

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01T

Fall Term 2004

In-Class Problems 20-21: Work and Kinetic Energy Solutions

In-Class-Problem 20 *Calculating Work Integrals*

a) Work done by Gravity Near the Surface of the Earth: Consider an object of mass m near the surface of the earth falling directly towards the center of the earth. The gravitational force between the object and the earth is nearly constant. Suppose the object starts from an initial point y_0 and moves to a final point y_f closer to the earth. How much work does the gravitational force do on the object as it falls?

Solution: The displacement of the body is negative, $\Delta y \equiv y_f - y_0 < 0$. The gravitational force is given by

$$\vec{\mathbf{F}}_{grav} = m\vec{\mathbf{g}} = F_{grav,y}\hat{\mathbf{j}} = -mg\hat{\mathbf{j}}.$$

The work done on the body is then

$$W_{gravity} = F_{gravity,y}\Delta y = -mg\Delta y.$$

For a falling body, the displacement of the body is negative, $\Delta y \equiv y_f - y_0 < 0$; therefore the work done by gravity is positive,

$$W_{gravity} = F_{gravity,y}\Delta y = -mg\Delta y > 0.$$

The gravitational force is pointing in the same direction as the displacement of the falling object so the work should be positive.

When an object is rising while under the influence of a gravitational force, $\Delta y \equiv y_f - y_0 > 0$. The work done by the gravitational force for a rising body is negative,

$$W_{gravity} = F_{gravity,y}\Delta y = -mg\Delta y < 0,$$

since gravity is pointing in the opposite direction as the object is displaced.

b) Work Done by the Spring Force: Connect one end of a spring with spring constant k to an object resting on a smooth table and fix the other end of the spring to a wall. Stretch the spring a distance x_0 and release the object. How much work does the spring do on the object as a function of the stretched or compressed length of the object?

Solution: We first begin by choosing a coordinate system with origin at the position of the body when the spring is at rest in the equilibrium position. We choose the \hat{i} unit vector to point in the direction the body moves when the spring is being stretched. We choose the coordinate function $x(t)$ to denote the position of the body with respect to the origin (equilibrium position) at time t . In Figure 8.5.2 we illustrate the coordinate system by showing the equilibrium position and the position of the body at time t when it is stretched, $x(t)$. Note that at $t = t_0$, the position of the body is $x_0 \equiv x(t = t_0)$, and at time $t = t_f$ the position of the body is denoted by $x_f \equiv x(t = t_f)$. Note that x_0 and x_f can be positive, zero, or negative.

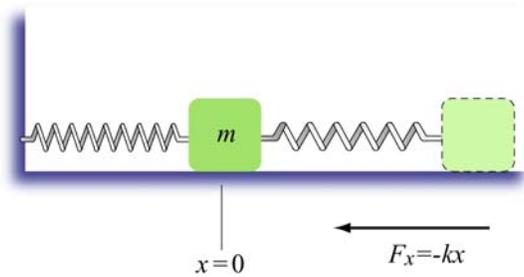


Figure 1 Equilibrium position and position at time t

The spring force on the body is given by

$$\vec{F} = F_x \hat{i} = -kx \hat{i}.$$

In Figure 8.5.3 we show the graph of the x -component of the spring force as a function of x for both positive values of x corresponding to stretching, and negative values of x corresponding to compressing of the spring.

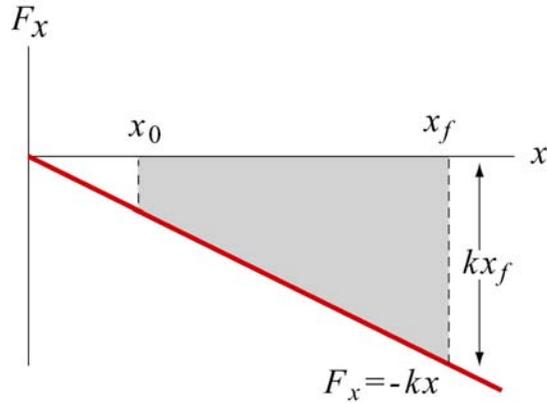


Figure 2 The x -component of the spring force as a function of x .

