma} .$$

This becomes using $\gamma = 5/3$

$$V_b^{2/3} = \frac{T_1}{T_2} V_a^{2/3}$$

The volume V_b is then

$$V_b = \left(\frac{T_1}{T_2} \right)^{3/2} V_a .$$

Thus the ratio of the volumes is

$$\frac{V_a}{V_b} = \left(\frac{T_2}{T_1} \right)^{3/2}$$

So the pressure P_b is

$$P_b = P_a \left(\frac{T_2}{T_1} \right)^{5/2}$$

b) What is the change in internal energy of the gas during this change of state?

Answer: The change in internal energy is

$$U_b - U_a = \frac{3}{2} n_m R \Delta T = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$$

c) What is the work done by the gas during this compression?

Answer: Since no heat is exchanged $Q_{ba} = 0$

$$U_b - U_a = -W_{ba} + Q_{ba} = -W_{ba} = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$$

So

$$W_{ba} = -\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1} < 0 .$$

The surroundings do work compressing the gas.

(2) The gas is now allowed to expand isothermally from b to c, from volume V_b to volume V_c .

d) Express the work done by the gas in this process W_{cb} and the amount of heat Q_{cb} that must be added from the heat source at T_2 in terms of P_a, V_a, T_2, T_1, and V_c. Is this heat positive or negative? Explain whether it is added to the system or removed.

Answer: This is an isothermal expansion so the temperature does not change $\Delta T = 0$. Thus the internal energy is constant,

$$U_c - U_b = \frac{3}{2} n_m R \Delta T = 0 .$$

The gas does work on the surroundings because it is expanding. The pressure is not constant during this expansion. Since the gas is expanding by an isothermal process, the Ideal Gas Law relates the pressure and volume variation according to

$$P = \frac{n_m RT}{V} .$$

Therefore the work done by the gas on the surroundings is the integral

$$W_{cb} = n_m RT_2 \int_{V_b}^{V_c} \frac{dV}{V} = n_m RT_2 \ln(V_c / V_b) .$$

Using the result for the volume V_b from part a)

$$V_b = \left(\frac{T_1}{T_2} \right)^{3/2} V_a ,$$

the work is

$$W_{cb} = n_m RT_2 \int_{V_b}^{V_c} \frac{dV}{V} = n_m RT_2 \ln(\left(\frac{T_2}{T_1} \right)^{3/2} V_c / V_a)$$

Recall that the volumes are related according to $R_V = V_c / V_a > 0$ and $n_m R = P_a V_a / T_1$ so the work done is positive and given by

$$W_{cb} = n_m R T_2 \ln(V_c / V_a) = P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V) > 0$$

From The First Law of Thermodynamics,

$$0 = U_c - U_b = -W_{cb} + Q_{cb} ,$$

Thus the heat that flows into the system from the heat source at temperature T_2 is equal to the work done by the expanding gas.

$$Q_{cb} = W_{cb} = P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V) > 0 ,$$

Note that this heat flow must flow from the higher temperature heat source into the system because as the gas expands it should lose internal energy and would decrease its temperature unless heat flows into the system keeping the internal energy and hence the temperature constant.

e) What is the pressure P_c of the gas after the expansion is finished?

Answer: $P_c V_c = n_m R T_2 = \frac{P_a V_a}{T_1} T_2$. Thus

$$P_c = \frac{P_a V_a}{V_c} \frac{T_2}{T_1} = \frac{P_a}{R_V} \frac{T_2}{T_1} .$$

(3) When the gas has reached point c is expands from V_c to V_d while no heat is allowed to flow into or out of the system. The expansion of the gas lowers the temperature and pressure from an initial temperature T_2 to a final temperature T_1 . During this process the quantity $PV^{\gamma} = \text{constant}$.

f) What is the pressure P_d and the volume V_d of the state d of the gas after the expansion is finished?

Answer: This calculation is identical to part a), with state d replacing state a, and state c replacing state b. So the volume V_b is then

$$V_c = \left(\frac{T_1}{T_2}\right)^{3/2} V_d .$$

Thus the ratio of the volumes is

$$\frac{V_d}{V_c} = \left(\frac{T_2}{T_1}\right)^{3/2}$$

So the pressure P_c is

$$P_c = P_d \left(\frac{T_2}{T_1}\right)^{5/2}$$

hence

$$P_d = P_c \left(\frac{T_1}{T_2}\right)^{5/2}$$

g) What is the change in internal energy of the gas during this change of state?

Answer: The decrease in the internal energy is due to the temperature decrease of the ideal gas during expansion

$$U_d - U_c = \frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$$

h) What is the work done by the gas during this expansion?

Answer: Since no heat is exchanged $Q_{dc} = 0$

$$U_d - U_c = -W_{dc} + Q_{dc} = -W_{dc} = \frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$$

So

$$W_{dc} = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1} > 0 .$$

The gas does work on the surroundings since the gas is expanding.

(4) The gas is now compressed isothermally from d to a at constant T_1 from volume V_d back to V_a.

i) Find the work done by the system on the surroundings W_{ad} and the amount of heat Q_{ad} that flows between the system and the surroundings. Are these quantities positive or negative? Explain whether heat is added to the system or removed from the heat source at T_1.

