MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T **Fall Term 2004**

Problem Set 12: Kinetic Theory; Mechanical Equivalent of Heat Solutions

Problem 1: Isothermal Ideal Gas Atmosphere

a) Find the root mean square (rms) speed of an oxygen molecule at T = 300K at sea level.

b) Find the root mean square z-component of the velocity of an oxygen molecule at T = 300K at sea level.

c) If a molecule with the above z-velocity were in a large evacuated container, to what maximum height H would it rise?

d) Find the rms z-component of velocity of a nitrogen molecule at T = 300K and height H assuming that the atmosphere extends that high (which it does)?

a) According to the equipartition theorem, each translational degree of freedom get the amount of energy equal to $k_BT/2$. Then the root mean square (RMS) speed is

$$v = \sqrt{\bar{v_x^2} + \bar{v_y^2} + \bar{v_z^2}} = \sqrt{\frac{3k_BT}{m}} \approx 483\, m/s$$

where the over-bar denotes the average.

b) For each component of the velocity, we find RMS values as

$$\bar{v}_z = \sqrt{\bar{v_z^2}} = \sqrt{\bar{v_x^2}} = \sqrt{\bar{v_y^2}} = \sqrt{\frac{1}{3}\bar{v^2}} = \sqrt{\frac{k_BT}{m}} \approx 279\, m/s$$

c) Using conservation of energy, $m\bar{v}_z^2/2 + mgh = Const$, we obtain

$$H = \frac{\bar{v}_z^2}{2g} = \frac{k_BT}{2mg} \approx 4\, km$$

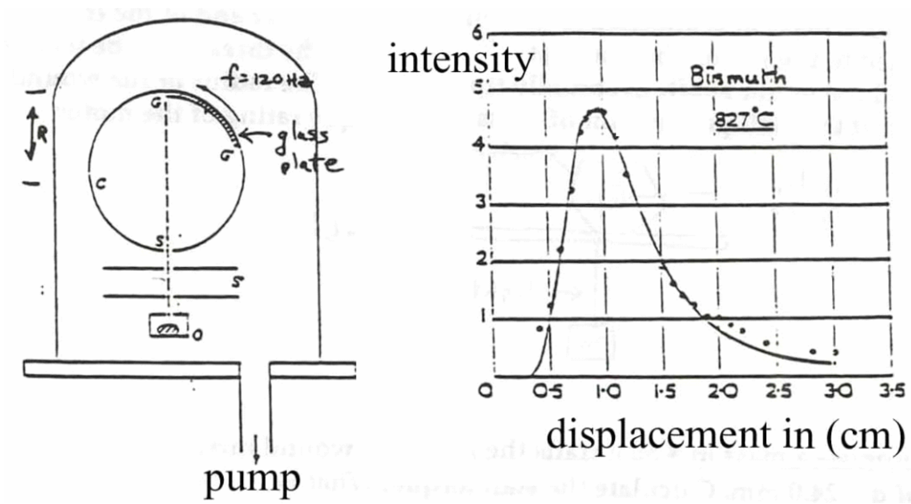
d) Repeating the same calculation for the nitrogen molecule, we obtain

$$v_z \approx 298\, m/s$$

and

$$H \approx 4.5\, km$$

Problem 2: ***Measuring Speeds of Gas Molecules*** The actual speeds of atoms and molecules were first measured directly in 1930 by the following experiment. A beam of bismuth atoms, evaporated from an 'oven', passed through some defining slits in a hollow cylinder of radius $R = 4.7 \times 10^{-2}\ m$, rotating with frequency $f = 1.2 \times 10^{2}\ Hz$. The cylinder had a narrow slit cut in one side parallel to its axis. A glass plate was mounted opposite the slit in the cylinder. (See sketch). The fastest atoms would arrive at the leading edge of the plate and the slower atoms would distribute themselves further back along the plate. The following graph shows the results for Bismuth atoms at a temperature $T = 827\ ^{0}C$. In this problem we shall calculate the velocity of the Bismuth atoms at the maximum of the distribution curve. The Bismuth atoms have a molecular weight $M = 2.09 \times 10^{2}\ g/mole$.



