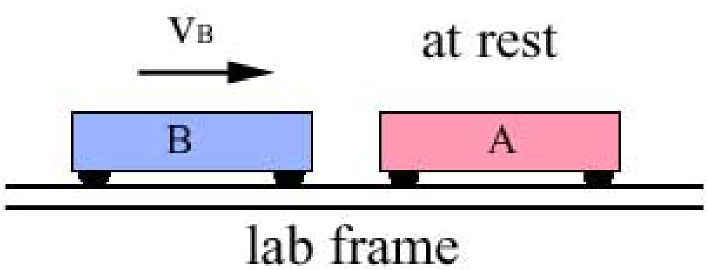
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T **Fall Term 2004**

Practice Exam Three Solutions

Problem 1a) (5 points) ***Collisions and Center of Mass Reference Frame***

In the lab frame, Cart A is at rest. An identical cart B, of the same mass, moving to the right, collides *elastically* with cart A. After the collision, describe the motion of the carts in a reference frame moving with the center of mass. (You may find a picture may help clarify your explanation.)



Lab Frame: Since the masses are equal, cart B will be at rest after the collision and cart A will move to the right with speed v_B.

Center-of-mass Reference frame: $v_{cm} = \frac{v_B}{2}$

After the collision, cart B moves to the left with speed $\frac{v_B}{2}$ and cart A moves to the right with speed $\frac{v_B}{2}$

O_{cm}: after collision

Problem 1b) (5 points) *Collisions*

An object P has an initial velocity $\vec{\mathbf{v}}$. It strikes an initially stationary object Q which is attached to a massless spring, as shown.

Assume that the masses of P and Q are equal. At the point of maximum compression of the spring, is the momentum of Q greater, equal, or less than the initial momentum of P. Explain your reasoning.

the point of maximum compression occurs when P and Q are moving to the right with the same speed $\frac{v}{2} = v_{cm}$.

Lab Frame:

max compression $\boxed{P}$~~$\boxed{Q} \rightarrow \frac{v}{2}$

(Reason: they are at rest in center-of-mass frame, so all initial KE has been stored in the spring)

Since they are moving with the same speed and their masses are equal

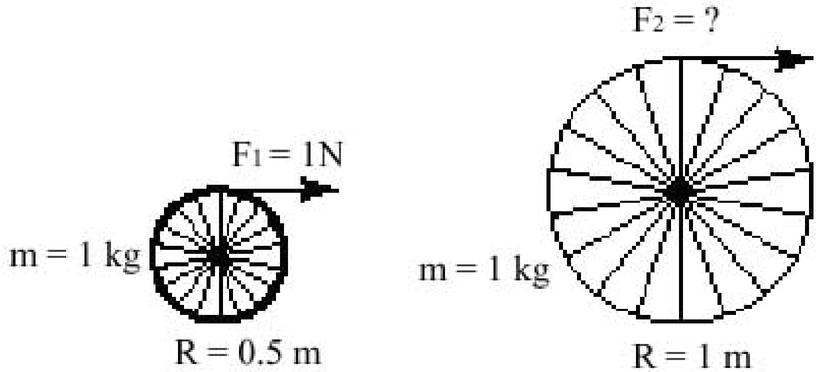
$$\vec{p}_{f,P} = m\frac{\vec{v}}{2} = \vec{p}_{f,Q} = m\frac{\vec{v}}{2}$$

the momentums are equal after the collision

so $\vec{p}_{f,I} < \vec{p}_{0,I} = m\vec{v}$

Problem 1c) (5 points) ***Torque and Moment of Inertia***

Two wheels with fixed hubs, each having a mass of $1kg$, start from rest, and forces are applied: $F_1 = 1N$ to the first wheel and F_2 to the second wheel, as shown. Assume the hubs and spokes are massless, so that the moment of inertia about the center of mass is $I_{cm} = mR^2$. How large must the force F_2 be in order to impart identical angular accelerations on each wheel? Explain how you arrived at your result.



