Two-Dimensional Rotational Dynamics

Recall: Fixed Axis Rotation Kinematics

Angle variable

$$\theta$$

Angular velocity

$$\omega \equiv d\theta / dt$$

Angular acceleration

$$\alpha \equiv d^2\theta / dt^2$$

Mass element

$$\Delta m_i$$

Radius of orbit

$$r_{\perp,i}$$

Moment of inertia

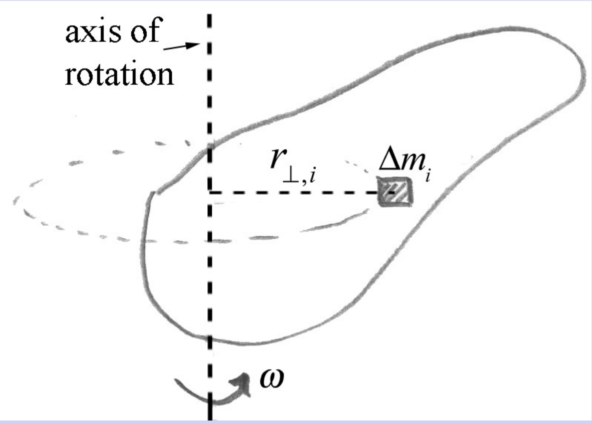
$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \rightarrow \int_{body} dm (r_\perp)^2$$

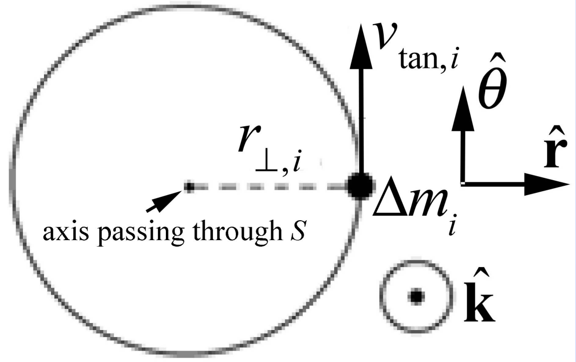
Parallel Axis Theorem

$$I_S = Md^2 + I_{cm}$$

Z-component of Angular Momentum

$$L_{S,z} = I_S \omega$$





Main Idea: Fixed Axis Rotation of Rigid Body

Torque produces angular acceleration about center of mass

$$\boldsymbol{\tau}_{\mathrm{cm}}^{\mathrm{total}} = I_{\mathrm{cm}}\alpha_{\mathrm{cm}}$$

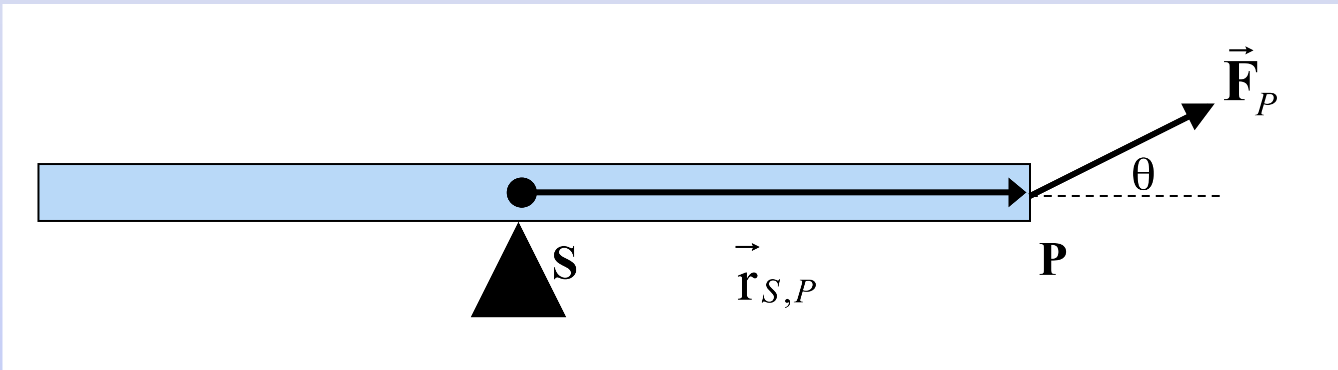
I_{cm} is the moment of inertial about the center of mass

α_{cm} is the angular acceleration about center of mass

Torque as a Vector

Force $\vec{\mathbf{F}}_P$ exerted at a point P on a rigid body.

