

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Physics

Physics 8.01T

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**Problem Set 11: Angular Momentum, Rotation and Translation Solutions**

**Problem 1 : Bohr hydrogen atom**

The Bohr hydrogen atom models the atom as an electron orbiting the proton under the influence of an electric force producing uniform circular motion with radius  $a_0$ . The mass of the electron is  $m_e = 9.1 \times 10^{-31} \text{ kg}$ ; the electric charge is  $e = 1.6 \times 10^{-19} \text{ C}$ ; the Planck constant is  $h = 6.63 \times 10^{-34} \text{ J}\cdot\text{s}$ , and the magnitude of the electric force is given by Coulomb's Law  $|\vec{F}| = ke^2/r^2$  where  $k = 9.0 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2$ . The angular momentum is quantized according to  $L = nh/2\pi$  where  $n = 1, 2, \dots$

- a) Write down the equation that arises from the application of Newton's 2nd Law to the electron.

Since the electron is much less massive than the proton the reduced mass is the mass of the electron. The radial component of Newton's Second Law is then

$$-\frac{ke^2}{a_0^2} = -m_e \frac{v^2}{a_0}.$$

You can solve this equation for the velocity in terms of the radius of the electron orbit,

$$v = \sqrt{\frac{ke^2}{m_e a_0}}$$

- b) What is the angular momentum of the electron about the center of the atom?

$$L = n \frac{h}{2\pi} = m_e a_0 v$$

- c) Using your results from parts a) and b) derive an equation for the radius  $a_0$  of the atom as a function of  $n, e, h, m_e$ , and  $k$ .

Using the result for the velocity of the orbit, the angular momentum equation becomes

$$n \frac{h}{2\pi} = m_e \sqrt{\frac{ke^2 a_0}{m_e}}$$

Square this equation and solve for the radius  $a_0$  of the atom

$$n^2 \frac{h^2}{4\pi^2} = m_e^2 \frac{ke^2 a_0}{m_e}$$

Thus

$$a_0 = \frac{n^2 h^2}{4\pi^2 m_e k e^2}$$

d) What is the energy  $E_n$  for the atom? Express your answer in terms of  $n, e, h, m_e$ , and  $k$ .

Using the value for the velocity of the orbit, the energy is given by

$$E = \frac{1}{2} m_e v^2 - \frac{ke^2}{a_0} = \frac{1}{2} m_e \frac{ke^2}{m_e a_0} - \frac{ke^2}{a_0} = -\frac{ke^2}{2a_0}$$

Now use the value for the radius of the orbit

$$E = -\frac{1}{2} \frac{ke^2}{a_0} = -\frac{ke^2}{2n^2 h^2 / 4\pi^2 m_e k e^2} = -\frac{2\pi^2 m_e k^2 e^4}{n^2 h^2}$$

Set

$$A = \frac{2\pi^2 m_e k^2 e^4}{h^2} = 2\pi^2 \frac{(9.11 \times 10^{-31} \text{ kg})(8.99 \times 10^9 \text{ N} \cdot \text{m}^2 \text{C}^{-2})^2 (1.60 \times 10^{-19} \text{ C})^4}{(6.63 \times 10^{-34} \text{ J} \cdot \text{sec})^2}$$

$$A = (2.17 \times 10^{-18} \text{ J}) \frac{1 \text{ eV}}{1.60 \times 10^{-19} \text{ J}} = 13.6 \text{ eV}$$

Then the energy is

$$E_n = -\frac{ke^2}{2n^2 h^2 / 4\pi^2 \mu k e^2} = -\frac{2\pi^2 \mu k^2 e^4}{h^2} \frac{1}{n^2} = -\frac{A}{n^2}$$

When  $n = 1$ , the constant  $A = 13.6 \text{ eV} = -E_1$  is minus the ground state energy.

- e) A hydrogen atom emits a photon which arise from an energy transition from the  $n = 3$  to  $n = 1$  energy level. Calculate the frequency of the light emitted

$$f = -\Delta E/h = -(E_1 - E_3)/h.$$

Using our result for the energy levels, we have that

$$f = \frac{A}{h} \left( 1 - \frac{1}{9} \right) = \frac{(8)(13.6 \text{ eV})(1.6 \times 10^{-19} \text{ J} \cdot \text{eV}^{-1})}{(9)(6.63 \times 10^{-34} \text{ J} \cdot \text{s})} = 3.46 \times 10^{15} \text{ J}$$

- f) What is the wavelength of the emitted light from part e)?