the torques $\tau_1 = I_1 \alpha = mR_1^2 \alpha$

$\tau_2 = I_2 \alpha = mR_2^2 \alpha$

but the torques $\tau_1 = R_1 F_1$ so

$\tau_2 = R_2 F_2$

$$R_1 F_1 = mR_1^2 \alpha \Rightarrow F_1 = mR_1 \alpha \Rightarrow \alpha = \frac{F_1}{mR_1}$$

$$R_2 F_2 = mR_2^2 \alpha \Rightarrow F_2 = mR_2 \alpha$$

$$F_2 = \frac{mR_2}{mR_1} F_1 = \frac{R_2}{R_1} F_1 = 2F_1 = 2N$$

Problem 1d) (5 points) ***Angular Momentum and Rotational Energy***

A figure skater stands on one spot on the ice (assumed frictionless) and spins around with her arms extended. When she pulls in her arms, she reduces her moment of inertia about the spin axis passing through her center of mass and her angular speed increases so that her angular momentum is conserved. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she has pulled in her arms must be

1. the same.
2. larger because she's rotating faster.
3. smaller because her moment of inertia is smaller.

Explain your reasoning.

Angular momentum is conserved

so $L_i = L_f \Rightarrow I_i \omega_i = I_f \omega_f \Rightarrow \omega_f = \frac{I_i \omega_i}{I_f}$

Rotational kinetic energy is not conserved

$K_f = \frac{1}{2} I_f \omega_f^2 \quad \frac{1}{2}\left(\frac{I_i}{I_f}\omega_i\right)^2 = \left(\frac{1}{2} I_i \omega_i^2\right)\frac{I_i}{I_f} = K_i \frac{I_i}{I_f}$

since pulling arms in decreases moment of inertia, $I_f < I_i$

the rotational kinetic energy increases.

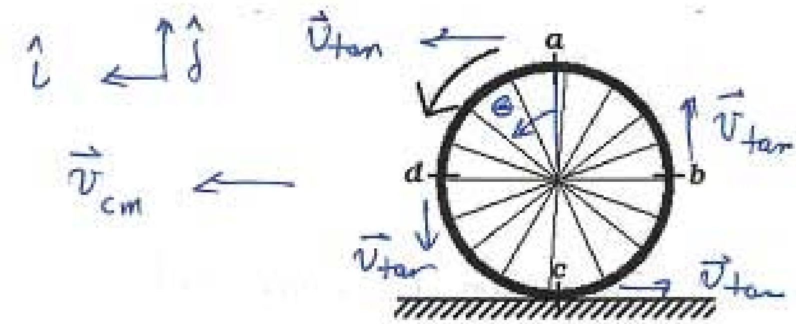
note. $\frac{I_i}{I_f} = \frac{\omega_f}{\omega_i}$ so

$K_f = \frac{\omega_f}{\omega_i} K_i$ and $\omega_f > \omega_i$ so $K_f > K_i$

kinetic energy increases because she is rotating faster

Problem 1e) (5 points) ***Rolling without slipping***

A bicycle tire rolls without slipping across the ground (moving to the left in the figure below). Describe the direction and magnitude of the velocity at each of the four points shown in the figure as seen from an observer at rest on the ground.



You need to add $\vec{v}_{tan} = R\omega\hat{\theta}$ from $\vec{v}_{cm}$ at each point. Since $R\omega = v_{cm}$

a) $\vec{v}_{cm} + \vec{v}_{tan} = 2 v_{cm}\,\hat{i}$, to left with twice v_{cm}

b) $\vec{v}_{cm} + \vec{v}_{tan} = v_{cm}(\hat{i} + \hat{j})$

mag = $\sqrt{2}\, v_{cm}$, dir: $\nwarrow \vec{v}_{tot}$ b 45° upper left

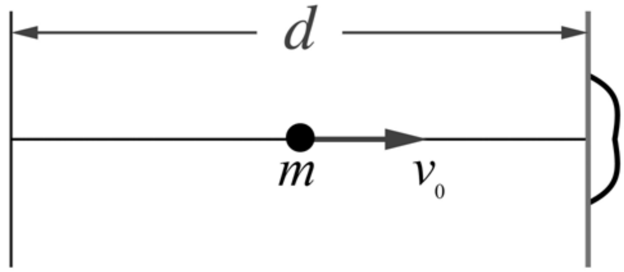
c) $\vec{v}_{cm} + \vec{v}_{tan} = \vec{0}$

