Torque and Simple Harmonic Motion

Recall: Fixed Axis Rotation Kinematics

Angle variable

$$\theta$$

Angular velocity

$$\omega \equiv d\theta / dt$$

Angular acceleration

$$\alpha \equiv d^2\theta / dt^2$$

Mass element

$$\Delta m_i$$

Radius of orbit

$$r_{\perp,i}$$

Moment of inertia

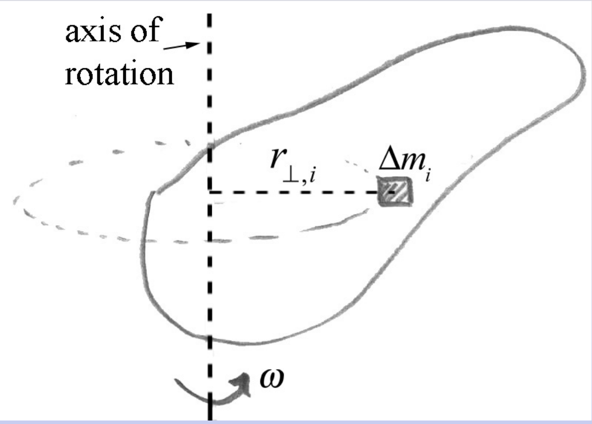
$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \rightarrow \int_{body} dm (r_\perp)^2$$

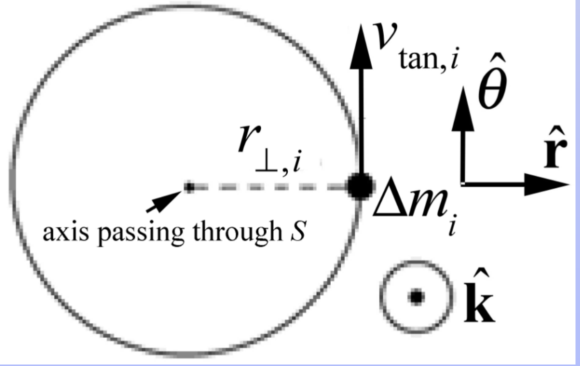
Parallel Axis Theorem

$$I_S = Md^2 + I_{cm}$$

Z-component of Angular Momentum

$$L_{S,z} = I_S \omega$$





Recall: Torque, Moment of Inertia and Angular Acceleration

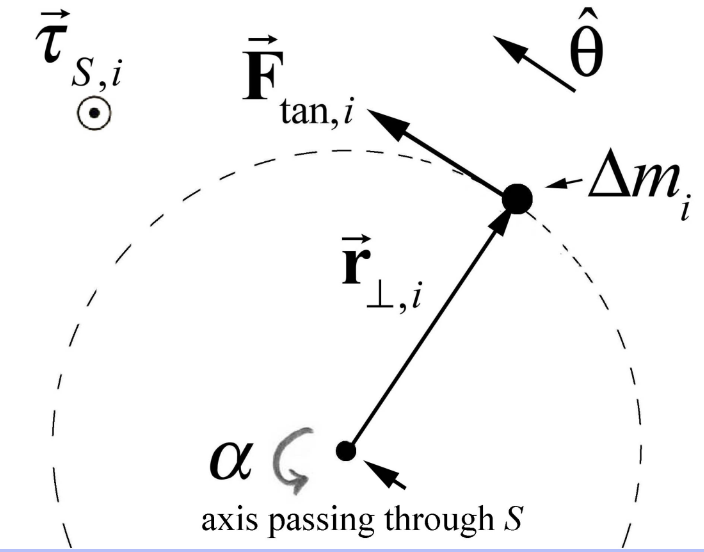
z-component of total torque about S

$$\tau_{S,z}^{total} = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \alpha$$

Recall: Moment of Inertia about S :

$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2$$

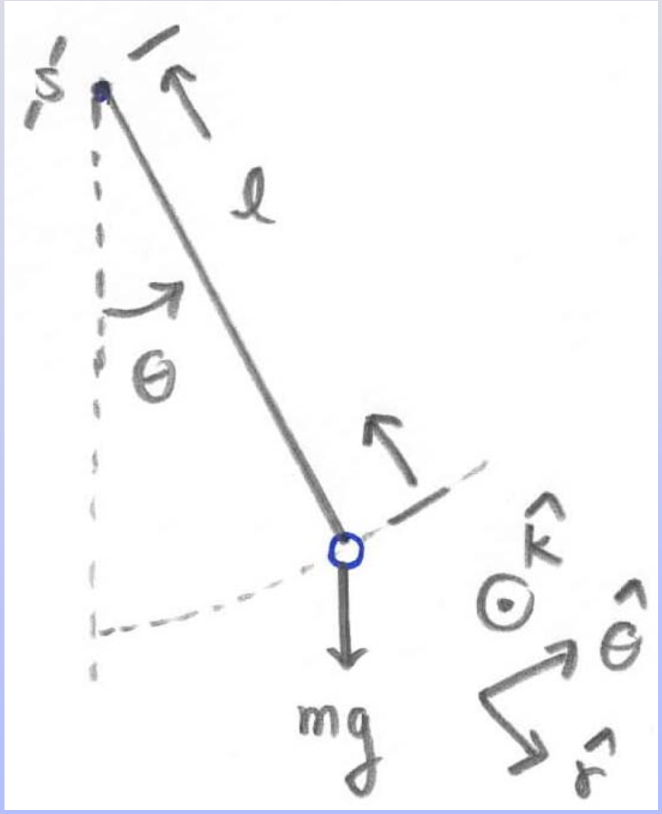
Summary: $\tau_{S,z}^{total} = I_S \alpha$