a) What is the displacement backwards along the glass of the maximum of the curve?

b) What angle (in radians) does this correspond to?

c) How long does it take for the cylinder to rotate through this angle? (This corresponds to the time it takes the Bismuth atom to cross the diameter of the cylinder).

d) What is the velocity of these Bismuth atoms?

e) Using the equipartition theorem $v_{rms}^{2} = 3RT/M$, where the universal gas constant $R = 8.31\ J/mole-K$, calculate the root mean square velocity of the Bismuth atoms. How does your answer compare to part d)?

Problem 2. Measuring Speeds of Gas Molecules

a) The displacement of the maximum is $\delta = 1$ cm.

b) This corresponds to angle

$$\theta = \frac{\delta}{R} \approx 0.21 \approx 12^\circ$$

c) The time is takes the cylinder to rotate through this angle is

$$\tau = \frac{\theta}{\omega} = \frac{\delta}{R 2\pi f} \approx 3 \times 10^{-4} s$$

d) Over the time τ the atoms travel distance $2R$. Thus their velocity is

$$v = \frac{2R}{\tau} = \frac{4R^2 \pi f}{\delta} \approx 333\, m/s$$

e) The RMS speed is

$$v_{\mathrm{RMS}} = \sqrt{\frac{3RT}{M}} \approx 362\, m/s$$

Problem 3. Burning Calories

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A racing bicyclist, traveling at an average speed of 25 mph, has a metabolic rate of

$$dE/dt = 1.36 \times 10^3 \text{ W}.$$

The average power output is

$$dW/dt = 3.0 \times 10^2 \text{ W}.$$

The person cycles for 4 hours and covers 100 miles. The person has mass $m = 60 \text{ kg}$.

a) What is the average force that the person applies?

b) What is the ratio of power output to catabolic rate, $\varepsilon = \dfrac{dW/dt}{dE/dt}$?

c) How many kilocalories did the person burn up?

d) What rate did the cyclist generate heat?

e) What happens to this heat?

Answer:

a) The cyclist applies an average force by pedaling. Suppose the bicyclist pedals at an angular velocity ω_0. Then the power output of the cyclist is

$$P = \tau\omega_0 = 3.0 \times 10^2 \text{ W}.$$

Now assume that the cyclist applies an average force F_{ave} and the radius of the crankshaft is r_0. Then the cyclist applies a torque

$$\tau = F_{ave} r_0 .$$

The crankshaft on my bicycle is $r_0 \cong 0.17 \text{ m}$.

So the average force exerted by the cyclist is

$$F_{ave} = P / r_0 \omega_0$$

The angular velocity of the cyclist legs and the angular velocity of the wheel depends on the gear ratio. Let's assume that at this gear ration at 25 mph is about 5**.** The angular velocity ω_1 of the bicycle is given by

$$\omega_1 = v_{cm} / r_1$$

where r_1 is the radius of the wheel. The radius of the wheel on bicycle is $r_1 \cong 0.342 \text{ m}$.

So the velocity of the bicycle is approximately

$$v_{cm} = (25 \text{ mph})(0.45 \text{ m} \cdot \text{s}^{-1} / \text{mph}) = 11.3 \text{ m} \cdot \text{s}^{-1}.$$

So the angular velocity is

$$\omega_1 = v_{cm} / r_1 = (11.3 \text{ m} \cdot \text{s}^{-1}) / (0.345 \text{ m}) = 3.3 \times 10^1 \text{ rad} \cdot \text{s}^{-1}.$$

The ratio of angular velocities is then

$$\omega_1 = (\textit{gear ratio})\omega_0 .$$

So the spin rate of the cyclist is

$$\omega_0 = \omega_1 / (\textit{gear ratio}) = (3.3 \times 10^1 \text{ rad} \cdot \text{s}^{-1}) / 5 = 6.5 \text{ rad} \cdot \text{s}^{-1}.$$

