

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 TEAL

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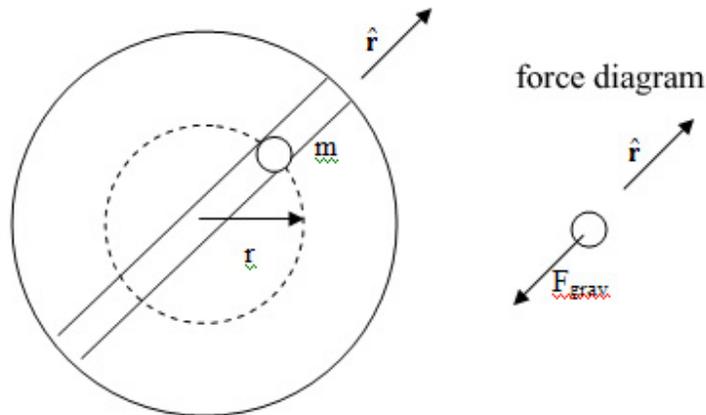
In-Class Problems 23-24: Harmonic Oscillation and Mechanical Energy: Solution

Problem 22: Journey to the Center of the Earth

Imagine that one drilled a hole with smooth sides straight through the center of the earth. If the air is removed from this tube (and it doesn't fill up with water, liquid rock, or iron from the core) an object dropped into one end will have enough energy to just exit the other end after an interval of time. Your goal is to find that interval of time.

- a) The gravitational force on an object of mass m , located inside the earth a distance $r < R_e$, from the center (R_e is the radius of the earth), is due only to the mass of the earth that lies within a solid sphere of radius r . What is the gravitational force as a function of the distance r from the center? Express your answer in terms of g and R_e . Note: you do not need the mass of the earth to answer this question. You only need to assume that the earth is of uniform density.

Answer:



Choose a radial coordinate with unit vector \hat{r} pointing outwards. The gravitational force on an object of mass m at the surface of the earth is given by two expressions

$$\vec{\mathbf{F}}_{grav} = -\frac{Gmm_e}{R_e^2}\hat{r} = -mg\hat{r}.$$

Therefore we can solve for the gravitational constant acceleration

$$g = \frac{Gm_e}{R_e^2}.$$

When the object is a distance r from the center of the earth, the mass of the earth that lies outside the sphere of radius r does not contribute to the gravitational force. The only contribution to the gravitational force is due to the mass enclosed in the sphere of radius r ,

$$m_{\text{enclosed}} = \rho \frac{4}{3} \pi r^3.$$

Since the mass density is given by

$$\rho = \frac{m_e}{(4/3)\pi R_e^3},$$

the mass enclosed is

$$m_{\text{enclosed}} = \frac{m_e}{(4/3)\pi R_e^3} (4/3)\pi r^3 = \frac{m_e r^3}{R_e^3}.$$

Therefore the gravitational force on the object of mass m when it is a distance r from the center of the earth is given by

$$\vec{\mathbf{F}}_{\text{grav}} = -\frac{Gmm_{\text{enclosed}}}{r^2} \hat{\mathbf{r}} = -\frac{Gmm_e r^3}{r^2 R_e^3} \hat{\mathbf{r}} = -\frac{Gmm_e}{R_e^3} r.$$

We can use our expression for the $g = Gm_e / R_e^2$ to find that the gravitational force on a mass m at a distance r from the center of the earth is given by

$$\vec{\mathbf{F}}_{\text{grav}} = -\frac{mg}{R_e} r \hat{\mathbf{r}}$$

- b) Using the concept of force explain how the object-earth system for objects inside the earth is analogous to an object-spring system.

Answer: The minus sign indicates that the force is always directed towards the center of the earth (restoring force) and proportional to the distance from the center of the earth. This is analogous to the restoring force of a spring,

$$\vec{\mathbf{F}}_{\text{spring}} = -kx\hat{\mathbf{x}},$$

where the 'spring constant' for gravitation is given by

$$k_{grav} = \frac{mg}{R_e}.$$

This means that the object will undergo simple harmonic motion just like a spring.