Answer: When the gas undergoes compression it will increase its internal energy but heat flows out of the system maintaining constant internal energy, $\Delta U = 0$ and hence the compression is isothermal. The calculation of the work and heat is similar to step (2) except the temperature is held at T_1 . The work done by the system on the surroundings is negative and is given by the integral

$$W_{ad} = n_m R T_1 \int_{V_d}^{V_a} \frac{dV}{V} = n_m R T_1 \ln(V_a / V_d) = P_a V_a \ln(V_a / V_d) = -P_a V_a \ln(R_V) .$$

From part f) the volume $V_d = \left(\frac{T_2}{T_1}\right)^{3/2} V_c$ so the work done is

$$W_{ad} = n_m R T_1 \int_{V_d}^{V_a} \frac{dV}{V} = n_m R T_1 \ln(V_a / V_d) = P_a V_a \ln(V_a / \left(\frac{T_2}{T_1}\right)^{3/2} V_c) = -P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$$

According to the First Law this is equal to the heat that flows into the system which is also negative which means that it actually flows out of the system into the surroundings at temperature T_1,

$$Q_{ad} = W_{ad} = -P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V) .$$

Total Cycle:

j) What is the total work W_{cycle} done by the gas during this cycle?

Answer: The work done by the heat engine on the surroundings during the cycle is positive and given by

$$W_{cycle} = P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V) - P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V) = P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)\left(\frac{T_2}{T_1} - 1\right).$$

k) What is the total heat $Q_{cycle}(from\, T_2)$ drawn from the higher temperature heat source during this cycle?

Answer: The heat that flowed from the higher temperature heat source T_2 occurred during step (2) $b \to c$ isothermal expansion,

$$Q^{total}\ taken\ from\ heat\ source\ at\ T_2 = P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V).$$

l) What is the efficiency of this cycle $\varepsilon_{\max} = W_{cycle} / Q_{cycle}(from\ T_2)$?

Answer: The efficiency is given by ratio of the work done divided by the ehat flowing into the system from the higher temperature heat source

$$\varepsilon_{\max} = W_{cycle} / Q_{cycle}(from\ T_2) = P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)\left(\frac{T_2}{T_1} - 1\right) / P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$$

$$\varepsilon_{\max} = \left(\frac{T_2}{T_1} - 1\right) / \frac{T_2}{T_1} = \frac{T_2 - T_1}{T_2} = \frac{\Delta T}{T_2}.$$

Table 1: Summary of Heat Engine

Process	$U_f - U_i$	$W_{f,i}$	$Q_{f,i}$
$a \to b$ adiabatic compression	$\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ positive	$-\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ negative	0
$b \to c$ isothermal expansion	0	$P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive	$P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive from T_2
$c \to d$ adiabatic expansion	$\frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$ negative	$\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ positive	0
$d \to a$ isothermal compression	0	$-P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ negative	$-P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ negative, into T_1
Total	0	$P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ $-P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive	$P_a V_a \frac{T_2}{T_1} \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ $-P_a V_a \ln(\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive, from T_2 into T_1

Problem 4 Heat pump

A reversible heat engine can be run in the other direction, in which case it does negative work W_{cycle} on the world while "pumping" heat $Q_{cycle}(into\, T_2)$ into a reservoir at an upper temperature, T_2, from a lower temperature, T_1. The heat gain of this cycle, defined to be

$$g \equiv Q_{cycle}(into\, T_2)/W_{cycle} = (1/\varepsilon_{\max})$$

where $\varepsilon_{\max} = (T_2 - T_1)/T_2$ is the maximum thermodynamic efficiency of a heat engine. The refrigerator performance is defined to be

$$K \equiv Q_{cycle}(\,from\, T_1)/W_{cycle} = T_1/(T_2 - T_1)$$

Consider that you have a large swimming pool and plan to heat your house with a heat pump that pumps heat from the pool into your house. A large plate in the water will remain at $0\ ^oC$ due to the formation of ice. You pick T_2 to be $50\ ^oC$, which will be the temperature of the (large) radiators used to heat your house. Assume that your heat pump has the maximum efficiency allowed by thermodynamics.

a) What is the heat gain and the refrigerator performance for this cycle? Be careful to use units of Kelvin for temperature.

b) If your house formerly burned 1200 gallons of oil in a winter (at $2.00/gallon), how much will the electricity cost (at $0.10 per kilowatt-hour) to replace this heat using the heat pump? A gallon of oil has mass 3.4 kg and contains $1.4\times10^8\ \mathrm{J\cdot gal^{-1}}$.

c) The ice cube that appears in your pool over the winter will be how many meters on each side? (It takes $3.35\times10^6\ \mathrm{J}$ to melt one kg of ice; it takes up this much heat when freezing.)

This would be great for cooling your house in the summer – even if the pool warmed up enough to swim in it, you could still cool your house by running the heat pump in reverse as an air conditioner! More practically, you might be able to use ground water (and the dirt around it) as the heat sink.

a) the heat gain with $T_2 = 323$ K, $T_1 = 273$ °K

$$g = (1/\varepsilon_{max}) = \frac{T_2}{\Delta T} = \frac{T_2}{T_2 - T_1} = \frac{323 \text{ K}}{(323-273)\text{K}} = 6.46$$

the refrigerator performance

$$K = \frac{T_1}{T_2 - T_1} = \frac{273 \text{ K}}{323 \text{ K} - 273 \text{ K}} = 5.46$$

b) if you burned 1200 gallons of oil, the energy content

$$\Delta U = (1200 \text{ gal})(1.4 \times 10^8 \tfrac{\text{J}}{\text{gal}}) = 1.68 \times 10^{11} \text{ J}$$