d) $\vec{v}_{cm} + \vec{v}_{tan} = v_{cm}(\hat{i} - \hat{j})$

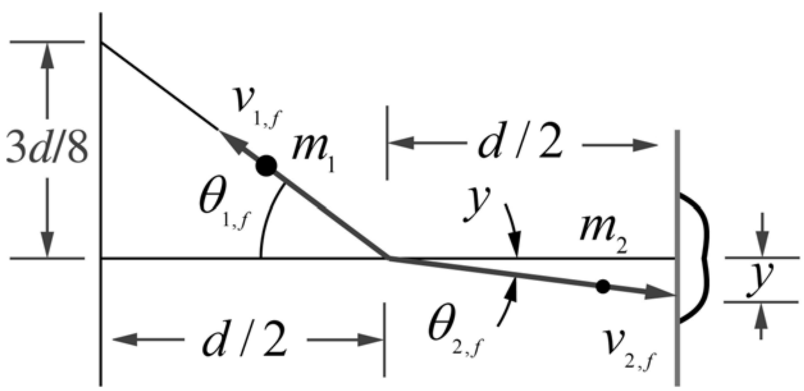
mag = $\sqrt{2}\, v_{cm}$, dir: 45° lower left

Problem 2: ***Exploding Puck***

A hockey player shoots a "trick" hockey puck along the ice towards the center of the goal from a position d directly in front of the goal. The initial speed of the puck is v_0 and the puck has a mass m.



Half way to the goal the puck explodes into two fragments. One piece of mass $m_1 = (3/5)m$ comes back towards the player and passes $3d/8$ to the side of the spot it was initially shot from with a speed $v_{1,f} = (5/12)v_0$. The other piece of the puck with mass $m_2 = (2/5)m$ continues on towards the goal with a speed $v_{2,f}$.



Assume that there is no friction as the puck slides along the ice and that the mass of explosive in the puck is negligible.

a) Write down the equations for conservation of momentum of the puck and fragments in terms of the quantities shown in the figure above.

b) By what distance, y, does the piece that continues towards the goal miss the center of the goal? Express your answer in terms of d.

a) $\hat{i}$: $m v_0 = -m_1 v_{1,f} \cos\theta_{1,f} + m_2 v_{2,f} \cos\theta_{2,f}$

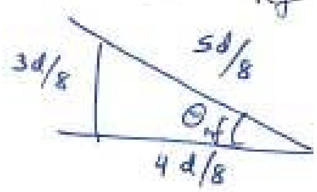
$\hat{j}$: $0 = m_1 v_{1,f} \sin\theta_{1,f} - m_2 v_{2,f} \sin\theta_{2,f}$

$m_1 = \frac{3}{5} m$, $m_2 = \frac{2}{5} m$, $v_{1,f} = \frac{5}{12} v_0$

$\Rightarrow \hat{i}$: $v_0 = -\left(\frac{3}{5}\right)\left(\frac{5}{12} v_0\right) \cos\theta_{1,f} + \frac{2}{5} v_{2,f} \cos\theta_{2,f}$

$\hat{j}$: $0 = \frac{3}{5} \frac{5}{12} v_0 \sin\theta_{1,f} - \frac{2}{5} v_{2,f} \sin\theta_{2,f}$

$\cos\theta_{1,f} = \frac{4}{5}$, $\sin\theta_{1,f} = \frac{3}{5}$



$\hat{i}$: $v_0 = \left(-\frac{1}{4} v_0\right)\left(\frac{4}{5}\right) + \frac{2}{5} v_{2,f} \cos\theta_{2,f}$

$\hat{j}$: $0 = \left(\frac{1}{4}\right)\left(\frac{3}{5}\right) v_0 - \frac{2}{5} v_{2,f} \sin\theta_{2,f}$

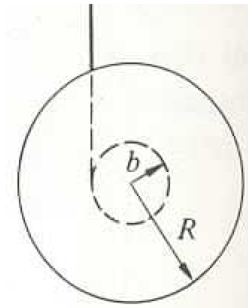
solve for $\tan\theta_{2,f}$

$$\tan\theta_{2,f} = \frac{\frac{2}{5} v_{2,f} \sin\theta_{2,f}}{\frac{2}{5} v_{2,f} \cos\theta_{2,f}} = \frac{\frac{3}{20} v_0}{\frac{6}{5} v_0} = \frac{1}{8} = \frac{y}{d/2}$$

$\Rightarrow$ $y = d/16$

Problem 3: Torque, Rotation and Translation

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.