- c) What is the potential energy inside the earth as a function of r for the object-earth system? Can you think of a natural point to choose a zero point for the potential energy? Find out how long this journey will take.

Answer: We can define a potential energy for gravitation inside the earth that is analogous to the spring potential energy function with zero point for potential energy chosen at the center of the earth and

$$U(r) = \frac{1}{2} k_{grav} r^2 = \frac{1}{2} \frac{mg}{R_e} r^2.$$

If we release the object from rest at the surface of the earth, the initial mechanical energy is all potential energy and is given by

$$E_i = U(R_e) = \frac{1}{2} \frac{mg}{R_e} R_e^2 = \frac{1}{2} mgR_e.$$

When the object reaches the center of the earth, the mechanical energy is all kinetic energy,

$$E_f = K_f = \frac{1}{2} mv_c^2$$

Since there are no external work acting on the system, the mechanical energy is constant

$$E_i = E_f$$

and

$$\frac{1}{2} mgR_e = \frac{1}{2} mv_c^2$$

The velocity v_c just before the object reaches the center of the earth is then

$$v_c = -\sqrt{gR_e}.$$

Note we choose ‘just before’ so that the velocity is radially inward in polar coordinates: there is no well defined direction when the object is located at the origin.

From Newton’s Second Law, $\vec{F} = m\vec{a}$, in the radial direction becomes

$$\hat{r} : -\frac{mg}{R_e} r = m \frac{d^2 r}{dt^2}.$$

This simplifies to

$$\frac{d^2 r}{dt^2} = -\frac{g}{R_e} r$$

The position of the object satisfies

$$r(t) = R_e \cos\left(\frac{2\pi}{T} t\right).$$

where T is the period of oscillation. The velocity of the object is

$$v(t) = -\frac{2\pi}{T} R_e \sin\left(\frac{2\pi}{T} t\right).$$

When $t = T/4$, $\cos\left(\frac{2\pi}{T} t\right) = \cos\left(\frac{\pi}{2}\right) = 0$ hence the object is at the center of the earth. Also

since $\sin\left(\frac{2\pi}{T} t\right) = \sin\left(\frac{\pi}{2}\right) = 1$, the velocity at the center of the earth is given by

$$v_c = v(T/4) = -\frac{2\pi}{T} R_e.$$

From the condition that mechanical energy is constant we found that $v = -\sqrt{gR_e}$.

Therefore

$$\frac{2\pi}{T} R_e = \sqrt{gR_e}.$$

We can solve for the period and find that

$$T = 2\pi(R_e / g)^{1/2}.$$

d) How does this time compare to a satellite’s period in low earth orbit (i.e. orbits just above the surface of the earth)?

Answer: If an object undergoes uniform circular motion around the earth the Newton's Second Law in the radial direction becomes

$$\hat{\mathbf{r}} : -mg = mR_e \left(\frac{2\pi}{T_c} \right)^2.$$

We can solve for the period of the circular orbit and find that the period is identical to the period of oscillation,

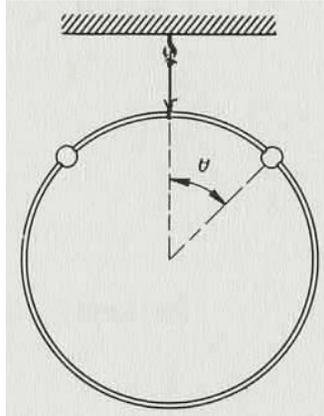
$$T_c = 2\pi(R_e / g)^{1/2}$$

e) In order to shorten the travel time, the sender travels down the hole a distance until she is only a fraction of the radius of the earth, βR_e , (R_e is the radius of the earth) from the center. The recipient travels down a similar distance from the far end in order to catch the object. By how much if any is the travel time thereby reduced (as a function of β)?