Since you are gaining heat according to

$$Q_{cycle}(\text{into } T_2) = (g)\, W_{cycle}$$

you only need to put in

$$W_{cycle} = \frac{Q_{cycle}(\text{into } T_2)}{g} = \frac{\Delta U}{g} = \Delta U \frac{(T_2 - T_1)}{T_2}$$

$$= (1.68 \times 10^{11} \text{ J}) \frac{(50 \text{ K})}{(323 \text{ K})} = 2.6 \times 10^{10} \text{ J}$$

$$\$(\text{cost}) = (W_{cycle})\left(\frac{\$0.10}{\text{KW-hr}}\right) = (W_{cycle})\left(\frac{\$0.10}{3.6\times10^6 J}\right)$$

since $1\ \text{kw-hr} = (10^3\ W)(3.6\times10^3\ \text{sec}) = 3.6\times10^6\ J$

$$\$(\text{cost})_{elec} = \frac{(2.6\times10^{10} J)(\$0.10)}{3.6\times10^6 J} = \$7.2\times10^2 \simeq \$72.$$

the cost of the oil at $2/gal is

$$\$(\text{cost})_{oil} = (1200\ \text{gal})(\$2/\text{gal}) = \$2400$$

B) the heat withdrawn from the water is

$$Q_{cycle}(\text{from } T_1) = K\, W_{cycle} = \frac{T_1}{(T_2 - T_1)} W_{cycle}$$

$$= \left(\frac{273\ K}{50 K}\right)\left(2.6\times10^{10} J\right) = 1.42\times10^{11} J$$

(less than the energy content of the oil).
So ice will develope
the amount of ice in kilograms is

$$m_{ice} = \frac{\Delta Q}{\left(\frac{\Delta Q}{\Delta m}\right)_{fusion}} = \frac{1.42\times10^{11} J}{3.35\times10^6 J\cdot kg^{-1}}$$

$$= 4.23\times10^4\ kg$$

assume the density of ice is

$$\rho_{ice} = 0.931 \times 10^3 \ kg/m^3$$

then

$$\rho_{ice} = \frac{m_{ice}}{V_{ice}} \Rightarrow V_{ice} = \frac{m_{ice}}{\rho_{ice}} = \frac{4.23 \times 10^4 \ kg}{0.931 \times 10^3 \ kg/m^3}$$

$$= 4.55 \times 10^1 \ m^3$$

if the ice forms a cube, the dimension on each side

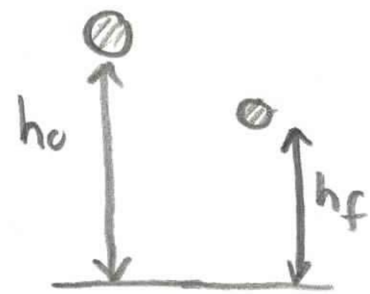
$$s = (V_{ice})^{1/3} = (4.55 \times 10^1 m^3)^{1/3}$$

$$= 3.57 \ m$$

Part Two: Earlier Material

Problem 1: ***(Momentum and Impulse)***

A superball of $m_1 = 0.08kg$, starting at rest, is dropped from a height falls $h_0 = 3.0m$ above the ground and bounces back up to a height of $h_f = 2.0m$. The collision with the ground occurs over $\Delta t_c = 5.0ms$.



a) What is the momentum of the ball immediately before the collision?

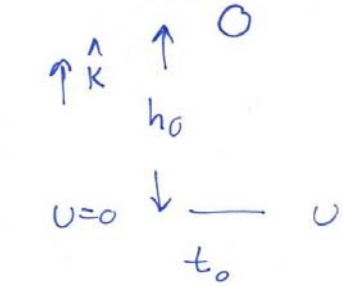
b) What is the momentum of the ball immediately after the collision?

c) What is the average force of the table on the ball?

d) What impulse is imparted to the ball?

e) What is the change in the kinetic energy during the collision?

Assume that the rubber has a specific heat capacity of $c_r = 0.48 cal \cdot g^{-1} \cdot {}^0C^{-1}$ and that all the lost mechanical energy goes into heating up the rubber. What is the change in temperature of the superball?



$U_0 = mgh_0 \quad , \quad K_0 = 0$

$E_0 = mgh_0$

$U_1 = 0 \quad , \quad K_1 = \frac{1}{2} m v_1^2$

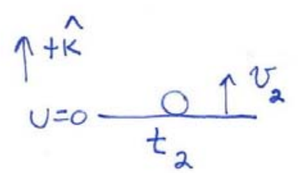
$E_1 = \frac{1}{2} m v_1^2$

Conservation of Energy: $E_0 = E_1 \Rightarrow$

$mgh_0 = \frac{1}{2} m v_1^2 \Rightarrow v_1 = \sqrt{2gh_0}$

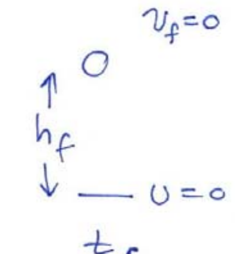
$\vec{P}_1 = -m v_1 \hat{k} = m\sqrt{2gh_0}(-\hat{k}) = (0.08\,kg)((2)(9.8\frac{m}{s^2})(3.0\,m))^{1/2}(-\hat{k})$

$\vec{P}_1 = (6.1 \times 10^{-1}\ kg\text{-}m\text{-}s^{-1})(-\hat{k})$



$U_2 = 0 \quad , \quad K_2 = \frac{1}{2} m v_2^2 \quad , \quad E_2 = \frac{1}{2} m v_2^2$