The wavelength of the emitted light  $\lambda$  is related to the frequency of light,  $f$ , and the speed of light  $c = 3.00 \times 10^8 \text{ m} \cdot \text{sec}^{-1}$  by

$$f\lambda = c$$

Thus the inverse wavelength of the light is given by

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{A}{hc} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

where the Rydberg constant,

$$R = \frac{A}{hc} = \frac{2\pi^2 m_e k^2 e^4}{h^3 c} = 1.09 \times 10^7 \text{ m}.$$

So the wavelength is

$$\lambda_{3,2} = \frac{1}{R} \left( \frac{n_i^2 n_f^2}{n_i^2 - n_f^2} \right)$$

When the electron in the energy level  $n = 3$  drops to the energy level  $n = 1$ . The wavelength is then

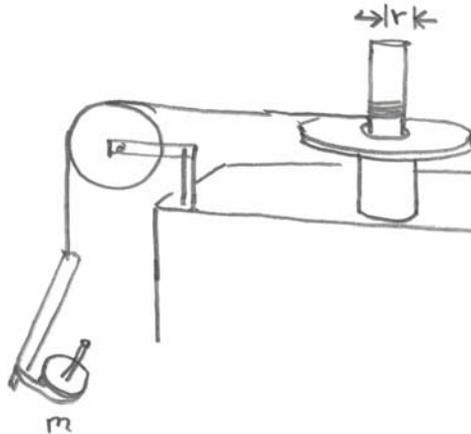
$$\lambda_{3,2} = \frac{9}{8R} = \frac{9}{8(1.09 \times 10^7 \text{ m})} = 1.03 \times 10^{-7} \text{ m},$$

**Problem 2: Experiment 11 Angular Momentum: Analysis**

**Part One: Rotor Moment of Inertia**

Enter the results from your experiment report into the table below.

$\alpha_{down}$ [rad·s <sup>-2</sup> ]	$\alpha_{up}$ [rad·s <sup>-2</sup> ]	$a$ [m·s <sup>-2</sup> ]	$T$ [N]	$\tau_{up}$ [N·m]	$\tau_f$ [N·m]	$I_R$ [kg·m <sup>2</sup> ]
-34.2	82.5	1.0505	.482	$6.12 \times 10^{-3}$	$-1.79 \times 10^{-3}$	$5.24 \times 10^{-5}$



Note: The rotational dynamics for the two stages are given by

$$RT - \tau_f = I_R \alpha_{up}$$

$$-\tau_f = I_R \alpha_{down}$$

Note that  $\alpha_{down} < 0$ .

The first two equations above imply that

$$RT + I_R \alpha_{down} = I_R \alpha_{up}$$

The force equation for the first stage is given by

$$mg - T = ma_{up}$$

The linear acceleration and angular acceleration are constrained by

$$a_{up} = R\alpha_{up}.$$

Combining these last two equations and solving for the tension yields

$$T = mg - m\alpha_{up}R$$

Substituting the tension into the combined torque equation yields

$$(mg - m\alpha_{up}R)R + I_R\alpha_{down} = I_R\alpha_{up}$$

We can solve this for the moment of inertia

$$I_R = \frac{mgR - m\alpha_{up}R^2}{\alpha_{up} - \alpha_{down}} = \frac{mgR - m\alpha_{up}R^2}{\alpha_{up} + |\alpha_{down}|}$$

(Here  $m = 0.055$  kg is the mass of the weight,  $r = 0.0127$  m,  $\alpha_{up}$  and  $\alpha_{down}$  are obtained from your measurements).