Vector $\vec{\mathbf{r}}_{S,P}$ from a point *S* to the point *P.*



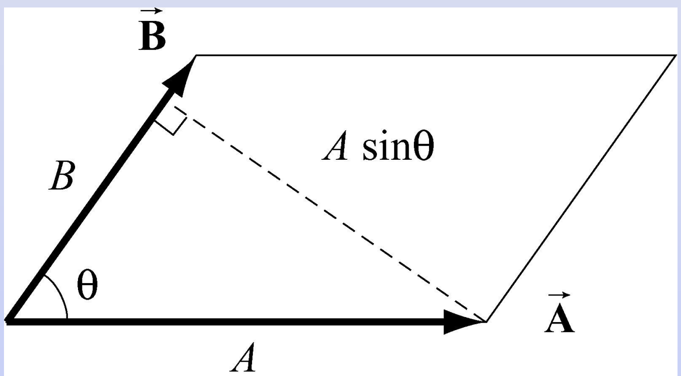
Torque about point S due to the force exerted at point P:

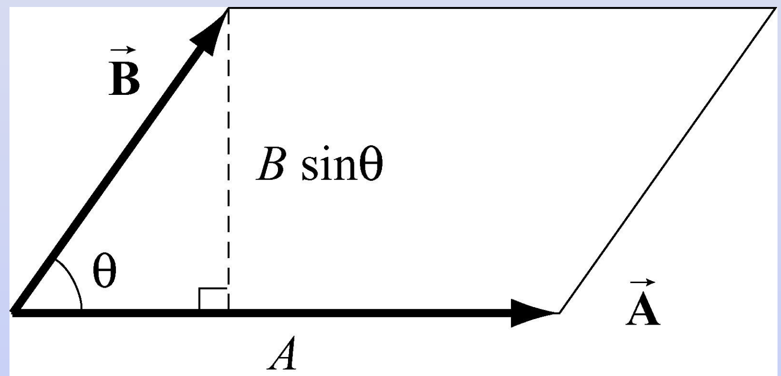
$$\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P$$

Summary: Cross Product

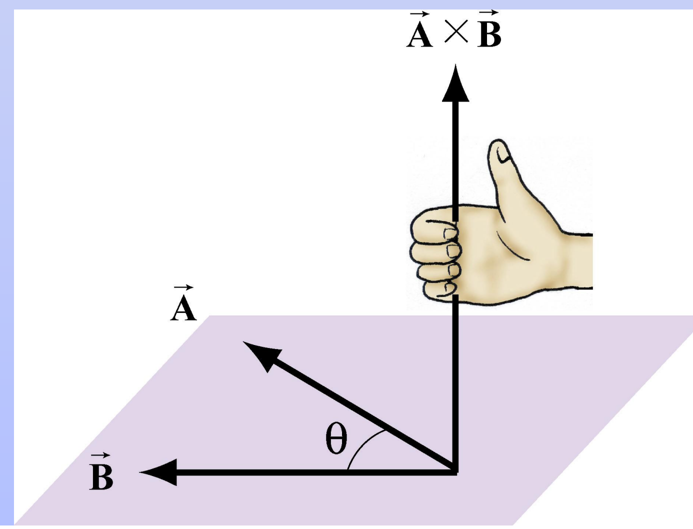
Magnitude: equal to the area of the parallelogram defined by the two vectors

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right| \left|\vec{\mathbf{B}}\right| \sin\theta = \left|\vec{\mathbf{A}}\right| \left(\left|\vec{\mathbf{B}}\right| \sin\theta\right) = \left(\left|\vec{\mathbf{A}}\right| \sin\theta\right) \left|\vec{\mathbf{B}}\right| \quad (0 \le \theta \le \pi)$$