$U_f = mgh_f \quad K_2 = 0 \quad E_f = mgh_f$



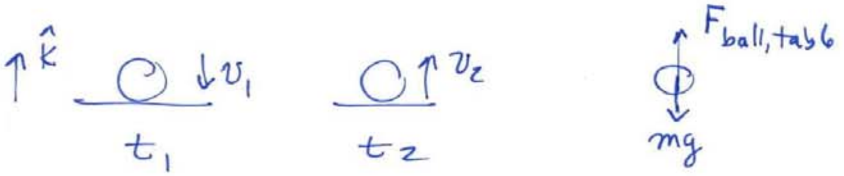
Conservation of Energy: $E_2 = E_f$

$\frac{1}{2} m v_2^2 = mg\,h_f \Rightarrow v_2 = \sqrt{2gh_f}$

$\vec{P}_2 = m v_2 \hat{k} = m\sqrt{2gh_f}\,\hat{k} =$

$\vec{P}_2 = 0.08\,kg\sqrt{(2)(9.8\frac{m}{s^2})(2.0\,m)}\ \hat{k} = 5.0 \times 10^{-1}\,kg\,\frac{m}{s}\,\hat{k}$

During the collision: $\Delta t_c = t_2 - t_1$



$$\overline{\vec{F}}_{ext}\,\Delta t_c = \Delta \vec{p}$$

$$\overline{F}_{ball,table} - mg = \left(\frac{m v_2 - - m v_1}{\Delta t_c}\right)\hat{k}$$

$$\overline{F}_{ball,table} = mg + \frac{m(v_2+v_1)}{\Delta t_c}\hat{k} = m\left(g + \frac{(v_2+v_1)}{\Delta t_c}\right)\hat{k}$$

$$\overline{F}_{ball,table} = (0.08\ kg)\left(\left(9.8\frac{m}{s^2}\right) + \sqrt{2}\sqrt{9.8\frac{m}{s^2}}\,\frac{(\sqrt{h_0}+\sqrt{h_f})}{(5.0\times10^{-3}s)}\right) = (2.2\times10^{2}N)\hat{k}$$

$$\vec{Impulse} = \Delta\vec{p} = m(v_2+v_1)\hat{k}$$

$$= \left(1.1\ kg\cdot\frac{m}{s}\right)\hat{k}$$

$$\Delta K = \frac{1}{2}m v_2^2 - \frac{1}{2}m v_1^2 = mg(h_f - h_0) = (0.08kg)\left(9.8\frac{m}{s^2}\right)(-1.0\ m)$$

$$\Delta K = -.78\ J$$

$$-\Delta K = c\,m\,\Delta T \Rightarrow \Delta T = \frac{-\Delta K}{c\,m} = \frac{mg(h_0 - h_f)}{c\,m}$$

$$1\ cal = 4.186\ J$$

$$\Delta T = \frac{(-.78\ J)}{(0.08kg)(0.48\ cal\cdot g^{-1}\cdot C^{-1})} = \frac{(-.78J)\left(\frac{1cal}{4.186\ J}\right)}{(80\ g)(0.48\ cal\cdot g^{-1}\cdot C^{-1})}$$

$$\Delta T = 4.85\times10^{-3}\ {}^{\circ}C$$

Problem 2: ***(Conservation of Energy and Momentum)***

An object of mass $m_1 = 1.5 kg$ is initially moving with a velocity v_0 . It collides completely inelastically with a block of mass $m_2 = 2.0 kg$. The second block is attached to a spring with constant $k = 5.6 \times 10^3 N \cdot m^{-1}$. The block and spring lie on a frictionless horizontal surface. The spring compresses a distance $d = 2.0 \times 10^{-1} m$.

a) What is the velocity of the object of mass m_1 and the block immediately after the collision?

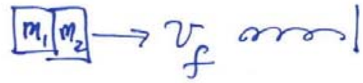
b) What is the initial velocity of the object of mass m_1 immediately before the collision?

c) If the block were attached to a very long string and hung as a pendulum, how high would the block and object of mass m_1 rise after the collision? Let $g = 9.8 m \cdot s^{-2}$.

initial

$\rightarrow \hat{i}$

after



$$\vec{P}_0 = \vec{P}_f \Rightarrow m_1 v_0 = (m_1 + m_2) v_f$$

$$v_f = \frac{m_1 v_0}{m_1 + m_2}$$

$$\frac{1}{2}(m_1 + m_2) v_f^{\,2} = \frac{1}{2} k d^2$$

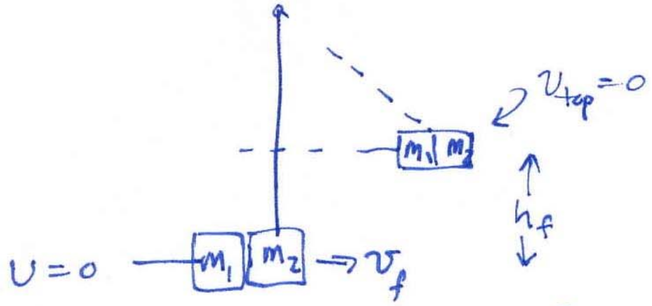
final

$$\frac{1}{2}(m_1 + m_2)\left(\frac{m_1^{\,2} v_0^{\,2}}{(m_1 + m_2)^2}\right) = \frac{1}{2} k d^2$$

$$v_0 = \left(\frac{(k d^2)(m_1 + m_2)}{m_1^{\,2}}\right)^{1/2}$$

$$v_0 = \left(\frac{(5.6\times 10^3\,\mathrm{N\cdot m^{-1}})(2.0\times 10^{-1}\,\mathrm{m})^2(1.5\,\mathrm{kg} + 2.0\,\mathrm{kg})}{(1.5\,\mathrm{kg})^2}\right)^{1/2}$$

$$v_0 = 1.87\times 10^1\,\mathrm{m\cdot s^{-1}}$$



$$E_f = mgh_f$$

$$E_f = (m_1 + m_2) g h_f$$

$$(m_1 + m_2) g h_f$$

$$E_i = \frac{1}{2}(m_1 + m_2) v_f^2 = mgh_f = E_f$$

$$\Rightarrow \quad h_f = \frac{v_f^2}{2g} = \frac{1}{2g}\left(\frac{m_1}{m_1 + m_2} v_0\right)^2$$

