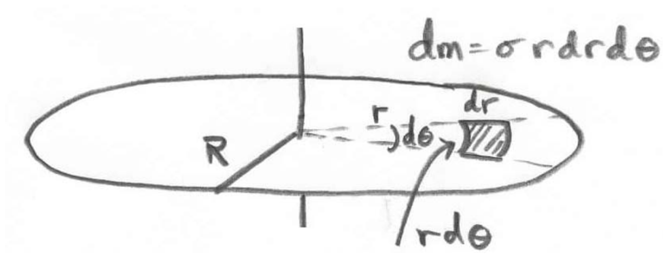
Two Dimensional Rotational Kinematics
Challenge Problem Solutions

Problem 1: Moment of Inertia: *Uniform Disc*

A thin uniform disc of mass M and radius R is mounted on an axis passing through the center of the disc, perpendicular to the plane of the disc. In this problem, you will calculate the moment of inertia about two different axes that pass perpendicular to the disc. One passes through the center of mass of the disc and the second passes through a point on the rim of the disc a distance R from the center.

As a starting point, consider the contribution to the moment of inertia from the mass element dm show in the figure below.



Problem 1 Solution:

Axis through center: Take the point S to be the center of mass of the disc. The axis of rotation passes through the center of the disc, perpendicular to the plane of the disc. Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r\,dr\,d\theta \tag{1.1}$$

can be thought of as the product of arc length $r\,d\theta$ and the radial width dr. Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}. \tag{1.2}$$

Therefore the mass in the infinitesimal area element as given in Equation (1.1), a distance r from the axis of rotation, is given by

$$dm = \sigma\,r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta. \tag{1.3}$$

When the disc rotates, the mass element traces out a circle of radius $r_\perp^2 = r^2$; that is, the distance from the center is the perpendicular distance from the axis.

The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus the limits of the integral are

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2 \, dm = \frac{M}{\pi R^2} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} r^3 \, d\theta \, dr \,. \qquad \textbf{(1.4)}$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2} \int_{r=0}^{r=R} \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) r^3 dr = \frac{M}{\pi R^2} \int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2} \int_{r=0}^{r=R} r^3 dr \qquad \textbf{(1.5)}$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2} \int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2} \frac{r^4}{4} \bigg|_{r=0}^{r=R} = \frac{2M}{R^2} \frac{R^4}{4} = \frac{1}{2} M R^2 \,. \qquad \textbf{(1.6)}$$

Remark: Instead of taking the area element as a small patch $da = r\,dr\,d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}} = \pi (r + dr)^2 - \pi r^2 = \pi r^2 + 2\pi r \, dr + \pi (dr)^2 - \pi r^2 = 2\pi r \, dr + \pi (dr)^2 \,. \qquad \textbf{(1.7)}$$

In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}} = r \, dr \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) = 2\pi r \, dr \,. \qquad \textbf{(1.8)}$$

Then the mass element is

$$dm_{\text{ring}} = \sigma da_{\text{ring}} = \frac{M}{\pi R^2} 2\pi r \, dr \,. \qquad \textbf{(1.9)}$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2 \, dm = \frac{2\pi M}{\pi R^2} \int_{r=0}^{r=R} r^3 dr = \frac{1}{2} M R^2 \,. \qquad \textbf{(1.10)}$$

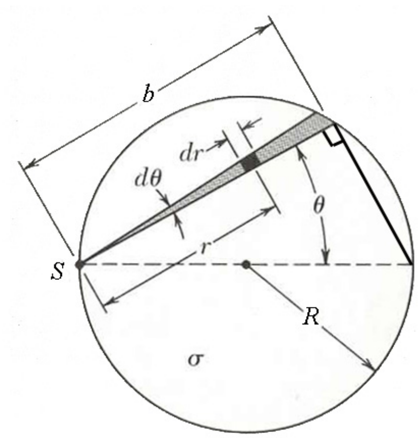
Axis at Rim – Parallel Axis Theorem: Use the parallel axis theorem to find the moment of inertia about the rim,

$$I_S = I_{\text{cm}} + M\, d_{S,\text{cm}}^2 \,. \tag{1.11}$$

The distance from the center of mass to the rim is the radius of the turntable, $d_{S,\text{cm}} = R$, so the moment of inertia about a parallel axis passing through a point on the rim is

$$I_S = \frac{1}{2} MR^2 + M\, R_{S,\text{cm}}^2 = \frac{3}{2} MR^2 \,. \tag{1.12}$$

Axis on Rim – Direct Integration: If we want to find the moment of inertia about the point S by direct integration, we can choose coordinates shown in the figure below.