**Answer:** What is your numerical value for  $I_R$ ?

$$I_R = \frac{mR(g - \alpha_{up}R)}{\alpha_{up} - \alpha_{down}} = \frac{(0.055 \text{ kg})(0.0127 \text{ m})((9.805 \text{ m} \cdot \text{s}^{-2}) - (82.5 \text{ rad} \cdot \text{s}^{-2})(0.0127 \text{ m}))}{((82.5 \text{ rad} \cdot \text{s}^{-2}) - (-34.2 \text{ rad} \cdot \text{s}^{-2}))}$$

$$I_R = 5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

The frictional torque is then

$$\tau_f = I_R\alpha_{down} = (5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(-34.2 \text{ rad} \cdot \text{s}^{-2}) = -1.79 \times 10^{-3} \text{ J}$$

### Part Two: Inelastic Collision:

Write your measurement results into the table below. Note you may not have included the last two columns since the report do not specify to include these. I included my values.

$\omega_i$ [rad · s <sup>-1</sup> ]	$\omega_f$ [rad · s <sup>-1</sup> ]	$\delta t$ [s]	$\alpha_i$ [rad · s <sup>-2</sup> ]	$\alpha_f$ [rad · s <sup>-2</sup> ]
281	147	0.003	41.3	18.9

What is your numerical value for  $I_W = \frac{1}{2}m_w(r_0^2 + r_i^2)$ ?

**Answer:** A 1" US Standard Washer has inner radius  $r_1 = 1.35 \times 10^{-2}$  m and an outer radius  $r_2 = 3.20 \times 10^{-2}$  m. The moment of inertia of a washer is given by

$$I_{cm} = \frac{1}{2}m_w(r_0^2 + r_i^2).$$

What was the mass of the washer you dropped? The mass is printed on the washer. The mass of my washer is  $m_w = 84.4$  g.

What was the moment of inertia of the washer you dropped?

$$I_W = \frac{1}{2}m_w(r_0^2 + r_i^2) = 5.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2$$

1. Use the moments of inertia  $I_R$  and  $I_W$  along with  $\omega_i$  and  $\omega_f$  to calculate the angular momentum before and after the collision and compare them.

**Answer:**

$$L_{cm,before} = 1.48 \times 10^{-2} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}, \quad L_{cm,after} = 1.52 \times 10^{-2} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

This is reasonably good agreement and shows that angular momentum is essentially conserved.

2. Use the values you found for the friction torque  $\tau_f$  and  $\delta t$  to estimate the angular impulse of  $\tau_f$  during the collision. Compare it to the angular momentum difference that you just calculated.

**Answer:**  $J = \tau_f \Delta t = (-1.79 \times 10^{-3} \text{ J})(0.003 \text{ s}) = -5.3 \times 10^{-6} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$ . **This does not agree with my difference in angular momentum**

$$L_{cm,after} - L_{cm,before} = 4 \times 10^{-4} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

3. Calculate the rotational kinetic energies  $K_1 = \frac{1}{2}I_R\omega_1^2$ , before, and  $K_2 = \frac{1}{2}(I_R + I_W)\omega_2^2$ , after the collision.

**Answer:**

$$K_1 = \frac{1}{2} I_R \omega_1^2 = \frac{1}{2} (5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2) (281 \text{ rad} \cdot \text{s}^{-1})^2 = 2.07 \text{ J}$$

$$K_2 = \frac{1}{2} (I_R + I_W) \omega_2^2 = \frac{1}{2} ((5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2) + (5.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2)) (147 \text{ rad} \cdot \text{s}^{-1})^2 = 1.12 \text{ J}.$$

$$\Delta K = K_2 - K_1 = \frac{1}{2} (I_R + I_W) \omega_2^2 - \frac{1}{2} I_R \omega_1^2 = 1.12 \text{ J} - 2.07 \text{ J} = 0.95 \text{ J}$$

Mechanical energy is not constant since the collision is inelastic.

### Part Three: Slow Inelastic Collision:

Write your measurement results into the table below. Note I included my values for  $\alpha_i$  and  $\alpha_f$  but the lab report did not ask you for these values.

$\omega_1$ [rad · s <sup>-1</sup> ]	$\omega_2$ [rad · s <sup>-1</sup> ]	$\delta t$ [s]	$\alpha_i$ [rad · s <sup>-2</sup> ]	$\alpha_c$ [rad · s <sup>-2</sup> ]	$\alpha_f$ [rad · s <sup>-2</sup> ]
271	127	0.79	-40.5	-185	-18.4

- Use the moments of inertia  $I_R$  and  $I_W$  along with  $\omega_1$  and  $\omega_2$  to calculate the angular momentum before and after the collision and compare them.