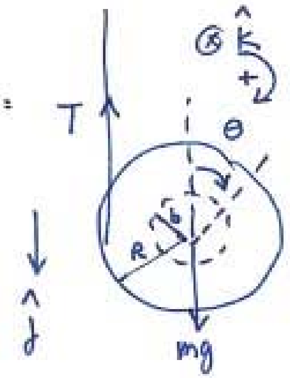


a) What is the tension in the cord as the Yo-Yo descends?

b) Use conservation of energy to find the angular velocity of the Yo-Yo when it reaches the bottom of the string.

c) What happens to the Yo-Yo at the bottom of the string?

Solution:



torque and force diagram; with this choice of sense of rotation (clockwise) and positive $\hat{j}$ direction

then $\quad a = b\alpha \qquad (1)$

Newton's 2nd Law: $\quad mg - T = ma \qquad (2)$

Torque Equation: $\quad bT = I_{cm}\alpha \qquad (3)$

moment of inertia: $\quad I_{cm} = \frac{1}{2} m R^2 \qquad (4)$

Use $mg - T = ma$ to solve for T.

$\Rightarrow \quad mg - T = m b \alpha \qquad$ eq (1)

$\Rightarrow \quad mg - T = m \dfrac{b\, bT}{I_{cm}} \qquad$ eq (3)

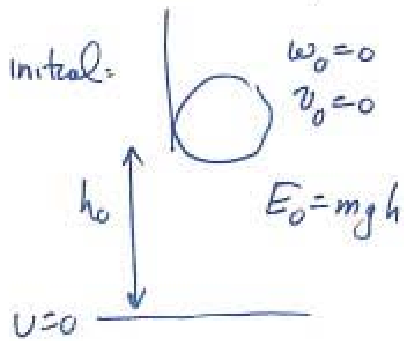
$$mg = T\left(1 + \frac{mb^2}{I_{cm}}\right)$$

$$\Rightarrow \quad T = \frac{mg}{1 + \frac{mb^2}{I_{cm}}} = \frac{mg}{1 + \frac{mb^2}{\frac{1}{2}mR^2}}$$

note:
$$\alpha = \frac{bT}{I_{cm}} = \frac{bT}{\frac{1}{2}mR^2} = \frac{b}{\frac{1}{2}mR^2} \frac{mg}{1 + \frac{mb^2}{\frac{1}{2}mR^2}}$$

$$a = b\alpha = \frac{b^2 T}{\frac{1}{2}mR^2}$$

Conservation of Energy



$$E_f = \tfrac{1}{2} m v_f^2 + \tfrac{1}{2} I_{cm} \omega_f^2$$

$$v_f = b\,\omega_f, \quad I_{cm} = \tfrac{1}{2} m R$$

ω_f

$$E_f = \tfrac{1}{2} m v_f^2 + \tfrac{1}{2}\left(\tfrac{1}{2} m R^2\right)\omega_f^2$$

v_f

$$= \tfrac{1}{2} m b^2 \omega_f^2 + \tfrac{1}{2}\left(\tfrac{1}{2} m R^2\right)\omega_f^2$$

$$E_0 = m g h_0 = E_f = \left(\tfrac{1}{2} m b^2 + \tfrac{1}{2}\left(\tfrac{1}{2} m R^2\right)\right)\omega_f^2$$

$$\Rightarrow \quad \omega_f = \left(\frac{m g h_0}{\left(\tfrac{1}{2} m b^2 + \tfrac{1}{2}\left(\tfrac{1}{2} m R^2\right)\right)} \right)^{1/2}$$