The mass element is still

$$dm = \sigma\, r\, dr\, d\theta = \frac{M}{\pi R^2} r\, dr\, d\theta \,. \tag{1.13}$$

Note that the variables r and θ are not the same as in the previous integrations. The integral for the moment of inertia becomes

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2\, dm = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=b(\theta)} r^3 dr\, d\theta \,. \tag{1.14}$$

We can perform the integration in the radial direction first ,yielding

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2 \, dm = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{b^4}{4} d\theta \, . \qquad \textbf{(1.15)}$$

However, the variable b is a non-constant function of θ. The triangle formed by the diameter of the circle and two chords is a right triangle; thus

$$b(\theta) = 2R\cos\theta \qquad \textbf{(1.16)}$$

and the integral in Equation (1.15) becomes

$$I_{cm} = \frac{M}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{(2R\cos\theta)^4}{4} d\theta \, . \qquad \textbf{(1.17)}$$

Using the known result (not hard to derive)

$$\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^4\theta d\theta = \frac{3\pi}{8} \, , \qquad \textbf{(1.18)}$$

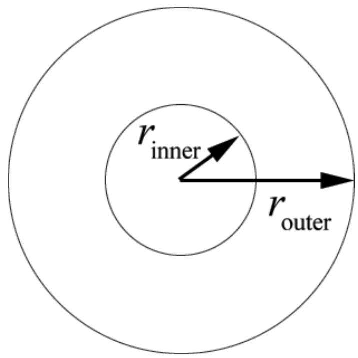
the moment of inertia about the point S is found to be

$$I_{\text{cm}} = \frac{4MR^4}{\pi R^2} \int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^4\theta d\theta = \frac{4MR^4}{\pi R^2} \frac{3\pi}{8} = \frac{3}{2} MR^2 \, , \qquad \textbf{(1.19)}$$

the same result found from the parallel axis theorem.

Problem 2: Rotational Dynamics: ***Moment of Inertia of a Washer***

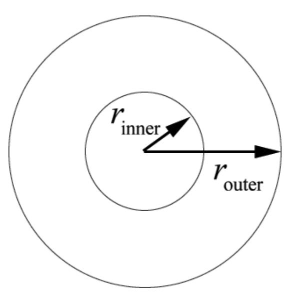
A 1" US Standard Washer has inner radius $r_{\text{inner}} = 13.5\text{ mm}$ and an outer radius $r_{\text{outer}} = 31.0\text{ mm}$. The washer is approximately 4.0 mm thick. The density of the washer is $\rho = 7.8\times10^{3}\text{ kg}\cdot\text{m}^{-3}$. Calculate the moment of inertia of the washer about an axis that is perpendicular to the plane of the washer and passes through its center of mass.



Problem 2 Solution:

Consider a small area element shown in the figure below.

A 1" US Standard Washer has inner radius $r_{\text{inner}} = 13.5\text{ mm}$ and an outer radius $r_{\text{outer}} = 31.0\text{ mm}$. The washer is approximately $d = 4.0\text{ mm}$ thick. The density of the washer is $\rho = 7.8\times10^{3}\text{ kg}\cdot\text{m}^{-3}$. Calculate the moment of inertia of the washer about an axis that is perpendicular to the plane of the washer and passes through its center of mass.



The moment of inertia about an axis is given by the integral formula

$$I_{\text{cm}} = \int_{\text{body}} (r_{\perp})^2\, dm. \qquad \textbf{(2.1)}$$

The axis of rotation passes through the center of the disc, perpendicular to the plane of the disc. Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The volume element

$$dV = r\,dr\,d\theta dz \qquad \textbf{(2.2)}$$

can be thought of as the product of arc length $r\,d\theta$, radial width dr and thickness dz. Since the washer is uniform, the mass per unit volume is a constant,

$$\rho = \frac{dm}{dV} = \frac{m_{\text{total}}}{\text{Volume}} = \frac{M}{\pi(r_{outer}{}^2 - r_{inner}{}^2)d}\,. \qquad \textbf{(2.3)}$$

where the mass M of the washer is

$$\begin{aligned} M &= \rho\pi(r_{outer}{}^2 - r_{inner}{}^2)d \\ &= (7.8\times10^3\ \text{kg}\cdot\text{m}^{-3})(\pi)((31.0\times10^{-3}\text{m})^2 - (13.5\times10^{-3}\text{m})^2)(4.0\times10^{-3}\text{m})\,. \\ &= 7.6\times10^{-2}\ \text{kg} \end{aligned} \qquad \textbf{(2.4)}$$