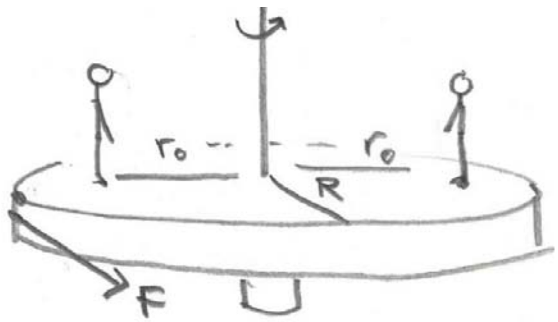
$$h_f = \frac{1}{2g} \frac{m_1^2}{(m_1 + m_2)^2} \quad \frac{kd^2 (m_1 + m_2)}{m_1^2}$$

$$h_f = \frac{1}{2} \frac{kd^2}{g(m_1 + m_2)} = \frac{\left(\frac{1}{2}\right)\left(5.6 \times 10^3 \, N \cdot m^{-1}\right)\left(2.0 \times 10^{-1} \, m\right)^2}{(2)\left(9.8 \, \frac{m}{s^{-2}}\right)(1.5 \, kg + 2.0 \, kg)}$$

$$h_f = 3.3 \ m$$

Problem 3: ***(Angular Dynamics)***

A playground merry-go-round has a radius of $R = 4.0m$ and has a moment of inertia $I_{cm} = 7.0 \times 10^3 kg \cdot m^2$ about an axis passing through the center of mass. There is negligible friction about its vertical axis. Two children each of mass $m = 25kg$ were standing on opposite sides a distance $r_0 = 3.0m$ from the central axis. The merry-go-round is initially at rest. A person on the ground applied a constant tangential force of $F = 2.5 \times 10^2 N$ at the rim of the merry-go-round for a time $\Delta t = 1.0 \times 10^1 s$.



a) What was the angular acceleration of the merry-go-round?

b) What was the angular velocity of the merry-go-round when the person stopped applying the force?

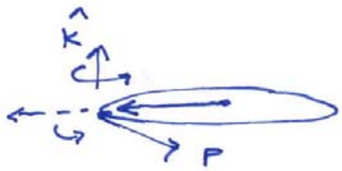
c) What average power did the person put out while pushing the merry-go-round?

d) What was the rotational kinetic energy of the merry-go-round when the person stopped applying the force?

The two children then walked inward and stop a distance of $r_1 = 1.0m$ from the central axis of the merry-go-round.

e) What was the angular velocity of the merry-go-round when the children reached their final position?

f) What was the change in rotational kinetic energy of the merry-go-round when the children reached their final position?



$$\vec{\tau} = I_{cm}^{total}\alpha$$

$$\hat{k}: \quad RF = (I_{cm} + 2mr_0^2)\alpha$$

$$\Rightarrow \alpha = \frac{RF}{(I_{cm}+2mr_0^2)} = \frac{(4.0\,m)(2.5\times10^2\,N)}{(7.0\times10^3\,kg\cdot m^2 + (2)(25\,kg)(3.0\,m)^2)}$$

$$\alpha = 1.3\times10^{-1}\ rad\cdot s^{-2}$$

two approaches: $\tau = RF$

1) Angular impulse $\equiv \tau\Delta t = \Delta L_{cm} = (I_{cm}+2mr_0^2)\,\omega_a$

$$\Rightarrow \omega_a = \frac{RF}{(I_{cm}+2mr_0^2)}\Delta t = \alpha\Delta t$$

2) $$\omega_a = \alpha\Delta t = (1.3\times10^{-1}\,rad\cdot s^{-2})(1.0\times10^{+1}\,s)$$

$$= 1.3\ rad\cdot s^{-1}$$

$$\text{Power} = \tau\,\omega_{ave} = \frac{RF\,\omega_a}{2} = \frac{(4.0\,m)(2.5\times10^2\,N)}{2}(1.3\,rad\cdot s^{-1})$$

$$P = \frac{1}{2}\frac{(RF)^2}{(I_{cm}+2mr_0^2)}\Delta t = 6.7\times10^2\ W$$

$$K_{rotational} = \frac{1}{2}(I_{cm}+2mr_0^2)\,\omega_a^2$$

$$= \frac{1}{2}\frac{(RF)^2\Delta t^2}{(I_{cm}+2mr_0^2)} = 6.7\times10^3\ J = P\Delta t$$

$$L_a = L_f$$

$$(I_{cm} + 2mr_0^2)\,\omega_a = (I_{cm} + 2mr_1^2)\,\omega_f$$

$$\omega_f = \frac{(I_{cm} + 2mr_0^2)\,\omega_a}{I_{cm} + 2mr_1^2}$$

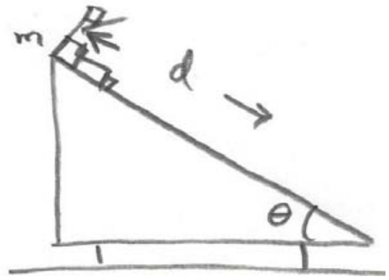
$$= \frac{RF\Delta t}{(I_{cm} + 2mr_1^2)}$$

$$= \frac{(4.0\,\mathrm{m})(2.5\times10^2\,\mathrm{N})(1.0\times10^1\,\mathrm{s})}{((7.0\times10^3\,\mathrm{kg\cdot m^2}) + (2)(25\,\mathrm{kg})(1.0\,\mathrm{m})^2)}$$

$$= 1.4 \times \quad \mathrm{rad\cdot s^{-1}}$$

Problem 4: ***(Energy, Force, and Kinematics)***

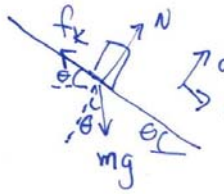
A child's playground slide is $d = 5.0m$ in length and is at an angle of $\theta = 2.0\times10^{1} deg$ with respect to the ground. A child of mass $m_b = 2.0\times10^{1} kg$ starts from rest at the top of the slide. The coefficient of sliding friction for the slide is $\mu_k = 0.2$.



a) What is the total work done by the friction force on the child?

b) What is the speed of the child at the bottom of the slide?

c) How long does the child take to slide down the ramp?



