

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

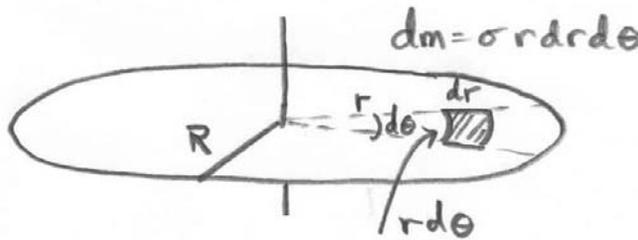
Physics 8.01 TEAL

Fall Term 2004

**In-Class Problems 30-32: Moment of Inertia, Torque, and Pendulum:
Solutions**

Problem 30 Moment of Inertia of a Uniform disc.

A uniform disc of mass m and radius R is mounted on an axis passing through the center of the disc, perpendicular to the plane of the disc. In this problem, you will calculate the moment of inertia about two different axes that pass perpendicular to the disc. One passes through the center of mass of the disc and the second passes through a point on the rim of the disc a distance R from the center. As a starting point, consider the contribution to the moment of inertia from the mass element dm show in the figure below.



Step 1: We can take the point S to be the center of mass of the disc. The axis of rotation passes through the center of the disc, perpendicular to the plane of the disc.

Step 2: We choose cylindrical coordinates with the coordinates (r, θ) in the plane and the z -axis perpendicular to the plane.

Step 3: The area element $da = r dr d\theta$ can be thought of as the product of arc length $r d\theta$ with the radial width dr , (Figure 9). Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{total}}{Area} = \frac{m}{\pi R^2}.$$

Therefore the mass in an infinitesimal area element a distance r from the axis of rotation is given by

$$dm = \sigma r dr d\theta.$$

Step 4: When the disc rotates, the mass element traces out a circle of radius $r_{\perp}^2 = r^2$.

Step 5: The integral is now an integral in two dimensions as the angle θ varies from $\theta=0$ to $\theta=2\pi$, and the radial coordinate r varies from $r=0$ to $r=R$. Thus the limits of the integral are

$$I_{cm} = \int_{body} (r_{\perp})^2 dm = \sigma \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} (r^2) r dr d\theta = \frac{m}{\pi R^2} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} (r^2) r dr d\theta.$$

Step 6: The integral can now be explicitly calculated by first integrating the θ -coordinate:

$$I_{cm} = \frac{m}{\pi R^2} \int_{r=0}^{r=R} \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) (r^2) r dr = \frac{m}{\pi R^2} \int_{r=0}^{r=R} \left(\theta \Big|_{\theta=0}^{\theta=2\pi} \right) r^3 dr = \frac{m}{\pi R^2} \int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2m}{R^2} \int_{r=0}^{r=R} r^3 dr.$$

and then integrating the r -coordinate:

$$I_{cm} = \frac{2m}{R^2} \int_{r=0}^{r=R} r^3 dr = \frac{2m}{R^2} \frac{r^4}{4} \Big|_{r=0}^{r=R} = \frac{2m}{R^2} \frac{R^4}{4} = \frac{1}{2} m R^2$$

Problem 30 Turntable

A turntable is a uniform disc of mass 1.2 kg and radius $1.3 \times 10^{-1} \text{ m}$. The turntable is spinning at an initial constant frequency of $f_0 = 33 \text{ cycles/min}$. The motor is turned off and the turntable slows to a stop in 8.0 s due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the disc?

Answer: Initially, the disc is spinning with a frequency

$$f_0 = (33 \frac{\text{cycles}}{\text{min}}) (\frac{1 \text{ min}}{60 \text{ s}}) = 0.55 \frac{\text{cycles}}{\text{s}} = 0.55 \text{ Hz} .$$

So the initial angular velocity is

$$\omega_0 = 2\pi f_0 = (2\pi \frac{\text{rad}}{\text{cycle}}) (0.55 \frac{\text{cycles}}{\text{s}}) = 3.5 \frac{\text{rad}}{\text{s}} .$$

The final angular velocity is zero, so the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5 \text{ rad/s}}{0.8 \text{ s}} = -4.3 \text{ rad/s}^2 .$$

Since the angular acceleration is negative, we confirm that the disc is slowing down.