Direction: determined by the Right-Hand-Rule



Properties of Cross Products

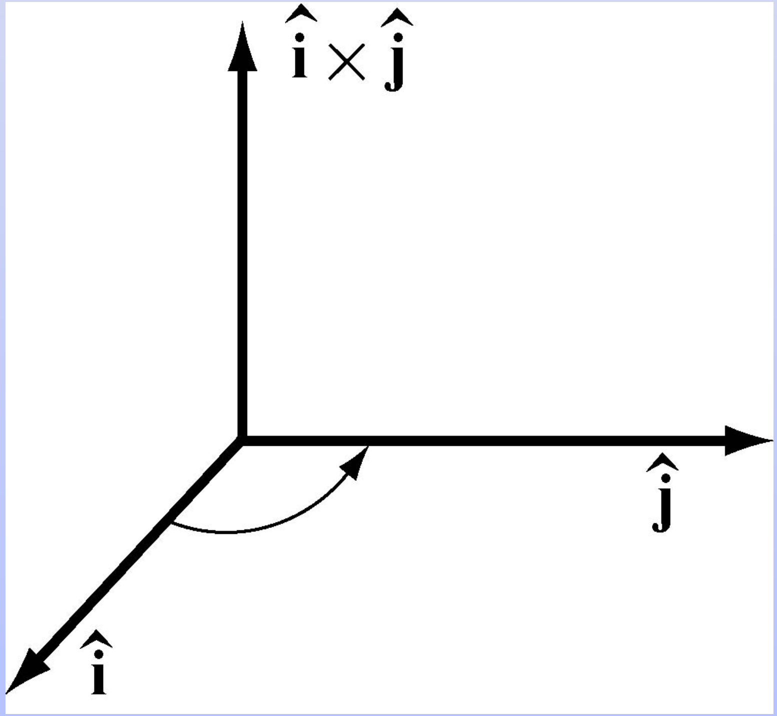
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c\vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}}$$

Cross Product of Unit Vectors

Unit vectors in Cartesian coordinates



$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \sin\left(\pi/2\right) = 1$$

$$\left|\hat{\mathbf{i}} \times \hat{\mathbf{i}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \sin(0) = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Components of Cross Product

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

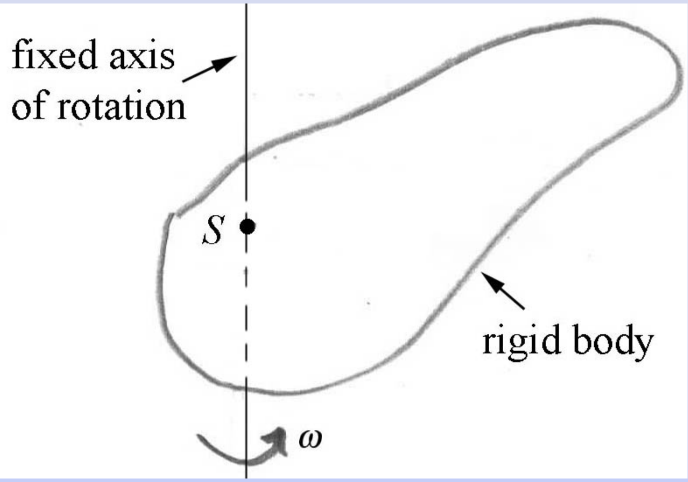
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Checkpoint Problem: Torque

Consider two vectors $\vec{\mathbf{r}} = x\hat{\mathbf{i}}$ with $x > 0$ and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$. What is the direction of the cross product $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$?