Note that this corresponds to a frequency of 1 cycle per second,

$$f = \omega_0 / 2\pi = (6.5 \text{ rad} \cdot \text{s}^{-1}) / 2\pi \cong 1 \text{ Hz}.$$

Thus the average force exerted by the bicyclist is

$$F_{ave} = P / r_0 \omega_0 = (3.0 \times 10^2 \text{ W})/(0.17 \text{ m})(6.5 \text{ rad} \cdot \text{s}^{-1}) = 2.7 \times 10^2 \text{ N}.$$

Note that cyclist pedal with two legs and they only apply a force per leg over an angle θ.

So the effective force on each leg is the average force divided by two divided by the fraction of the circle the force is applied angle $\theta / 2\pi$

$$F_{eff} = \frac{F_{ave}}{2} / \frac{\theta}{2\pi} = \frac{F_{ave} 2\pi}{2\theta}.$$

Suppose I apply a force for only 60 degrees, $\theta = \pi / 3$, then the effective force on each leg is

$$F_{eff} = \frac{F_{ave} 2\pi}{2\theta} = \frac{(2.7\times10^2 \text{ N})(2\pi)}{(2)(\pi/3)} = 8.1\times10^2 \text{ N}.$$

b)

$$\epsilon \approx 0.22$$

c) The amount of energy obtained through the metabolism is

$$E = \frac{dE}{dt}T \approx 1.96 \times 10^7 \, J \approx 4.7 \times 10^3 \, \text{kilocalories}$$

d) The cyclist generates heat at a rate

$$\frac{dQ}{dt} = \frac{dE}{dt} - \frac{dW}{dt} = 1060 \, W$$

e) This heat increases the cyclist's temperature and is also exchanged with the environment (air).

Problem 4: Experiment Mechanical Equivalent of Heat

This problem will be part of the experiment write-up (last page) and given out in class the day of the experiment.

Part One. Calibration

The results of our calibration where

$$B = 159°C$$
$$A = 29°C$$

The uncertainty in measurement of T is

$$\delta T = \frac{\delta R}{R} B \approx 0.2°C \quad \text{at } T \approx 20°C$$

Part Two. Analysis

Our fit gave

$$A = 6370\, s$$
$$B = 3.68 \times 10^{-3}\, °C/s$$
$$C = 19.2\, °C$$

The mass of water was $m_w = 60$ g. Using $P_f = W r_{pulley} \omega_{motor}$, we calculate

$$P_f = 1.19 \pm 0.06 J/s.$$

On the other hand, the fit gave

$$\frac{P_f}{m_w c_w} = 3.79 \times 10^{-3}\, °C\, s^{-1} (\pm 1.5\%).$$

Thus the mechanical equivalent of heat is

$$\frac{P_f}{P_f/(m_w c_w) \times m_w} = 5.2 \pm 0.3\, J/cal$$

Similarly, for the electric heating $P_e = V^2/R = 1.54 \pm 0.08\, J/s$ and the fit gave

$$\frac{P_e}{m_w c_w} = 5.25 \times 10^{-3}\, °C\, s^{-1} (\pm 2.5\%)$$

Thus the mechanical equivalent of heat is

$$\frac{P_e}{P_e/(m_w c_w) \times m_w} = 4.9 \pm 0.3\, J/cal$$

To get the result $P = 4.2\, J/cal$ we would have to use $m_w = 75$g for the friction experiment and $m_w = 70$g for the electric experiment.