The mass in the infinitesimal volume element is given by

$$dm = \rho\, r\,dr\,d\theta dz = \frac{M}{\pi(r_{outer}{}^2 - r_{inner}{}^2)d}\, r\,dr\,d\theta dz\,. \qquad \textbf{(2.5)}$$

The perpendicular distance from the mass element to the axis passing through the center is $r_\perp^2 = r^2$; The moment of inertia integral is now an integral in three dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, the radial coordinate r varies from $r = r_{inner}$ to $r = r_{outer}$, and the z coordinate varies from $z = 0$ to $z = d$. Thus the limits of the integral are

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2\, dm = \frac{M}{\pi(r_{outer}{}^2 - r_{inner}{}^2)d} \int_{r=r_{inner}}^{r=r_{outer}} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=d} r^3\, d\theta\, drdz\,. \qquad \textbf{(2.6)}$$

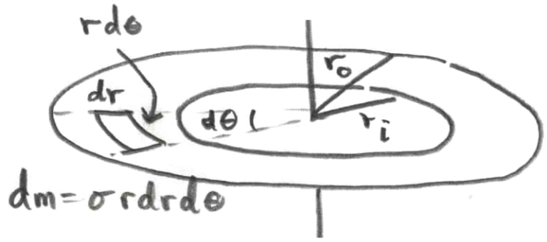
The integral can now be explicitly calculated by first integrating the z- coordinate

$$I_{\text{cm}} = \int_{\text{body}} (r_\perp)^2\, dm = \frac{M}{\pi(r_{outer}{}^2 - r_{inner}{}^2)} \int_{r=r_{inner}}^{r=r_{outer}} \int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr\,. \qquad \textbf{(2.7)}$$

If we had ignored the thickness of the washer then we could begin by choosing an infinitesimal area element with mass

$$dm = \sigma\, r\, dr\, d\theta = \frac{M}{\pi(r_{outer}^{\;2} - r_{inner}^{\;2})} r\, dr\, d\theta\, . \tag{2.8}$$

The perpendicular distance from the mass element to the axis passing through the center remains the same $r_\perp^2 = r^2$ and so the expression for the moment of inertia is given by Eq. (2.7).



We now continue the integration with the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2} \int_{r=r_{inner}}^{r=r_{outer}} \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) r^3 dr = \frac{2M}{(r_{outer}^{\;2} - r_{inner}^{\;2})} \int_{r=r_{inner}}^{r_{outer}} r^3 dr \tag{2.9}$$

and finally integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{(r_{outer}^{\;2} - r_{inner}^{\;2})} \int_{r=0}^{r=R} r^3 dr = \frac{2M}{\pi(r_{outer}^{\;2} - r_{inner}^{\;2})} \frac{r^4}{4} \Bigg|_{r=r_{inner}}^{r=r_{outer}} = \frac{2M}{(r_{outer}^{\;2} - r_{inner}^{\;2})} \frac{(r_{outer}^{\;4} - r_{inner}^{\;4})}{4}$$

$$= \frac{M(r_{outer}^{\;2} + r_{inner}^{\;2})(r_{outer}^{\;2} - r_{inner}^{\;2})}{2(r_{outer}^{\;2} - r_{inner}^{\;2})} = \frac{M(r_{outer}^{\;2} + r_{inner}^{\;2})}{2} \tag{2.10}$$