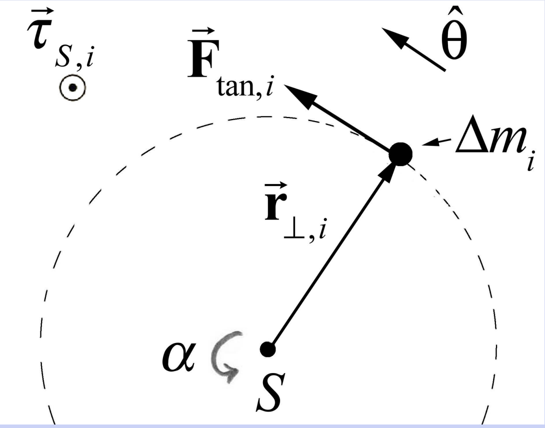
Recall: Rotational Kinematics

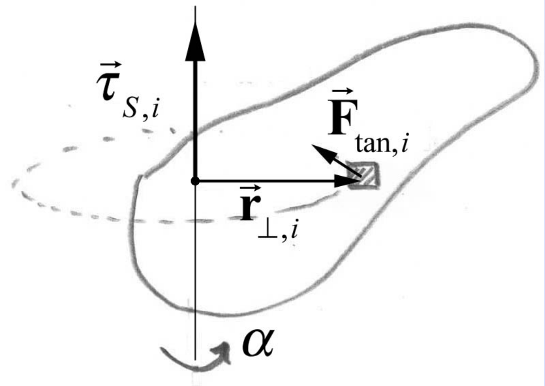
- Individual element of mass Δm_i
- Radius of orbit $r_{\perp,i}$
- Tangential velocity $v_{\tan,i} = r_{\perp,i}\omega$
- Tangential acceleration $a_{\tan,i} = r_{\perp,i}\alpha$
- Radial Acceleration $a_{\mathrm{rad},i} = \dfrac{v_{\tan,i}^2}{r_{\perp,i}} = r_{\perp,i}\omega^2$



Dynamics: Newton's Second Law and Torque about *S*

Tangential force on mass element produces torque





Newton's Second Law

$$\vec{\mathbf{F}}_{\tan,i} = F_{\tan,i}\hat{\theta} = \Delta m_i a_{\tan,i}\,\hat{\theta}$$

$$F_{\tan,i} = \Delta m_i r_{\perp,i}\alpha$$

Torque about S

$$\vec{\tau}_{S,i} = \vec{\mathbf{r}}_{\perp,i} \times \vec{\mathbf{F}}_{\tan,i}$$

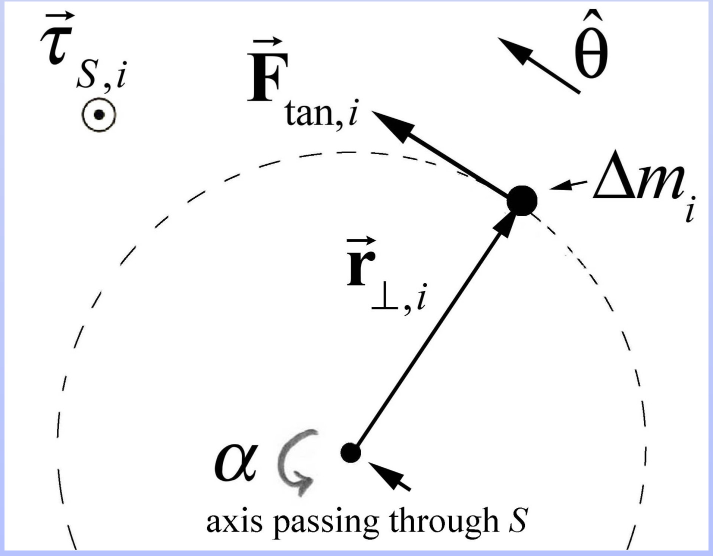
z-component of torque about S

$$(\tau_{z,S})_i = r_{\perp,i}F_{\tan,i} = \Delta m_i (r_{\perp,i})^2\alpha$$

Moment of Inertia and Torque

Component of the total torque about an axis passing through S is the sum over all elements

$$\tau_{z,S}^{\text{total}} = (\tau_{z,S})_1 + (\tau_{z,S})_2 + \cdots = \sum_{i=1}^{i=N} (\tau_{z,S})_i = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \alpha$$



Recall: Moment of Inertia about and axis passing through S :