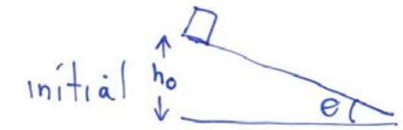
$\hat{i}: \; mg\sin\theta - f_k = ma_x$

$\hat{j}: \; N - mg\cos\theta = 0$

$f_k = \mu_k N = \mu_k mg\cos\theta$

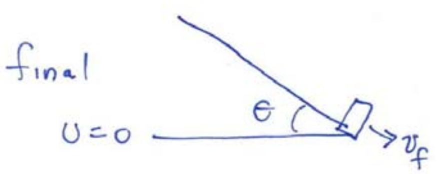
$$W_{nc} = -f_k d = -\mu_k mg\cos\theta\, d$$

$$W_{nc} = -(0.2)(2.0\times10^1\,kg)(9.8\tfrac{m}{s^2})(\cos 20^\circ)(5.0\,m) = 1.8\times10^2\,J$$



$U_0 = mgh_0 = mgd\sin\theta$

$K_0 = 0 \;, \; E_0 = mgd\sin\theta$



$U_f = 0 \qquad E_f = \frac{1}{2}mv_f^2$

$K_f = \frac{1}{2}mv_f^2$

$$W_{nc} = \Delta E = \frac{1}{2}mv_f^2 - mgd\sin\theta$$

$$-f_k d = \frac{1}{2}mv_f^2 - mgd\sin\theta$$

$$v_f = \left(2dg(\sin\theta - \mu_k\cos\theta)\right)^{1/2}$$

$$v_f = \left((2)(5.0\,m)(9.8\tfrac{m}{s^2})(\sin 20^\circ - (0.2)(\cos 20^\circ)\right)^{1/2} = 3.9\ m\cdot s^{-1}$$

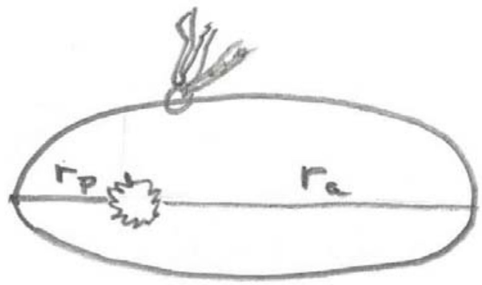
$$a_x = \frac{1}{m}(mg\sin\theta - f_k) = g(\sin\theta - \mu_k\cos\theta)$$

$$v_f = a_x t \Rightarrow t = \frac{v_f}{a_x} = \frac{(2dg(\sin\theta - \mu_k\cos\theta))^{1/2}}{g(\sin\theta - \mu_k\cos\theta)}$$

$$t = \left(\frac{2d}{g(\sin\theta - \mu_k\cos\theta)}\right)^{1/2} = \left(\frac{(2)(5.0\,m)}{(9.8\frac{m}{s^2})(\sin 20^\circ - (.2)(\cos 20^\circ))}\right)^{1/2} = 2.7\ s$$

Problem 5: ***(Planetary Orbits)***

Comet Encke was discovered in 1786 by Pierre Mechain and in 1822 Johann Encke determined that its period was 3.3 years. It was photographed in 1913 at the aphelion distance, $r_a = 6.1 \times 10^{11} m$, (furthest distance from the sun) by the telescope at Mt. Wilson. The distance of closest approach to the sun, perihelion, is $r_p = 5.1 \times 10^{10} m$. The universal gravitation constant $G = 6.7 \times 10^{-11} N \cdot m^2 \cdot kg^{-2}$. The mass of the sun is $m_s = 2.0 \times 10^{30} kg$.



a) Explain why angular momentum is conserved about the focal point and then write down an equation for the conservation of angular momentum between aphelion and perihelion.

b) Explain why mechanical energy is conserved and then write down an equation for conservation of energy between aphelion and perihelion.

c) Find the velocities at perihelion and aphelion.

Since the only forces are gravitational and point toward the focal point

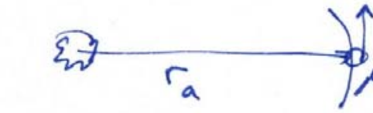
$$\vec{\tau}_{focal} = \vec{r}_{focal,comet} \times \vec{F}_{grav} = 0$$

therefore angular momentum is conserved

At perihelion. since v_p is tangent to the orbit

$$L_p = \mu r_p v_p$$

At aphelion:



$$L_a = \mu r_a v_a$$

$$L_a = L_p \Rightarrow \mu r_a v_a = \mu r_p v_p \Rightarrow$$

$$r_a v_a = r_p v_p$$

The gravitational force is conservative so mechanical energy is conserved

$$E_a = \frac{1}{2}\mu v_a^2 - \frac{G m_1 m_2}{r_a} \quad , \quad E_p = \frac{1}{2}\mu v_p^2 - \frac{G m_1 m_2}{r_p}$$

thus

$$E_a = E_p \Rightarrow$$

$$\frac{1}{2}\mu v_a^2 - \frac{G m_1 m_2}{r_a} = \frac{1}{2}\mu v_p^2 - \frac{G m_1 m_2}{r_p}$$