The moment of inertia of the turntable about an axis passing perpendicular to the disc and through the center of mass is

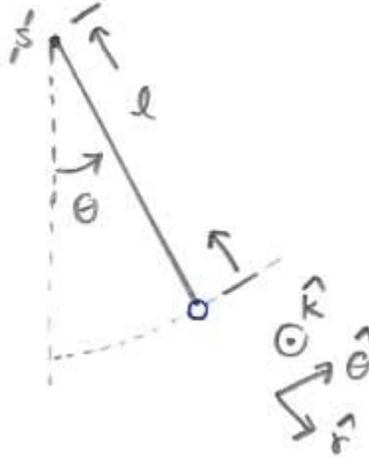
$$I_{cm} = \frac{1}{2} mR^2 = \frac{1}{2} (1.2 \text{ kg})(1.3 \times 10^{-1} \text{ m})^2 = 1.01 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

So the magnitude of the frictional torque is

$$|\tau_{fric}^{total}| = I_s |\alpha| = (1.01 \times 10^{-2} \text{ kg} \cdot \text{m}^2)(4.3 \text{ rad/s}^2) = 4.3 \times 10^{-2} \text{ N} \cdot \text{m} .$$

Problem 31: Simple Pendulum

A pendulum consists of an object hanging from the end of a string. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string is massless, then the pendulum is called a simple pendulum.



- Find the equation of motion for the object.
- Show that the object undergoes simple harmonic motion when the initial angle is small and you can approximate $\sin \theta \cong \theta$.
- For small angles, what is the period of oscillation?
- Is the angular velocity the same as the angular frequency for the object?

The torque about the pivot point is given by

$$\vec{\tau}_S = \vec{r}_{s,m} \times m\vec{g} = l\hat{r} \times mg(-\sin \theta \hat{\theta} + \cos \theta \hat{r}) = -lmg \sin \theta \hat{k}.$$

The angular acceleration is a vector pointing along the z-axis,

$$\vec{\alpha} = \frac{d^2\theta}{dt^2} \hat{k}.$$

Remark: The z-component of the angular acceleration vector is $\frac{d^2\theta}{dt^2}$ and can be positive or negative.

The moment of inertial of a point mass about the pivot point is

$$I_S = ml^2$$

The rotational dynamical equation is

$$\vec{\tau}_S = I_S \vec{\alpha}.$$

Therefore

$$-lmg \sin \theta \hat{\mathbf{k}} = ml^2 \frac{d^2 \theta}{dt^2} \hat{\mathbf{k}}.$$

Thus we have the equation of motion for the simple pendulum,

$$-lmg \sin \theta = ml^2 \frac{d^2 \theta}{dt^2}.$$

When the angle of oscillation is small, then we can use the small angle approximation

$$\sin \theta \cong \theta.$$

Then the pendulum equation becomes

$$\frac{d^2 \theta}{dt^2} \cong -\frac{g}{l} \theta.$$

This equation is similar to the mass-spring equation,

$$\frac{d^2 x}{dt^2} = -\frac{k}{m} x$$

that described the oscillation of a mass about the equilibrium point of a spring. Recall that the angular frequency of oscillation was given by

$$\omega_{spring} = \sqrt{\frac{k}{m}}.$$

So by comparison, the angular frequency of oscillation for the pendulum is approximately

$$\omega_{pendulum} \cong \sqrt{\frac{g}{l}},$$

with period

$$T = \frac{2\pi}{\omega_p} \cong 2\pi \sqrt{\frac{l}{g}}.$$

By analogy with the spring, if the pendulum has been displaced by a small angle θ_0 and released from rest, the angle that the pendulum makes with the vertical as a function of time is given by

$$\theta(t) = \theta_0 \cos((\sqrt{g/l})t).$$

The angular velocity of the pendulum is then given by the derivative

$$d\theta/dt = -(\sqrt{g/l})\theta_0 \sin((\sqrt{g/l})t).$$

This varies sinusoidally with time and is not equal to the angular frequency which is a constant associated with the periodic nature of the motion.