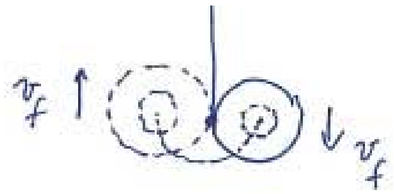
note: $\quad h_0 = \tfrac{1}{2} a t^2 \Rightarrow t = \left(\frac{2 h_0}{a}\right)^{1/2}$

$$\omega_f = \alpha t = \alpha \left(\frac{2 h_0}{a}\right)^{1/2}$$

$$= \left(2 h_0 \frac{T}{\tfrac{1}{2} m R^2} \right)^{1/2}$$

$$\omega_f = \left(\frac{2 h_0}{\tfrac{1}{2} m R^2} \; \frac{m g}{1 + \frac{m b^2}{\tfrac{1}{2} m R^2}} \right)^{1/2}$$

c) At the bottom of the descent, the Yo-Yo is spinning with angular velocity ω_f. the Yo-Yo pivots about the end of the string.



with angular velocity $\omega_f = \frac{d\theta}{dt} = \frac{\pi \text{ rad}}{\Delta t}$

where Δt is the time it takes the Yo-Yo to go π radians. the momentum has reversed so,

$$\vec{I} = \Delta \vec{p}$$

the impulse $\vec{I} = \int_0^{\Delta t} (-T+mg)\hat{j}\,dt = -mv_f\hat{j} - mv_f\hat{j}$

$$\Rightarrow \Delta t(mg - T) = -2mv_f$$

or $T = mg + \frac{2mv_f}{\Delta t} = mg + \frac{2mv_f}{\pi}\omega_f$

since $v_f = b\omega_f =$ we have

$$T = mg + \frac{2mb}{\pi}\omega_f^2$$

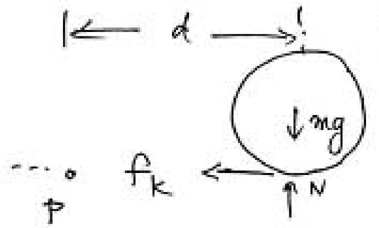
Problem 4: *Rolling Cylinder*

A solid cylinder of mass m and radius R is initially thrown along a wooden floor hallway with an initial speed v_0 and zero initial angular velocity $\omega_0 = 0$ as shown in the figure below.

a) What is the moment of inertia of the cylinder about the axis of rotation of the cylinder?

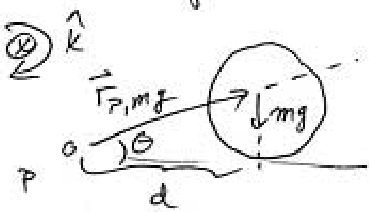
b) What effects does friction have on the cylinder before it reaches its final speed? Is the friction static or kinetic? Which direction does it point?

c) What is the angular speed, ω_f, of the cylinder when it just starts to roll without slipping? Express your answer in terms of m, R, and v_0.

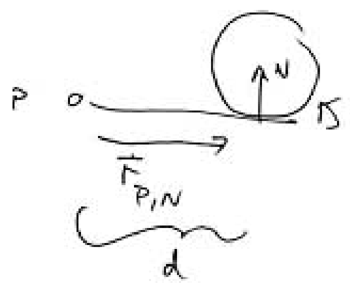


$$\vec{\tau}_P = \vec{r}_{P,f_k} \times \vec{f}_k + \vec{r}_{P,mg} \times m\vec{g} + \vec{r}_{P,N} \times \vec{N}$$

$\vec{\tau}_{P,f_k} = \vec{r}_{P,f_k} \times \vec{f}_k = 0$ because $\vec{r}_{P,f_k}$ and $\vec{f}_k$ are antiparallel.



$$\vec{\tau}_{P,mg} = \vec{r}_{P,mg} \times m\vec{g} = \underbrace{r_{P,mg}\sin\theta}\, mg\,\hat{k} = d\,mg\,\hat{k}$$



$$\vec{\tau}_{P,N} = \vec{r}_{P,N} \times \vec{N} = -dN\,\hat{k}$$

in the vertical direction

$$N - mg = 0$$

Thus $\vec{\tau}_{P,mg} + \vec{\tau}_{P,N} = (d\,mg - d\,N)\,\hat{k} = \vec{0}$

So $\vec{\tau}_P^{\,total} = 0 \Rightarrow \vec{L}_{P,0} = \vec{L}_{P,f}$

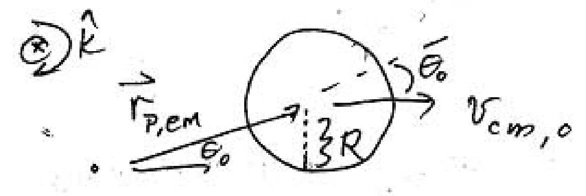
Angular momentum about any point lying on the line of contact is constant throughout motion.