Since the mass m_1 of the comet is much less than the mass of the sun m_2

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \simeq \frac{m_1 m_2}{m_2} = m_1$$

the energy conservation equation becomes

$$\frac{1}{2} m_1 v_a^2 - \frac{G m_1 m_2}{r_a} = \frac{1}{2} m_1 v_p^2 - \frac{G m_1 m_2}{r_p}$$

or $$\frac{1}{2} v_a^2 - \frac{G m_2}{r_a} = \frac{1}{2} v_p^2 - \frac{G m_2}{r_p}$$

the condition $v_p = \frac{r_a v_a}{r_p}$ from conservation of angular momentum can now be used in the energy equation

$$\frac{1}{2} v_a^2 - \frac{G m_2}{r_a} = \frac{1}{2}\left(\frac{r_a}{r_p} v_a\right)^2 - \frac{G m_2}{r_p}$$

Solve for v_a:

$$\frac{1}{2}\left(\left(\frac{r_a}{r_p}\right)^2 - 1\right) v_a^2 = G m_2 \left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$v_a = \left(\frac{2 G m_2 \left(\frac{1}{r_p} - \frac{1}{r_a}\right)}{\left(\left(\frac{r_a}{r_p}\right)^2 - 1\right)}\right)^{1/2}$$

$$v_a = \left(\frac{2 G m_2 \, r_p}{r_a (r_a + r_p)}\right)^{1/2}$$

$$v_a = \left(\frac{(2)(6.67\times10^{-11} N\cdot m^2\cdot kg^{-2})(2.0\times10^{30} kg)(5.1\times10^{10} m)}{(6.1\times10^{11} m)(6.1\times10^{11} m + 5.1\times10^{10} m)}\right)^{1/2} = 5.8\times10^{3}\, m\cdot s^{-1}$$

the velocity at perihelion is then

$$v_p = \frac{r_a}{r_p} v_a = \frac{(6.1\times10^{11} m)(5.8\times10^{3} m\cdot s^{-1})}{(5.1\times10^{10} m)} = 6.9\times10^{4}\, m\cdot s^{-1}$$

Problem 6: ***escape speed of moon***

Find the escape speed of a rocket from the moon. Ignore the rotational motion of the moon. The mass of the moon is $m = 7.36\times10^{22}\,kg$. The radius of the moon is $R = 1.74\times10^{6}\,m$.

at infinity: $U(\infty) = 0$

escape velocity condition $K(\infty) = 0$

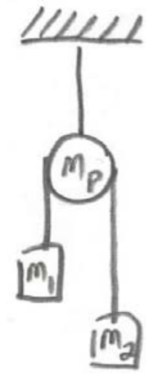
$E(\infty) = 0$

at surface of moon: $E = \frac{1}{2} m v_{esc}^2 - \frac{G m m_m}{R} = 0$

$$\Rightarrow \quad v_{esc} = \left(\frac{2 G m_m}{R}\right)^{1/2} = \left(\frac{(2)(6.67\times10^{-11}\,\frac{N\cdot m^2}{kg^2})(7.36\times10^{22}\,kg)}{1.74\times10^{6}\,m}\right)^{1/2}$$

$$v_{esc} = 2.38\times10^{3}\ m\cdot s^{-1}$$

Problem 7: (***Torque and angular acceleration)***



A pulley of mass m_p, radius R, and moment of inertia $I_{cm} = (1/2)m_pR^2$ about the center of mass is hung from a ceiling with a massless string. A massless inextensible rope is wrapped around the pulley an attached on one side to an object of mass m_1 and on the other side to an object mass $m_2 > m_1$. At time $t = 0$, the objects are released from rest.

a) Draw the free body diagram on the pulley and the two objects.

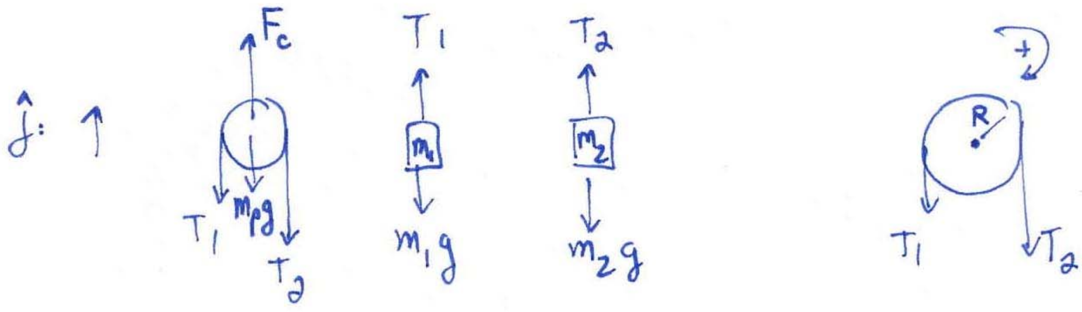
b) Write down Newton's Second Law for the pulley and the two objects.

c) Write down the rotational equation of motion for the pulley.

d) Find the direction and magnitude of the translational acceleration of the two objects.

e) How long does it take for the object of mass m_2 to fall a distance d ?

f) What is the tension on the two sides of the rope?