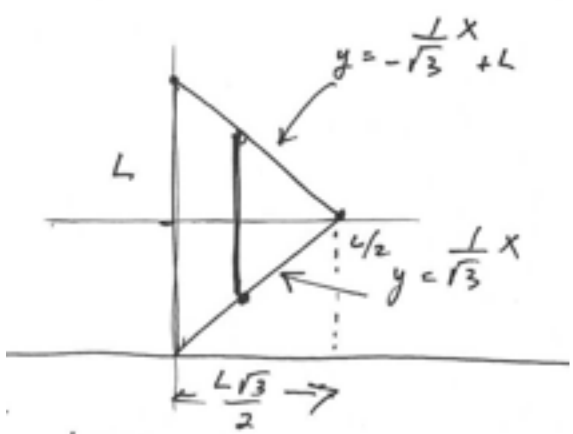
$$= \frac{1}{2}(7.6\times10^{-2}\text{ kg})((31.0\times10^{-3}\text{m})^2 + (13.5\times10^{-3}\text{m})^2) = 4.4\times10^{-5}\text{ kg}\cdot\text{m}^2$$

Problem 3: Moment of Inertia of a Sheet

a) Find the moment of inertia of a thin sheet of metal of mass m in the shape of an isosceles right triangle about an axis that passes through one vertex of the sheet, perpendicular to the plane of the sheet. The length of the two equal sides is s.

b) Find the moment of inertia of a thin sheet of metal of mass m in the shape of an isosceles right equilateral triangle about an axis that passes through the same vertex of the sheet, but aligned along one side of length s (in the plane of the sheet).

Problem 3 Solutions



$$\text{slope} = \frac{\frac{L}{2}}{\frac{L\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\int_{x=0}^{\frac{L\sqrt{3}}{2}} \int_{y=\frac{1}{\sqrt{3}}x}^{-\frac{1}{\sqrt{3}}x+L} \sigma\, dx\, dy\left(x^2+y^2\right)$$

$$= \sigma \int_0^{L\sqrt{3}/2} dx\left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{\frac{1}{\sqrt{3}}x}^{-\frac{1}{\sqrt{3}}x+L}$$

$$= \sigma \int dx\left(x^2\left(-\frac{1}{\sqrt{3}}x + L - \frac{1}{\sqrt{3}}x\right) + \left(-\frac{1}{\sqrt{3}}x + L\right)^3 - \frac{1}{\sqrt{3}}x^3\right)$$

$$= \sigma \int dx\left(\frac{2}{\sqrt{3}}x^3 + Lx^2\right) + \left(-\frac{2}{3\sqrt{3}}x^3 + x^2 L - \frac{3}{\sqrt{3}}L^2 x + L^3\right)$$

$$= \sigma \int dx\left(\frac{-8}{3\sqrt{3}x^3} + 2Lx^2 - \frac{3}{\sqrt{3}}L^2 x + L^3\right)$$

$$= \sigma\left(-\frac{8}{3\sqrt{3}}\frac{x^4}{4} + \frac{2Lx^3}{3} - \frac{3}{\sqrt{3}}\frac{L^2x^2}{2} + L^3 x\right)\Bigg|_0^{\frac{L\sqrt{3}}{2}}$$

$$= \sigma\left(-\frac{8}{3\sqrt{3}}\frac{\left(\frac{L\sqrt{3}}{2}\right)^4}{4} + \frac{2L}{3}\left(\frac{L\sqrt{3}}{2}\right)^3 - \frac{3}{\sqrt{3}}L^2\left(\frac{\frac{L\sqrt{3}}{2}}{2}\right)^2 + L^3\frac{L\sqrt{3}}{2}\right)$$

$$= \sigma\left(-\frac{8\sqrt{3}}{3}\frac{L^4}{64} + \frac{2}{3}L\frac{L^3 3\sqrt{3}}{8} - \sqrt{3}L^2\frac{L^2 3}{8} + \frac{L^4\sqrt{3}}{2}\right)$$

$$= \sigma\frac{L^4\sqrt{3}}{3}\left(-\frac{8}{64} + \frac{6}{8} - \frac{9}{8} + \frac{3}{2}\right)$$

$$= \sigma\frac{L^4\sqrt{3}}{3}(1)$$

Since $\sigma = \dfrac{4m}{L^2\sqrt{3}}$

$$I_{p,z} = \left(\frac{4m}{L^2\sqrt{3}}\right)\left(\frac{L^4\sqrt{3}}{3}\right)$$

$$= \frac{4}{3}mL^2$$

Problem 4:

A turntable is a uniform disc of mass 1.2 kg and radius 1.3×10^{-1} m. The turntable is spinning at a constant rate of $f_0 = 0.5$ Hz. The motor is turned off and the turntable slows to a stop in 8.0 s with constant angular deceleration.

a) What is the moment of inertia of the turntable?

b) What is the initial rotational kinetic energy?

c) What is the angular deceleration of the turntable while it is slowing down?

d) What is the total angle in radians that the turntable spins while slowing down?