$$\underbrace{\vec{L}_P^{\,total}}_{\substack{\text{total} \\ \text{ang. mom. about } P}} = \underbrace{\vec{L}_{P,cm}^{\,orbital}}_{\substack{\text{orbital ang. mom.} \\ \text{about } P}} + \underbrace{\vec{L}_{cm}^{\,spin}}_{\substack{\text{spin ang. mom} \\ \text{about cm}}}$$

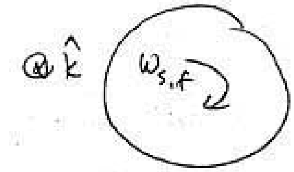
$$\vec{L}_P^{\,total} = \underbrace{\vec{r}_{P,cm} \times m\vec{v}_{cm}}_{\substack{\text{treat body as a} \\ \text{point particle located} \\ \text{at cm}}} + I_{cm}\,\vec{\omega}_{cm}$$

$$\vec{L}_{P,0}^{\,orbital} = \vec{r}_{P,cm} \times m\vec{v}_{cm,0} = \underbrace{r_{P,cm}\sin\theta_0}\, m\, v_{cm,0}\,\hat{k}$$

$$= R\, m\, v_{cm,0}\,\hat{k}$$



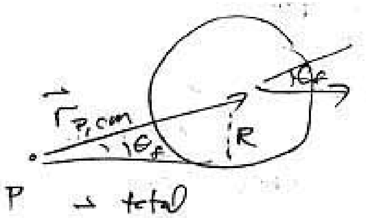
$$\vec{L}^{\,total}_{P,0} = R m v_{cm,0}\,\hat{k}$$



$$\vec{L}^{\,spin}_{cm,f} = I_{cm}\,\omega_{s,f}\,\hat{k} = \frac{1}{2} m R^2 \omega_{s,f}\,\hat{k}$$

$\vec{\omega}_f$ points in $+\hat{k}$ direction.

$$\vec{L}^{\,orbital}_{P,f} = R m v_{cm,f}\,\hat{k}$$



$\vec{v}_{cm,f}$ (same as above)

$$\vec{L}^{\,total}_{P,f} = R m v_{cm,f}\,\hat{k} + \frac{1}{2} m R^2 \omega_{s,f}\,\hat{k}$$

$$\vec{L}^{\,total}_{P,0} = \vec{L}^{\,total}_{P,f} \Rightarrow$$

$$\hat{k}: \quad R m v_{cm,0} = R m v_{cm,f} + \frac{1}{2} m R^2 \omega_{s,f} \quad (1)$$

rolling without slipping condition: (2)

$$v_{cm,f} = R\,\omega_{s,f}$$

substitute into eq (1) $\Rightarrow$

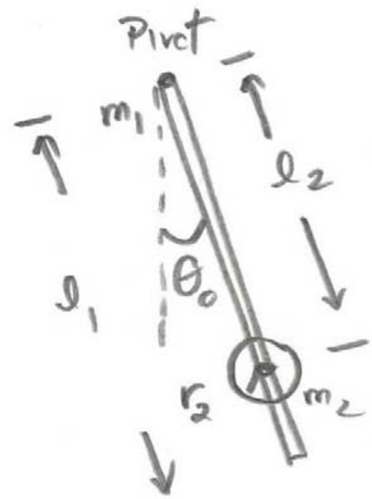
$$R m v_{cm,0} = R m v_{cm,f} + \frac{1}{2} m R^2 \frac{v_{cm,f}}{R}$$

$$\Rightarrow \quad R m v_{cm,0} = \frac{3}{2} R m v_{cm,f}$$

$$\Rightarrow \quad v_{cm,f} = \frac{2}{3} v_{cm,0}$$

Problem 5: Experiment 7: Physical Pendulum

A physical pendulum consists of a rod of mass m_1 pivoted at one end. The rod has length l_1 and moment of inertia I_1 about the pivot point. A disc of mass m_2 and radius r_2 with moment of inertia I_{cm} about it's center of mass, is rigidly attached a distance l_2 from the pivot point. The pendulum is initially displaced to an angle θ_0 and then released from rest.



a) What is the moment of inertia of the physical pendulum about the pivot point?

b) What is the angular acceleration of the pendulum about the pivot point?