**Answer:**

$$L_{cm, \text{before}} = I_R \omega_1 = 1.42 \times 10^{-2} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}, \quad L_{cm, \text{after}} = (I_R + I_W) \omega_2 = 1.31 \times 10^{-2} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

These are close to each other in value.

- Use the value you found for the friction torque  $\tau_f$  and  $\delta t$  to estimate the angular impulse of  $\tau_f$  during the collision. Compare it to the difference in angular momentum before and after the collision.

**Answer:**

$$J = \tau_f \Delta t = (-1.79 \times 10^{-3} \text{ J}) (0.79 \text{ s}) = -1.41 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1},$$

$$L_{cm, \text{after}} - L_{cm, \text{before}} = -1.1 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

This gives even better agreement than in the fast collision.

3. Use the value you found for the angular acceleration during the collision,  $\alpha_c$ , to estimate the total torque  $\tau_{total}$  on the rotor during the collision.

**Answer:** The total torque on the rotor during the collision is

$$\tau_{total} = I_R \alpha_c = (5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(-185 \text{ rad} \cdot \text{s}^{-2}) = -9.70 \times 10^{-3} \text{ J}$$

The angular impulse is

$$J = \tau_{total} \Delta t = I_R \alpha_c \Delta t$$

$$J = I_R \alpha_c \Delta t = (5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(-185 \text{ rad} \cdot \text{s}^{-2})(0.79 \text{ s}) = -7.66 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$$

The change in angular momentum is

$$\Delta L = -6.5 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$$

Reasonable agreement!

4. The torque  $\tau_c$  is made of two parts: the friction torque  $\tau_f$  from the bearings and the torque  $\tau_{RW}$  due exerted by the washer you dropped on the rotor. By the 3<sup>rd</sup> law, the rotor exerts an equal and opposite torque on the washer. Since you know  $\tau_f$ , subtract it from  $\tau_c$  to find an estimate for  $\tau_{RW}$ .

**Answer:** The total torque is equal to the frictional torque and the torque  $\tau_{RW}$  exerted by the washer

$$\tau_{total} = \tau_f + \tau_{RW}$$

So the torque  $\tau_{RW}$  exerted by the washer is

$$\tau_{RW} = \tau_{total} - \tau_f = (-9.70 \times 10^{-3} \text{ J}) - (-1.79 \times 10^{-3} \text{ J}) = -7.91 \times 10^{-3} \text{ J}.$$

**Note:** this torque causes the dropped washer to increase its angular speed. It produces an angular impulse

$$J_W = -\tau_{RW} \Delta t = -(-7.91 \times 10^{-3} \text{ J})(0.79 \text{ s}) = 6.25 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.$$

The change in angular momentum of the dropped washer is

$$\Delta L_W = I_W \omega_2 = (5.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(127 \text{ rad} \cdot \text{s}^{-1}) = 6.46 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$$

in excellent agreement.

#### Part Four: Nonconservative Work in the Slow Collision

You can find  $\tau_{RW}$  a different way, because it produces the angular acceleration of the dropped washer, whose average value is  $\alpha_W = \omega_2 / \delta t$ . Use this relation to estimate  $\tau_{RW}$  and compare it to the previous estimate you made:

The torque  $\tau_{RW}$  comes from the sliding friction between the washer on the rotor and the washer you dropped. Thus there must be some non-conservative work. You may calculate it if you know the angular “distance” the washer slides over the rotor before it reaches the same angular velocity as the rotor, and you do have enough information to find that. The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_R = \omega_1 \delta t - \frac{1}{2} |\alpha_c| \delta t^2$$

while the dropped washer rotates through an angle

$$\Delta\theta_W = \frac{1}{2} |\alpha_W| \delta t^2.$$

Thus the non-conservative work done by the sliding friction between the two washers will be

$$W_{NC,W} = \tau_{RW} (\Delta\theta_R - \Delta\theta_W).$$

You can also calculate the non-conservative work done by the bearing friction during the collision

$$W_{NC,B} = \tau_f \Delta\theta_R.$$

Calculate these two amounts of non-conservative work and compare their sum to the change in rotational kinetic energy  $\frac{1}{2} I_R \omega_1^2 - \frac{1}{2} (I_R + I_W) \omega_2^2$  during the collision.