Problem 4 Solutions:

a) The moment of inertia of the turntable about an axis passing perpendicular to the disc and through the center of mass is

$$I_{\text{cm}} = \frac{1}{2}MR^2 = \frac{1}{2}(1.2\text{ kg})(1.3\times10^{-1}\text{m})^2 = 1.01\times10^{-2}\text{ kg}\cdot\text{m}^2. \qquad \textbf{(4.1)}$$

b) Initially, the disc is spinning with a frequency $f_0 = 0.5$ Hz, so the initial angular velocity is

$$\omega_0 = 2\pi f_0 = \left(2\pi\frac{\text{radian}}{\text{cycle}}\right)\left(0.5\frac{\text{cycles}}{\text{s}}\right) = \pi\text{ rad}\cdot\text{s}^{-1}. \qquad \textbf{(4.2)}$$

The initial rotational kinetic energy is

$$K_{\text{rot}} = \frac{1}{2}I_{\text{cm}}\omega_0^2 = \frac{1}{2}(1.01\times10^{-2}\text{ kg}\cdot\text{m}^2)(\pi\text{ rad}\cdot\text{s}^{-1})^2 = 5.0\times10^{-2}\text{ J}. \qquad \textbf{(4.3)}$$

c) The final angular velocity is zero, so the angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-\pi\text{ rad}\cdot\text{s}^{-1}}{8.0\text{ s}} = -3.9\times10^{-1}\text{ rad}\cdot\text{s}^{-2}. \qquad \textbf{(4.4)}$$

Because the angular acceleration is negative, the disc is slowing down.

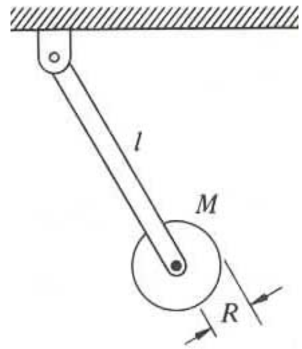
d) The disc turns through angle

$$\Delta\theta = \omega_0 \,\Delta t + \frac{1}{2}\alpha\,\Delta t^2 = (\pi \text{ rad}\cdot\text{s}^{-1})(8.0\text{ s}) + \frac{1}{2}(-3.9\times10^{-1}\text{ rad}\cdot\text{s}^{-2})(8.0\text{ s})^2$$

$$= 12.7\text{ rad} \qquad \textbf{(4.5)}$$

Problem 5:

A physical pendulum consists of a disc of radius R and mass m_d fixed at the end of a rod of mass m_r and length l.



a) What is the moment of inertia about the pivot point P?

b) How does the moment of inertia about the pivot point P change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin?

Problem 5 Solutions:

a) The physical pendulum consists of two pieces. A uniform rod of length d and a disk attached at the end of the rod. The moment of inertia about the pivot point P is the sum of the moments of inertia of the two pieces,

$$I_P = I_{P,rod} + I_{P,disk} \tag{5.1}$$

We calculated the moment of inertia of a rod about the end point P in class, and found that

$$I_{P,rod} = \frac{1}{3} m_r l^2 \tag{5.2}$$

We can use the parallel axis theorem to calculate the moment of inertia of the disk about the pivot point P,

$$I_{P,disk} = I_{cm,disk} + m_d l^2 \tag{5.3}$$

We calculated the moment of inertia of a disk about the center of mass in class, and found that

$$I_{cm,disk} = \frac{1}{2} m_d R^2 \quad \textbf{(5.4)}$$

So the total moment of inertia is

$$I_P = \frac{1}{3} m_r l^2 + m_d g l^2 + \frac{1}{2} m_d R^2 \quad \textbf{(5.5)}$$

b) If the disk is not fixed to the rod, then it will not rotate as the pendulum oscillates. Therefore it does not contribute to the moment of inertia. Notice that the pendulum is no longer a rigid body. So the total moment of inertia is only due to the rod and the disk treated as a point like object.

$$I_P = \frac{1}{3} m_r l^2 + m_d l^2 \quad \textbf{(5.6)}$$