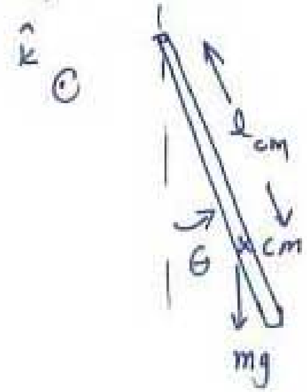
c) What is the angular velocity of the pendulum when the pendulum is at the bottom of it's swing?

d) If the disc is moved further from the pivot, will the period of the pendulum increase or decrease or stay the same?

Solution:

a)

$$I_p = \underbrace{I_1}_{\text{rod}} + \underbrace{m_2 l_2^2 + I_{cm}}_{\text{disc}}$$



$$l_{cm} = \frac{m_1 \frac{l_1}{2} + m_2 l_2}{m_1 + m_2}$$

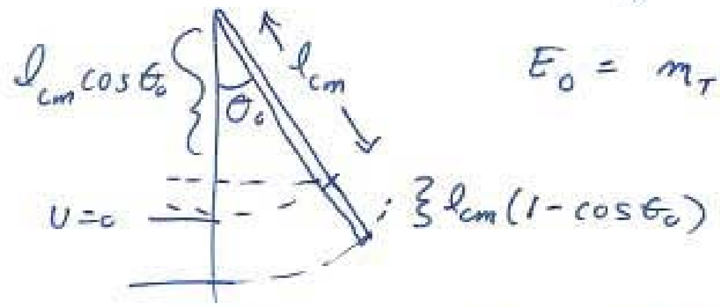
$\vec{\tau}_p =$	$I_p \vec{\alpha}$
$-m_T g l_{cm} \sin\theta$	$= I_p \alpha$

b)

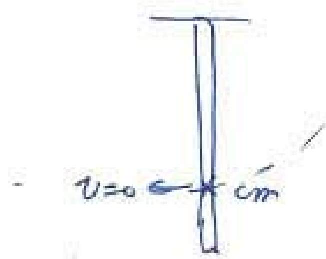
$$\alpha = \frac{-m_T g l_{cm} \sin\theta}{I_p} = \frac{-(m_1+m_2) g \left(\frac{m_1 \frac{l_1}{2} + m_2 l_2}{m_1+m_2}\right) \sin\theta}{I_1 + m_2 l_2^2 + I_{cm}}$$

$$\alpha = \frac{-g\left(\frac{m_1}{2}l_1 + m_2 l_2\right)\sin\theta}{I_1 + m_2 l_2^2 + I_{cm}}$$

c) Use conservation of energy about pivot



$$E_0 = m_T g\, l_{cm}(1-\cos\theta_0)$$



$$E_f = \frac{1}{2} I_{pivot}\, \omega_f^2$$

$$E_0 = E_f \Rightarrow$$

$$m_T g\, l_{cm}(1-\cos\theta_0) = \frac{1}{2} I_p\, \omega_f^2$$

$$\omega_f = \left(\frac{m_T g\, l_{cm}(1-\cos\theta_0)}{I_p}\right)^{1/2}$$

$$\omega_f = \left(\frac{g\left(\left(\frac{m_1}{2}l_1 + m_2 l_2\right)(1-\cos\theta_0)\right)}{I_1 + m_2 l_2^2 + I_{cm}}\right)^{1/2}$$

d) For the small angle approximation

$$T = \frac{2\pi}{\sqrt{\frac{m_T g\, l_{cm}}{I_p}}} = 2\pi\sqrt{\frac{I_p}{m_T g\, l_{cm}}}$$