The work done is just the area under the curve for the interval x_0 to x_f ,

$$W = \int_{x=x_0}^{x=x_f} F_x dx = \int_{x=x_0}^{x=x_f} (-kx) dx .$$

This integral is straightforward and the work done by the spring force on the body is

$$W = \int_{x=x_0}^{x=x_f} (-kx) dx = -\frac{1}{2}k(x_f^2 - x_0^2) .$$

When the absolute value of the final distance is less than the absolute value of the initial distance, $|x_f| < |x_0|$, the work done is positive. This means that if the spring is less stretched or compressed in the final state than in the initial state, the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of ‘greater tension’ to a state of ‘lesser tension’. We shall see that the positive work done by the spring force decreases the *potential energy* stored in the spring.

c) Work done by the Inverse Square Gravitational Force: Consider an object of mass m moving directly towards the sun (mass m_s). Initially the object is at a distance r_0 from the center of the sun. The object moves to a final distance r_f from the center of the sun. How much work does the gravitational force between the sun and the object do on the object during this motion?

Solution: The gravitational force between the sun and the body is given by

$$\vec{F}_g = F_g \hat{r} = -\frac{GMm}{r^2} \hat{r}.$$

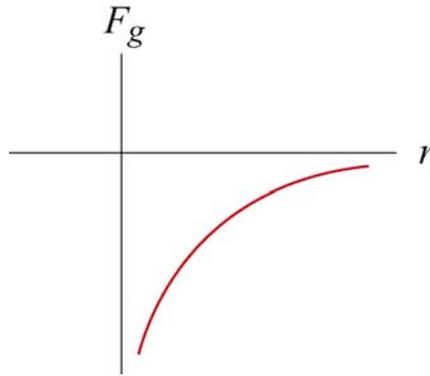


Figure 3 Graph of force between the sun and the body as a function of r

The work done by this gravitational force on the body is the area under the curve in Figure 8.5.4 and is given by the integral

$$W = \int_{r_0}^{r_f} F_g dr = \int_{r_0}^{r_f} \left(-\frac{GMm}{r^2} \right) dr.$$

Upon evaluation of this integral, we have for the work

$$W = \int_{r_0}^{r_f} \left(-\frac{GMm}{r^2} \right) dr = \frac{GMm}{r} \Big|_{r_0}^{r_f} = GMm \left(\frac{1}{r_f} - \frac{1}{r_0} \right).$$

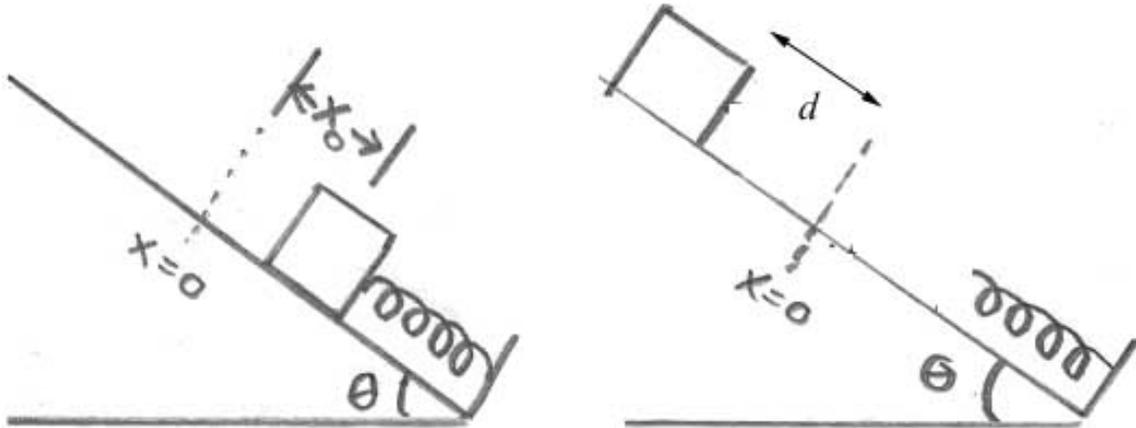
Since the body has moved closer to the sun, $r_f < r_0$, hence $1/r_f > 1/r_0$. Thus the work done by gravitational force between the sun and the body on the body is positive,