pulley: $F_c - T_1 - T_2 - m_p g = 0$

mass m_1: $T_1 - m_1 g = m_1 a_1$

mass m_2: $T_2 - m_2 g = m_2 a_2$

constraint condition $a \equiv a_1 = -a_2 > 0$

torque equation: $T_2 R - T_1 R = I_{cm} \alpha$

rope not slipping: $a = R\alpha > 0$ since $m_2 > m_1$

$$(T_2 - T_1) R = I_{cm} \frac{a}{R}$$

solve for acceleration:
$$\left.\begin{aligned} T_1 - m_1 g &= +m_1 a \\ T_2 - m_2 g &= -m_2 a \end{aligned}\right\} \text{ subtract}$$

$$(T_2 - T_1) - (m_2 - m_1) g = -(m_2 + m_1) a$$

moment of inertia: $I_{cm} = \frac{1}{2} m_p R^2$

$$\Rightarrow (T_2 - T_1) R = \frac{1}{2} m_p R^2 \frac{a}{R} = \frac{1}{2} m_p a R$$

$$\Rightarrow (T_2 - T_1) = \frac{1}{2} m_p a$$

$$(T_2 - T_1) - (m_2 - m_1) g = -(m_1 + m_2) a$$

$$\frac{1}{2} m_p a - (m_2 - m_1) g = -(m_1 + m_2) a$$

$$a = \frac{(m_2 - m_1) g}{\left(\frac{1}{2} m_p + (m_1 + m_2)\right)} = a_1 = -a_2$$

$$d = \frac{1}{2} a t^2 \Rightarrow t = \left(\frac{2d}{a}\right)^{1/2}$$

$$t = \left(\frac{2d}{(m_2 - m_1) g} \left(\frac{1}{2} m_p + (m_1 + m_2) \right) \right)^{1/2}$$

$$T_1 = m_1 a + m_1 g$$

$$T_1 = \left(\frac{m_1 (m_2 - m_1)}{\frac{1}{2} m_p + (m_1 + m_2)} + m_1 \right) g$$

$$T_1 = \frac{\left(\frac{1}{2} m_p m_1 + 2 m_1 m_2\right) g}{\frac{1}{2} m_p + (m_1 + m_2)}$$

$$T_2 = -m_2 a + m_2 g = \left(m_2 - \frac{m_2 (m_2 - m_1)}{\left(\frac{1}{2} m_p + (m_1 + m_2\right)} \right) g$$

$$T_2 = \left(\frac{\frac{1}{2} m_p m_2 + 2 m_1 m_2}{\frac{1}{2} m_p + (m_1 + m_2)} \right) g$$

Problem 8: Projectile Motion

A bat hits a baseball into the air with an initial speed, $v_0 = 5.0\times10^1 m/s$, and makes an angle $\theta = 3.0\times10^1 \deg$ with respect to the horizontal. How high does it go from the point where it was hit? How far does the ball travel if it is caught at exactly the same height that it is hit from? When the ball is in flight, ignore all forces acting on the ball except for gravitation.

the total travel time to return to the same height is

$$t^{total} = 2t_1 = 2 v_{y,0}/g$$

the horizontal distance traveled is

$$x(t^{total}) = v_{x,0} t^{total} = 2 v_{x,0} v_{y,0}/g$$

$$= 2 v_0^2 \cos\theta_0 \sin\theta_0 / g$$

since $v_{x,0} = v_0 \cos\theta_0$, $v_{y,0} = v_0 \sin\theta_0$

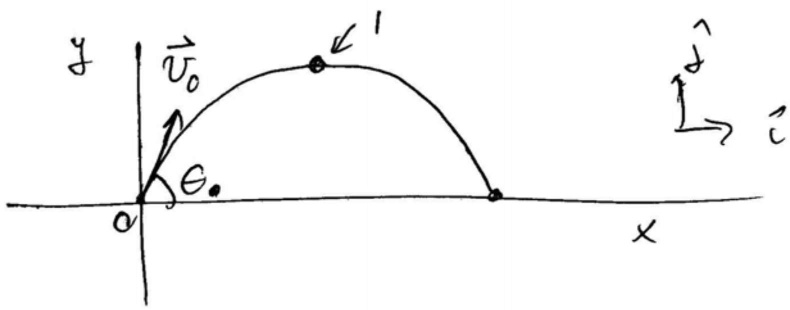
note: $2\cos\theta_0 \sin\theta_0 = \sin(2\theta_0)$ so

$$x(t^{total}) = \frac{v_0^2 \sin(2\theta_0)}{g} = \frac{(5.0\times10^1 m/s)^2 \sin(60^\circ)}{9.8 m/s^2}$$

$$x(t^{total}) = 2.21\times10^2 m$$

$$x(t^{total}) = 7.25\times10^2 ft$$

note: this is much further than the record distance for balls hit which shows that air resistance plays a large factor in this problem.



choose origin at the initial height the ball was hit above the ground

Equations of motion

$$x = (v_0)_x t \qquad v_x = v_{x,0} +$$

$$y = (v_0)_y t - \frac{1}{2} g t^2 \qquad v_y = v_{y,0} - g t$$

At the top of the flight $t = t_1$

$$0 = v_y(t_1) = v_{y,0} - g t_1 \Rightarrow$$

$$t_1 = v_{y,0}/g$$

The ball has traveled a vertical distance

$$y(t_1) = v_{0,y}\, t_1 - \frac{1}{2} g t_1^2$$

$$y(t_1) = v_{0,y} \frac{v_{0,y}}{g} - \frac{1}{2} g \left(\frac{v_{0,y}}{g}\right)^2 = \frac{1}{2} \frac{v_{0,y}^2}{g}$$

$$v_{0,y} = v_0 \sin\theta_0 = (5.0 \times 10^1 \tfrac{m}{s}) \sin(30^\circ)$$

$$v_{0,y} = 2.5 \times 10^1 \ m/s$$

$$y(t_1) = \left(\frac{1}{2}\right)\frac{(2.5 \times 10^1 \ m/s)^2}{9.8 \ m/s} = 31.9 \ m$$