If the disc is slid to a new position l_2'

$$I_p' = I_1 + I_{cm} + m_2 l_2'^2 = I_p + m_2(l_2' - l_2)$$

so this has changed

The center of mass has also changed

$$l_{cm}' = \frac{m_1 \frac{l}{2} + m_2 l_2'}{m_1 + m_2} = l_{cm} + \frac{m_2(l_2' - l_2)}{m_1 + m_2}$$

so

$$T' = 2\pi\sqrt{\frac{I_{cm}'}{m_T g l_{cm}'}} = 2\pi\sqrt{\frac{I_p + m_2(l_2'^2 - l_2^2)}{m_T g\left(l_{cm} + \frac{m_2(l_2' - l_2)}{m_1 + m_2}\right)}}$$

$$= 2\pi\sqrt{\frac{I_p\left(1 + \frac{m_2(l_2'^2 - l_2^2)}{I_p}\right)}{m_T g\, l_{cm}\left(1 + \frac{m_2(l_2' - l_2)}{(m_1 + m_2) l_{cm}}\right)}}$$

If $1 + \frac{m_2(l_2'^2 - l_2^2)}{I_p} > 1 + \frac{m_2(l_2' - l_2)}{(m_1 + m_2) l_{cm}}$

period increases!

Problem 6: Experiment 11 Angular Momentum

A steel washer, is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. Assume that the frictional torque on the axle remains the same throughout the slowing down. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down until it reaches an angular velocity of ω_b at time t_b. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t = t_a - t_b$. Assume the frictional torque on the axle remains the same. The two washers continue to slow down until at they stop at t_f.

a) Describe how you measured the moment of inertia in the experiment.

A torque was applied to the rotor by a string that was supporting a weight. When the weight was dropped, the rotor began to accelerate. The acceleration was measured by making a straight line fit to the voltage vs. time graph, and then using the calibration factor for turning voltage into angular frequency. Once the string unwound, the frictional deceleration was measured in a similar fashion.

The rotational dynamics for the two stages are given by

$$RT - \tau_f = I\alpha_{up}$$

$$-\tau_f = I\alpha_{down}$$

Note that $\alpha_{down} < 0$

The first two equations above imply that

$$RT + I\alpha_{down} = I\alpha_{up}.$$

The force equation for the first stage is given by

$$mg - T = ma_{up}.$$

The linear acceleration and angular acceleration are constrained by

$$a_{up} = R\alpha_{up}.$$

Combining these last two equations and solving for the tension yields

$$T = mg - m\alpha R_{up}$$

Substituting the tension into the combined torque equation yields

$$mg - m\alpha R_{up} + I\alpha_{down} = I\alpha_{up}$$

We can solve this for the moment of inertia

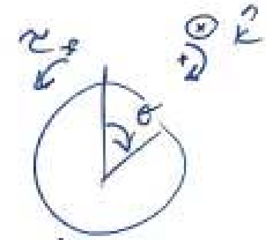
$$I = \frac{mgR - m\alpha R_{up}}{\alpha_{up} - \alpha_{down}} = \frac{mgR - m\alpha R_{up}}{\alpha_{up} + |\alpha_{down}|}$$

b) What is the angular deceleration while the first washer and motor is slowing down?

c) What is the angular velocity of the two washers immediately after the collision is finished?

d) What is the angular deceleration after the collision?

Solution: a) $\alpha = \dfrac{\omega_b - \omega_0}{t_b}$



b) $\vec{J} = \int_{t_b}^{t_a} \vec{\tau}\, dt = -\left(\tau_f \quad\right) \Delta t\, \hat{k} = \quad \Delta \vec{L}$

$$\Delta \vec{L} = L_a - L_b = \left((I_0 + I_w)\omega_a - I_0 \omega_b\right)\hat{k}$$

$$-\tau_f = I_0 \alpha = I_0 \left(\frac{\omega_b - \omega_0}{t_b}\right)$$

$$\Rightarrow \quad -\tau_f \Delta t = (I_0 + I_w)\,\omega_a - I_0 \omega_b$$

$$\omega_a = \frac{-\tau_f \Delta t + I_0 \omega_b}{I_0 + I_w}$$

$$= \frac{-I_0\left(\frac{\omega_b - \omega_0}{t_b}\right)\Delta t + I_0 \omega_b}{I_0 + I_w}$$

$$\omega_a = \frac{I_0 \omega_b \left(1 - \frac{\Delta t}{t_b}\right) + I_0 \omega_0 \frac{\Delta t}{t_b}}{I_0 + I_w}$$