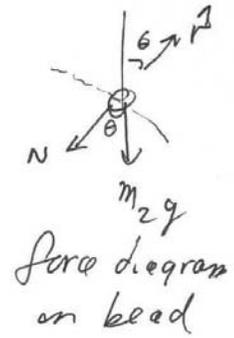
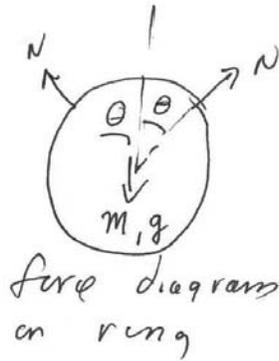
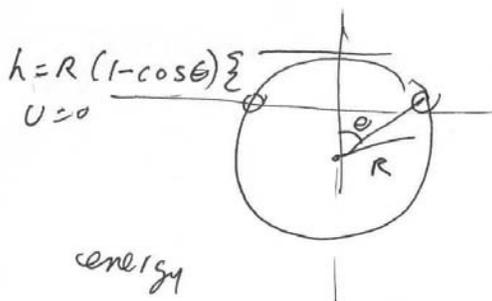
Answer: If the object starts closer to the earth the period is still the same because we have only changed the initial conditions we have not changes the 'spring constant' of gravitation.

Problem 23: Beads Sliding along a Ring

A ring of mass m_1 hangs from a thread, and two identical beads of mass m_2 slide on it without friction. The beads are released simultaneously from the top of the ring from rest (actually they need a very small initial speed but this can be ignored) and slide down opposite sides. Assume $m_2 > (3/2) m_1$. The ring will start to rise when the beads reach a critical angle θ_c with respect to the vertical.



- Draw free body force diagrams for the ring and the beads. What direction is the force of the bead on the ring pointing? Does it change as the bead moves? Can you still proceed with an analysis using Newton's Second Law if you are not sure which way this force points? Try to find a physical explanation for the direction of this force. What is the condition that the ring just starts to rise?
- Does the mechanical energy change between when the beads are released and the ring just starts to rise? Write down an equation that describes the changes in the mechanical energy.
- This is hard!** Can you show that the ring will start to rise if $m_2 > (3/2) m_1$? Find the angle θ_c with respect to the vertical direction that this occurs.



$$\vec{F}_1 = m_1 \vec{a}_1$$

$$2N \cos\theta - m_1 g = 0 \quad \Rightarrow \quad N = \frac{m_1 g}{2 \cos\theta} \quad (1)$$

condition that the ring just starts to rise

$$\vec{F}_2 = m_2 \vec{a}_2$$

$$-N - m_2 g \cos\theta = -\frac{m_2 v^2}{R} \quad \Rightarrow \quad N + m_2 g \cos\theta = \frac{m_2 v^2}{R} \quad (2)$$

$$E_0 = E_f \quad \text{for single bead}$$

$$R m_2 g (1 - \cos\theta) = \frac{1}{2} m_2 v^2 \quad (3)$$

substituting eq (1) into eq (2) yields

$$\frac{m_1 g}{2 \cos\theta} + m_2 g \cos\theta = \frac{m_2 v^2}{R} \quad (4)$$

solving eq (3) for $m_2 v^2$ and substituting into eq (4) yields

$$\frac{m_1 g}{2 \cos \theta} + m_2 g \cos \theta = 2 m_2 g (1 - \cos \theta) \quad (5)$$

This simplifies to

$$m_1 g + 2 m_2 g \cos^2 \theta = 4 m_2 g \cos \theta - 4 m_2 g \cos^2 \theta$$

or

$$m_1 g = 4 m_2 g \cos \theta - 6 m_2 g \cos^2 \theta$$

which is a quadratic eq for $\cos^2 \theta$

$$\cos^2 \theta - \frac{2}{3} \cos \theta + \frac{m_1}{6 m_2} = 0 \quad (6)$$

solving

$$\cos \theta = \frac{2}{3} \pm \left(\frac{4}{9} - \frac{2}{3} \frac{m_1}{m_2} \right)^{1/2} \quad (6a)$$

When $m_1 = 0$ choose ² pos. root to get

$$\cos \theta = \frac{2}{3}$$

Since $\left(\frac{4}{9} - \frac{2}{3} \frac{m_1}{m_2} \right) > 0$ or we would

have an imaginary root

$$\frac{2}{3} > \frac{m_1}{m_2}$$

$$m_2 > \frac{3}{2} m_1 \quad (7)$$