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2$$

Summary:

$$\boxed{\tau_{z,S}^{\text{total}} = I_S \alpha}$$

Torque due to Uniform Gravitational Force

The total torque on a rigid body due to the gravitational force can be determined by placing all the gravitational force at the center-of-mass of the object.

$$\vec{\boldsymbol{\tau}}_{\mathrm{S,grav}} = \sum_{i=1}^{N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{\mathrm{grav},i} = \sum_{i=1}^{N} \vec{\mathbf{r}}_{S,i} \times m_i \vec{\mathbf{g}} = \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{g}}$$

$$= \left(\frac{1}{m^{\mathrm{totat}}} \sum_{i=1}^{N} m_i \vec{\mathbf{r}}_{S,i} \right) \times m^{\mathrm{totat}} \vec{\mathbf{g}}$$

$$= \vec{\mathbf{R}}_{\mathrm{S,cm}} \times m^{\mathrm{totat}} \vec{\mathbf{g}}$$

Problem Solving Strategy: Two Dimensional Rotation

Step 1: Draw free body force diagrams for each object and indicate the point of application of each force

Step 2: Select point to compute torque about (generally select center of mass)

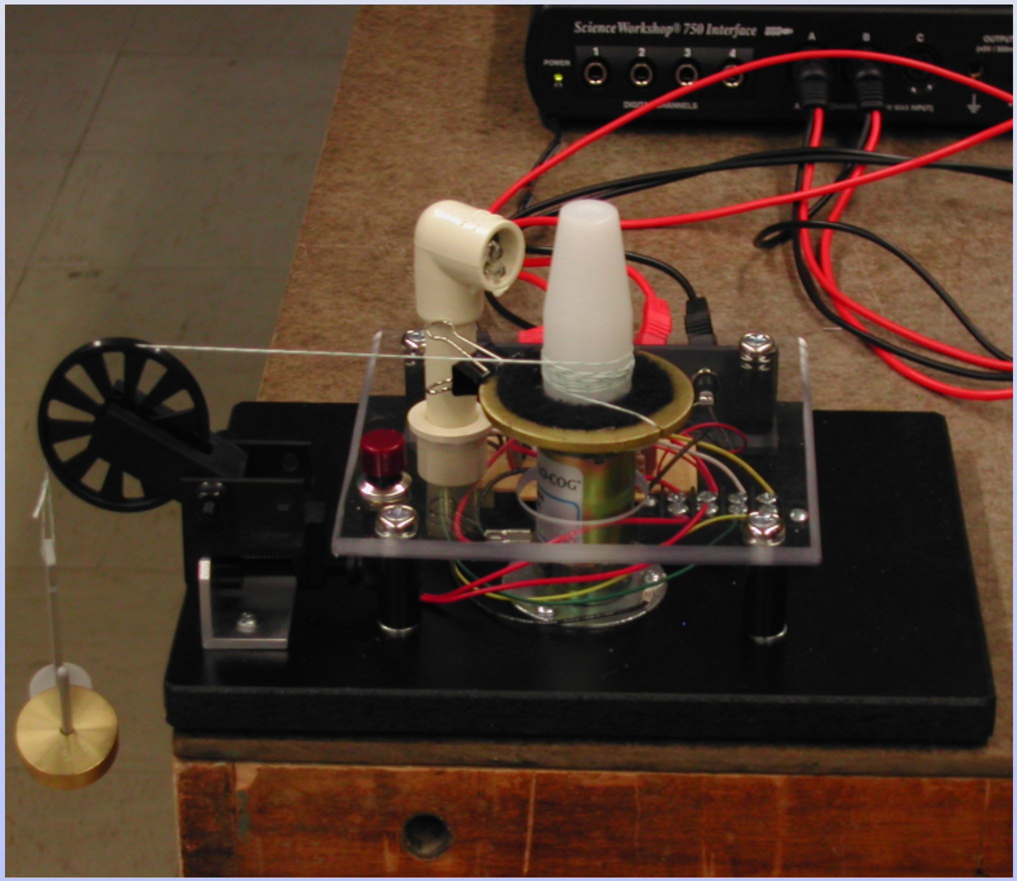
Step 3: Choose coordinate system. Indicate positive direction for increasing rotational angle.