Using the dependence of temperature on time, derived in the experiment write-up

$$T(t) = T_0 + \left(\frac{\tau_0 P_f}{m_w c_w} - T_0 + T_{room}\right)\left(1 - e^{-t/\tau_0}\right),$$

we can see that if the experiment is run for $t \gg \tau_0 = A/2$, the temperature should approach

$$T = T_{room} + \frac{\tau_0 P_f}{m_w c_w} = T_{room} + \frac{A P_f}{2 m_w c_w}.$$

For our friction experiment, for example,

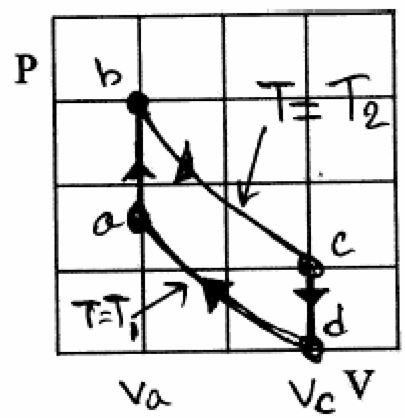
$$\frac{A P_f}{2 m_w c_w} \approx 12^\circ$$

so we would be able to heat the water by $12^\circ C$ compared to the initial (i.e. room) temperature.

Problem 5: Hand in the In-class problem Friday Dec 3 with your problem set.

Problem 43: Thermodynamic Cycle for an Ideal Gas Heat Engine

a)



b) The gas is heated at constant volume, thus there is no work done. The change in the internal energy is equal to

$$\Delta U = C_V \Delta T = 3/2nR(T_2 - T_1) = \frac{3}{2} p_a V_a \left(\frac{T_2}{T_1} - 1 \right)$$

and is equal to the heat flowing *from* the heat source *to* the system.

c) The work done in the process of isothermal expansion is

$$W_{cb} = \int_{V_b}^{V_c} p(V)\, dV.$$

Since the temperature is constant, we express

$$p(V) = \frac{nRT_2}{V} = \frac{p_b V_b}{V}$$

and find the work as

$$W_{cb} = p_b V_b \int_{V_b}^{V_c} \frac{1}{V}\, dV = p_b V_b \ln \frac{V_c}{V_b} = p_a V_a \frac{T_2}{T_1} \ln R_V$$

where we used the fact that $p_a V_a = nRT_1$ and $p_b V_b = nRT_2$. Since the temperature is kept constant during this process, the internal energy does not change, thus the amount of heat taken from the heat source is

$$Q_{cb} = W_{cb}$$

d) This process is completely analogous to the process $a \rightarrow b$, except that the gas is now cooled from T_2 to T_1 at constant volume V_c. Thus we find

$$\Delta U = C_V \Delta T = 3/2nR(T_1 - T_2) = \frac{3}{2} p_a V_a \left(1 - \frac{T_2}{T_1}\right)$$

This change in the internal energy is negative and is equal to the amount of heat Q_{dc} exchanged with the heat sink. Thus the heat is removed from the system.

e) Again, this process is analogous to $b \to c$. Thus

$$W_{ad} = p_d V_d \int_{V_d}^{V_a} \frac{1}{V}\, dV = p_d V_d \ln \frac{V_a}{V_d} = p_a V_a \ln(1/R_V) = -p_a V_a \ln R_V$$

The heat $Q_{ad} = W_{ad}$ is negative, so it is removed from the system.

f) The net work done by the system is

$$W^{net} = W_{cb} + W_{ad} = p_a V_a \ln R_V \left(\frac{T_2}{T_1} - 1 \right).$$

The total heat taken from the heat source at T_2 is

$$Q^{total} = \frac{3}{2} p_a V_a \left(\frac{T_2}{T_1} - 1 \right) + p_a V_a \frac{T_2}{T_1} \ln R_V.$$