$$W = Gmm_s \left(\frac{1}{r_f} - \frac{1}{r_0} \right) > 0.$$

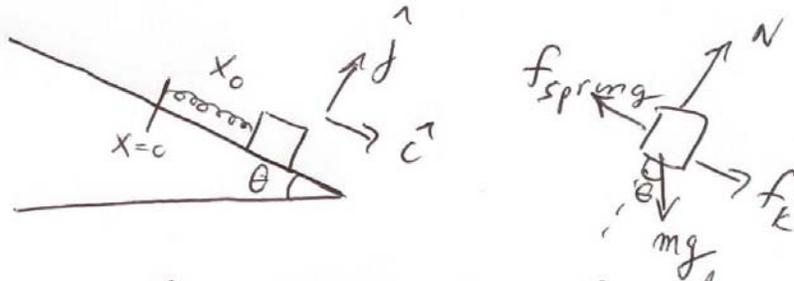
We expect this result because the gravitational force points along the direction of motion, so the work done by the force is positive. Also we expect that the sign of the work is the same for a body moving closer to the sun as a body falling towards the earth in a constant gravitational field. We shall see that the positive work done by the gravitational force will decrease the potential energy stored in the gravitational field.

In-Class Problem 21: Spring-Object on an Inclined Plane with Friction

A object of mass m is pushed against a spring at the bottom of a plane that is inclined at an angle θ with respect to the horizontal and held in place with a catch. The spring compresses a distance x_0 and has spring constant k . The catch is released and the object slides up the inclined plane. At $x = 0$ the object detaches from the spring and continues to slide up the inclined plane.



Assume that the incline plane has a coefficient of kinetic friction μ_k . How far up the inclined plane does the object move from the point where the object detaches from the spring?



Stage 1: The spring is released and the object moves back to equilibrium

$$\Delta K = \int \vec{F}^{total} \cdot d\vec{r}$$

$$\frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 = \int_{x_0}^{x=0} -kx dx + \int_{x_0}^{x=0} \mu_k N dx + \int_{x_0}^{x=0} mgsin\theta dx$$

$$\text{since } (\vec{F}^{total})_y = -kx \hat{i} + \mu_k N \hat{i} + mgsin\theta \hat{i}$$

note that $F_y = ma_y = 0 \Rightarrow$
 $N - mg \cos\theta = 0$ or $N = mg \cos\theta$

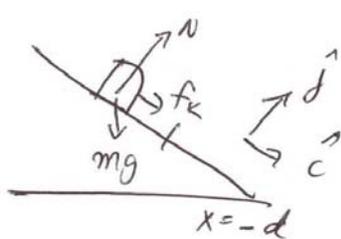
The object is released at rest $v_0 = 0$

$$\text{thus } \frac{1}{2} m v_1^2 = -\int_{x_0}^{x=0} kx dx + \int_{x_0}^{x=0} \mu_k mg \cos\theta dx + \int_{x_0}^{x=0} mgsin\theta dx$$

The three integrals yield

$$\frac{1}{2} m v_1^2 = \frac{1}{2} k x_0^2 - \mu_k m g \cos \theta x_0 - m g \sin \theta x_0 \quad (1)$$

stage 2: object slides up inclined plane until it comes to rest



$$\Delta K = \int_{x=0}^{x=-d} \vec{F} \cdot d\vec{r}$$

$$\Delta K = + \int_0^d \mu_k m g \cos \theta d\theta + \int_0^d m g \sin \theta d\theta$$

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_1^2 = (-\mu_k m g \cos \theta - m g \sin \theta) d$$

$$v_f = 0 \quad \text{so}$$

$$\frac{1}{2} m v_1^2 = (\mu_k m g \cos \theta + m g \sin \theta) d \quad (2)$$

Using eq (1) for $\frac{1}{2} m v_1^2$ yields

$$\frac{1}{2} k x_0^2 - (\mu_k m g \cos \theta + m g \sin \theta) x_0 = (\mu_k m g \cos \theta + m g \sin \theta) d$$

solve for $d = \frac{\frac{1}{2} k x_0^2}{(\mu_k m g \cos \theta + m g \sin \theta)} - x_0$