Step 4: Apply Newton's Second Law and Torque Law to obtain equations

Step 5: Look for constraint condition between rotational acceleration and any linear accelerations.

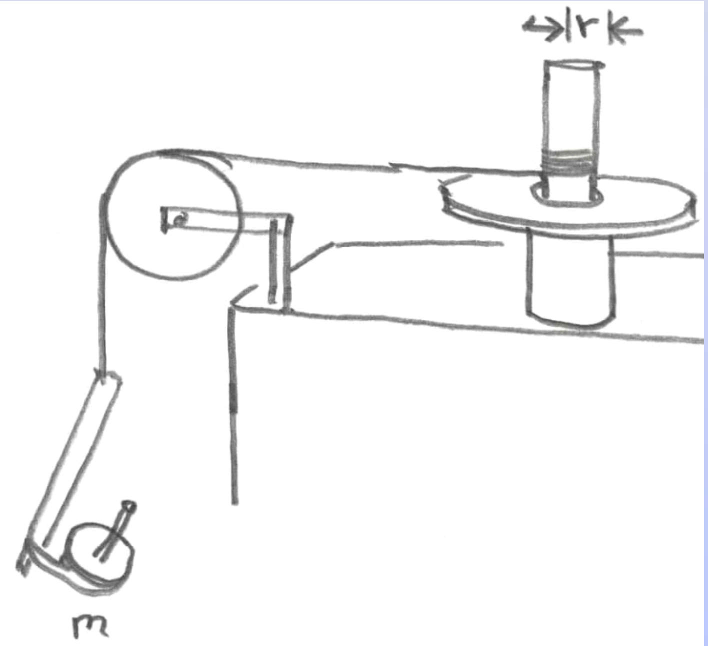
Step 6: Design algebraic strategy to find quantities of interest

Rotor Moment of Inertia



Worked Example: Moment of Inertia Wheel

A steel washer is mounted on a cylindrical rotor of radius *r*. A massless string, with an object of mass *m* attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley. Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the mass falls, the rotor undergoes an angular acceleration of magnitude α_1. After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude α_2. Let *g* denote the gravitational constant.



What is the moment of inertia of the rotor assembly (including the washer) about the rotation axis?

Worked Example Solution: moment of inertia of rotor

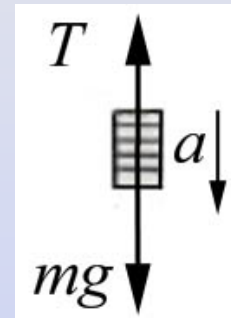
Force and rotational equations while weight is descending:

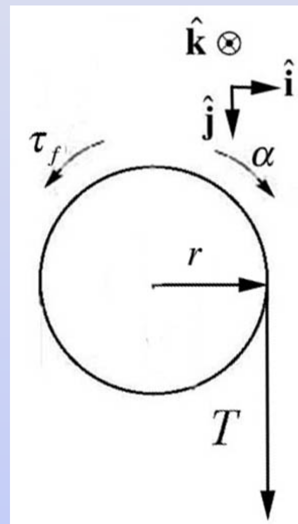
$$mg - T = ma_1$$

$$rT - \tau_f = I_R \alpha_1$$

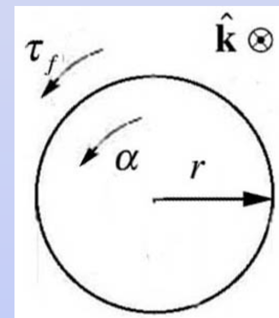
Constraint: $a_1 = r\alpha_1$

Rotational equation while slowing down $-\tau_f = I_R \alpha_2$





Speeding up



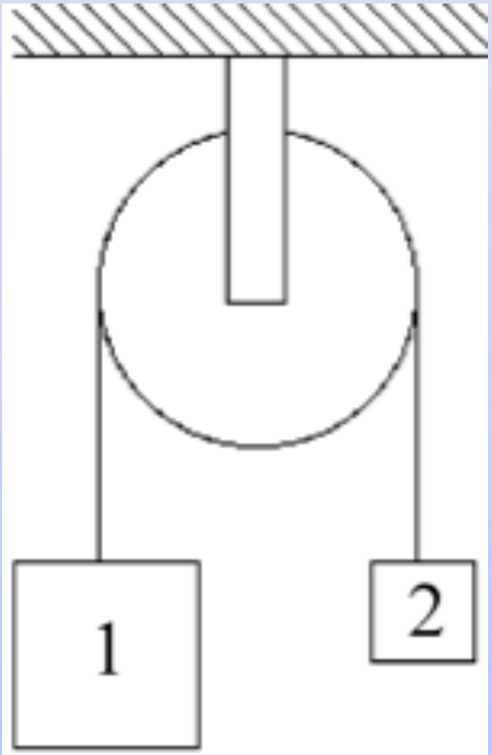
Slowing down

Solve for moment of inertia: $rm(g - r\alpha_1) + I_R\alpha_2 = I_R\alpha_1$

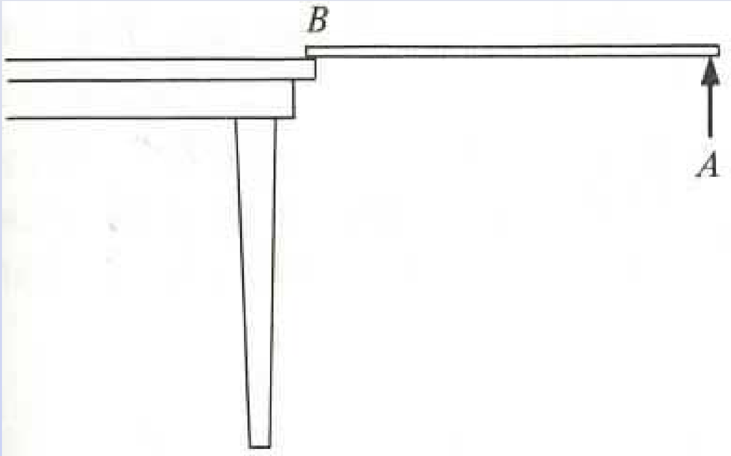
$$I_R = \frac{rm(g - r\alpha_1)}{(\alpha_1 - \alpha_2)}$$

Checkpoint Problem: Atwood's Machine

A pulley of mass m_{p}, radius R, and moment of inertia I_{cm} about the center of mass, is suspended from a ceiling. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to an object of mass m_1 and on the other end to an object of mass m_2 , with $m_1 > m_2$. At time $t = 0$, the objects are released from rest. Find the magnitude of the acceleration of the objects.



Checkpoint Problem: Falling Stick



A uniform stick of mass *m* and length *l* is suspended horizontally with end A at the edge of a table and the other end B is held by hand. End A is suddenly released. At the instant after release:

a) What is the torque about the end B on the table?

b) What is the angular acceleration about the end B on the table?

c) What is the vertical acceleration of the center of mass?

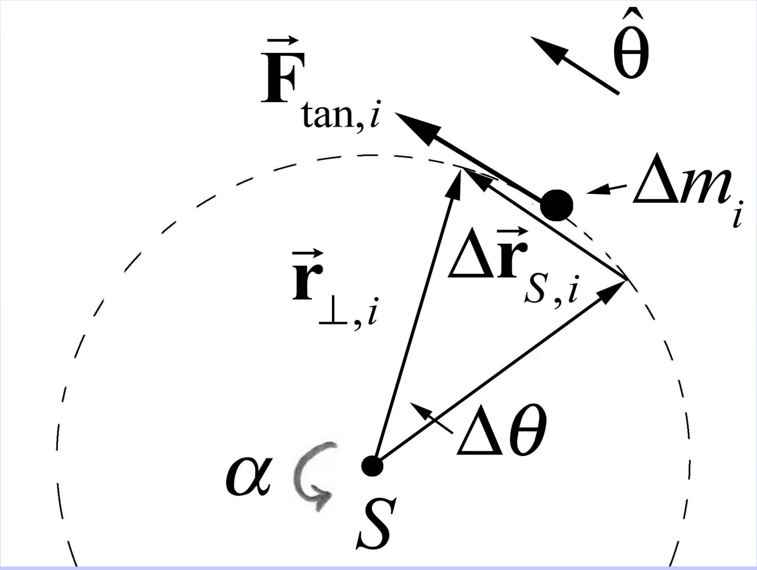
d) What is the vertical component of the hinge force at B?