Thus the efficiency is

$$\epsilon = \frac{W^{net}}{Q^{total}} = \frac{p_a V_a \ln R_V \left(\frac{T_2}{T_1} - 1 \right)}{\frac{3}{2} p_a V_a \left(\frac{T_2}{T_1} - 1 \right) + p_a V_a \frac{T_2}{T_1} \ln R_V} = \frac{1 - T_1/T_2}{3/2(1 - T_1/T_2)/\ln R_V + 1}$$

g) In the limit $R_V \gg 1$, $\ln R_V \gg 1$ and we can ignore the first term in the denominator of this expression. Thus

$$\epsilon \to 1 - \frac{T_1}{T_2} \quad \text{as} \quad R_V \to \infty$$

In-Class Problem 44 Power plant efficiency

A large coal-fired power plant generates about 10^9 W average power. This is enough to meet the load of approximately 10^6 Americans. In the plant, the coal is burned to create high pressure steam, which then runs a heat engine. Practical considerations limit the temperature of the steam to typically 750 K (~900 ^oF, this would be superheated steam; the vapor pressure of water at this temperature is impracticably high). Similarly, the heat has to be removed from the heat engine at well above room temperature if a cooling tower is used (these might have a temperature around 350 K).

a) Assuming that the plant achieves 75% of the thermodynamically allowed efficiency, estimate how many kilograms of coal the plant will consume per day. (Anthracite, the best coal, has roughly $2.8\times10^7\ \mathrm{J\cdot kg^{-1}}$.)

Answer: The ratio of the power output to the power input is the efficiency of the plant,

$$\varepsilon_{plant} = P_{out} / P_{in}$$

The maximum allowed efficiency is

$$\varepsilon = \frac{\Delta T}{T_2} = \frac{750\ K\text{-}350\ K}{750\ K} = .53.$$

The plant achieves 75% of this efficiency, so the efficiengy of the plant is

$$\varepsilon_{plant} = (.75)\frac{\Delta T}{T_2} = (.75)\frac{750\ K\text{-}350\ K}{750\ K} = 0.4$$

The power output per day is

$$P_{out} = (10^9\ \mathrm{J\cdot s^{-1}})(8.64\times10^4\ \mathrm{s\cdot day}) = 8.64\times10^{13}\ \mathrm{J\cdot day^{-1}}.$$

The necessary power input is therefore

$$P_{in} = P_{out} / \varepsilon_{plant} = (8.64\times10^{13}\ \mathrm{J\cdot day^{-1}})/(.4) = 2.16\times10^{14}\ \mathrm{J\cdot day^{-1}}.$$

b) The energy content of Anthracite per kilogram, the best coal, is roughly $2.8\times10^7\ \mathrm{J\cdot kg^{-1}}$. The amount of kilograms of coal the plant will consume per day will be

$$(mass\ of\ coal)/day = (energy\ input/day)/(energy\ of\ coal/kilogram)$$

$$(mass\ of\ \ coal)/day = (2.16\times10^{14}\ \mathrm{J\cdot day^{-1}})/(2.8\times10^{7}\ \mathrm{J\cdot kg^{-1}}) = 7.7\times10^{6}\ \mathrm{kg\cdot day^{-1}}$$

c) How many railroad cars full is this (1 loaded car's mass is $\sim10^5$ kg), and how long would the train be in miles (1 car is about 75 feet long)?

Answer:

The number of railroad cars per day full of coal is

$$cars/day = \frac{(mass\ of\ \ coal)/day}{(mass\ of\ \ coal/car)} = (7.7\times10^{6}\ \mathrm{kg\cdot day^{-1}})/(10^{5}\ \mathrm{kg}) \cong 77\,.$$

So we need about 80 cars per day.

The length of the train is then about 1 mile since

$$length = (\#\,number\ of\ \ cars)(length/car) \cong (80)(75\ \mathrm{ft}) = (6000\ \mathrm{ft})(1\ \mathrm{mile}/5280\ \mathrm{ft}) = 1.1\,\mathrm{miles}$$

The laws of thermodynamics therefore considerably complicate the generation of mechanical energy in today's society. One way to deal with this is cogeneration such as practiced in MIT's 20×10^{6} W=20 MW power plant: the waste heat from the generator is used to heat the campus (but what to do in the summer?). As you can see, a HUGE improvement in energy efficiency and concomitant reduction in waste heat awaits the development of economically practical alternatives (such as fuel cells) to burning fuel and using heat engines to transform this to mechanical energy.