**Answer:** The torque from the lower washer on the dropped washer causes the dropped washer to increase its angular speed.

If we average the angular acceleration

$$\alpha_W = \omega_2 / \Delta t = (127 \text{ rad} \cdot \text{s}^{-1}) / (0.79 \text{ s}) = 1.61 \times 10^2 \text{ rad} \cdot \text{s}^{-2}.$$

So the torque is then

$$\tau_w = I_w \alpha_w = (5.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(1.61 \times 10^2 \text{ rad} \cdot \text{s}^{-2}) = -8.20 \times 10^{-3} \text{ J}$$

The angular displacement for the washer is

$$\Delta\theta_w = \frac{1}{2} \alpha_w \Delta t^2 = \frac{1}{2} (1.61 \times 10^2 \text{ rad} \cdot \text{s}^{-2})(0.79 \text{ s})^2 = 5.02 \times 10^1 \text{ rad} .$$

The angular displacement for the rotor is

$$\Delta\theta_R = \omega_1 \Delta t - \frac{1}{2} \alpha_c \Delta t^2 = (271 \text{ rad} \cdot \text{s}^{-1})(0.79 \text{ s}) - \frac{1}{2} (185 \text{ rad} \cdot \text{s}^{-2})(0.79 \text{ s})^2 = 1.56 \times 10^2 \text{ rad} .$$

So the relative angular displacement between the washer and the rotor is

$$(\Delta\theta_R - \Delta\theta_w) = (1.56 \times 10^2 \text{ rad}) - (5.02 \times 10^1 \text{ rad}) = 1.06 \times 10^2 \text{ rad}$$

Thus the torque between the washers produces a negative non-conservative work

$$W_{NC,W} = \tau_{RW} (\Delta\theta_R - \Delta\theta_w) = (-7.91 \times 10^{-3} \text{ J})(1.06 \times 10^2 \text{ rad}) = -8.46 \times 10^{-1} \text{ J} .$$

The bearing introduces a frictional torque on the rotor which also contributes a negative non-conservative work slowing down the rotor

$$W_{NC,R} = \tau_f (\Delta\theta_R) = (-1.79 \times 10^{-3} \text{ J})(1.56 \times 10^2 \text{ rad}) = -2.79 \times 10^{-1} \text{ J}$$

So the total non-conservative work is

$$W_{NC,W} + W_{NC,R} = \tau_{RW} (\Delta\theta_R - \Delta\theta_w) + \tau_f (\Delta\theta_R) = (\tau_{RW} + \tau_f) (\Delta\theta_R) - \tau_{RW} \Delta\theta_w = \tau^{total} (\Delta\theta_R) - \tau_{RW} \Delta\theta_w .$$

As we can see, the non-conservative work can be divided into the negative work done on the rotor and the positive work done on the washer since  $\tau^{total} < 0$  and  $\tau_{RW} < 0$ .

This is equal to

$$W_{NC,W} + W_{NC,R} = (-8.46 \times 10^{-1} \text{ J}) + (-2.79 \times 10^{-1} \text{ J}) = -1.12 \text{ J} .$$

The rotational kinetic energy is given by

$$K_2 = \frac{1}{2}(I_R + I_W)\omega_2^2 = \frac{1}{2}((5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2) + (5.09 \times 10^{-5} \text{ kg} \cdot \text{m}^2))(127 \text{ rad} \cdot \text{s}^{-1})^2 = 8.3 \times 10^{-1} \text{ J}$$

$$K_1 = \frac{1}{2}I_R\omega_1^2 = \frac{1}{2}(5.24 \times 10^{-5} \text{ kg} \cdot \text{m}^2)(271 \text{ rad} \cdot \text{s}^{-1})^2 = 1.92 \text{ J}$$

So the change in rotational kinetic energy is given by

$$\Delta K = K_2 - K_1 = \frac{1}{2}(I_R + I_W)\omega_2^2 - \frac{1}{2}I_R\omega_1^2 = 0.83 \text{ J} - 1.92 \text{ J} = -1.10 \text{ J}$$

The agreement is excellent with the error due to rounding.