e) Does the hinge force have a horizontal component at the instant after release?

Rotational Work

Tangential force

$$\vec{\mathbf{F}}_{\tan,i} = F_{\tan,i}\hat{\boldsymbol{\theta}}$$



Displacement vector

$$\Delta\vec{\mathbf{r}}_{S,i} = r_{\perp,i}\Delta\theta\,\hat{\boldsymbol{\theta}}$$

work for a small displacement

$$\Delta W_i = \vec{\mathbf{F}}_{\tan,i}\cdot\Delta\vec{\mathbf{r}}_{S,i} = F_{\tan,i}\,\hat{\boldsymbol{\theta}}\cdot r_{\perp,i}\Delta\theta\,\hat{\boldsymbol{\theta}} = r_{\perp,i}F_{\tan,i}\Delta\theta$$

Rotational Work

Newton's Second Law $$F_{\tan,i} = \Delta m_i a_{\tan,i}$$

Tangential acceleration $$a_{\tan,i} = r_{\perp,i}\alpha$$

Work for small displacement $$\Delta W_i = \Delta m_i \, r_{\perp,i}{}^2 \, \alpha \, \Delta\theta$$

Summation becomes integration for continuous body

$$\Delta W = \left(\sum_i \Delta m_i r_{\perp,i}{}^2 \right) \alpha \, \Delta\theta = \left(\int_{body} dm (r_{S,\perp})^2 \right) \alpha \, \Delta\theta = I_S \, \alpha \, \Delta\theta$$

Rotational Work

Rotational work for a small displacement $\Delta W = I_S \, \alpha \, \Delta\theta$

Torque about S $\tau_S = I_S \, \alpha$

Infinitesimal rotational work $\Delta W = \tau_S \, \Delta\theta$

Integrate total work $W = \int_{\theta=\theta_0}^{\theta=\theta_f} dW = \int_{\theta=\theta_0}^{\theta=\theta_f} \tau_S \, d\theta$

Rotational Work-Kinetic Energy Theorem

Infinitesimal rotational work

$$dW_{\text{rot}} = I_S \alpha \, d\theta = I_S \frac{d\omega}{dt} d\theta = I_S d\omega \frac{d\theta}{dt} = I_S d\omega \, \omega$$

Integrate rotational work

$$W_{\text{rot}} = \int_{\omega=\omega_0}^{\omega=\omega_f} dW_{\text{rot}} = \int_{\omega=\omega_0}^{\omega=\omega_f} I_S d\omega \, \omega = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_0^2$$

Kinetic energy of rotation about S

$$W_{\text{rot}} = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_0^2 = K_{\text{rot},f} - K_{rot,0} \equiv \Delta K_{\text{rot}}$$

Rotational Power

Rotational power is the time rate of doing rotational work

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt}$$

Product of the applied torque with the angular velocity

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \boldsymbol{\tau}_S \frac{d\theta}{dt} = \boldsymbol{\tau}_S\, \omega$$

Checkpoint Problem: Rotational Work

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular speed ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down during an interval Δt_1 down until it reaches an angular speed of ω_a at time t_a . At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time Δt_{int} after which the two washers and rotor rotate with the angular speed ω_b . Assume the frictional torque τ_f on the axle is independent of speed, and remains the same when the second washer is dropped.

a) What angle does the rotor rotate through during the collision?

b) What is the work done by the friction torque τ_f from the bearings during the collision?