Simple Pendulum

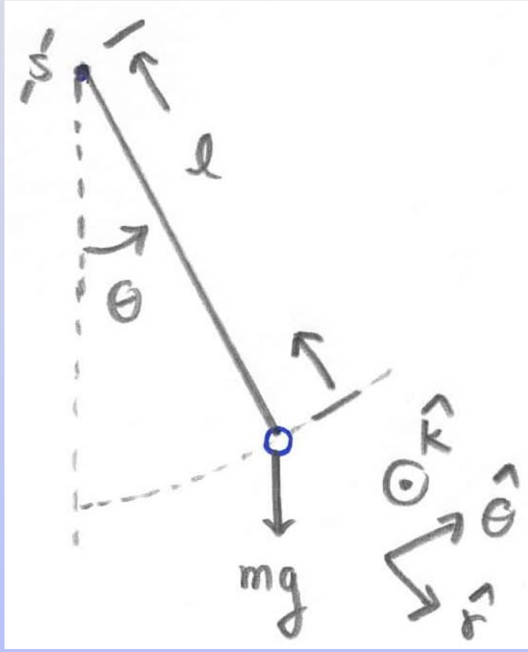
Pendulum: bob hanging from end of string

- Pivot point
- bob of negligible size
- massless string



Simple Pendulum

Simple Pendulum: bob of mass *m* hanging from end of massless string string pivoted at S.



Torque about S

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{s,m} \times m\vec{\mathbf{g}} = l\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -lmg\sin\theta\,\hat{\mathbf{k}}$$

Angular acceleration $\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}}$

Moment of inertial of a point mass about S, $I_S = ml^2$

Rotational Law of Motion $\vec{\boldsymbol{\tau}}_S = I_S\vec{\boldsymbol{\alpha}}$

Simple harmonic oscillator equation

$$-lmg\sin\theta = ml^2\frac{d^2\theta}{dt^2}$$

Simple Pendulum: Small Angle Approximation

Angle of oscillation is small

$$\sin\theta \cong \theta$$

Simple harmonic oscillator

$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta$$

Analogy to spring equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Angular frequency of oscillation

$$\omega_0 \cong \sqrt{\frac{g}{l}}$$

Period

$$T_0 = \frac{2\pi}{\omega_0} \cong 2\pi\sqrt{\frac{l}{g}}$$

Period and Angular Frequency

Equation of Motion:

$$-\frac{g}{l}\theta = \frac{d^2\theta}{dt^2}$$

Solution: Oscillatory with Period T

$$\theta(t) = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$$

angular velocity:

$$\frac{d\theta}{dt}(t) = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}B\cos\left(\frac{2\pi}{T}t\right)$$

angular acceleration:

$$\frac{d^2\theta}{dt^2}(t) = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right) - \left(\frac{2\pi}{T}\right)^2 B\sin\left(\frac{2\pi}{T}t\right) = -\left(\frac{2\pi}{T}\right)^2 x$$

Period:

$$-\frac{g}{l}\theta = \frac{d^2\theta}{dt^2} = -\left(\frac{2\pi}{T}\right)^2\theta \Rightarrow \frac{g}{l} = \left(\frac{2\pi}{T}\right)^2 \Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$$

Angular frequency

$$\omega_0 \equiv \frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

Simple Harmonic Motion: Initial Conditions

Equation of Motion:

$$-\frac{g}{l}\theta = \frac{d^2\theta}{dt^2}$$

Solution: Oscillatory with Period

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Angle:

$$\theta(t) = A\cos\left(\frac{2\pi}{T}t\right) + B\sin\left(\frac{2\pi}{T}t\right)$$

Angular Velocity:

$$\frac{d\theta}{dt}(t) = -\frac{2\pi}{T}A\sin\left(\frac{2\pi}{T}t\right) + \frac{2\pi}{T}B\cos\left(\frac{2\pi}{T}t\right)$$

Initial Angle at t = 0:

$$\theta_0 \equiv \theta(t=0) = A$$

Initial Angular Velocity at t = 0:

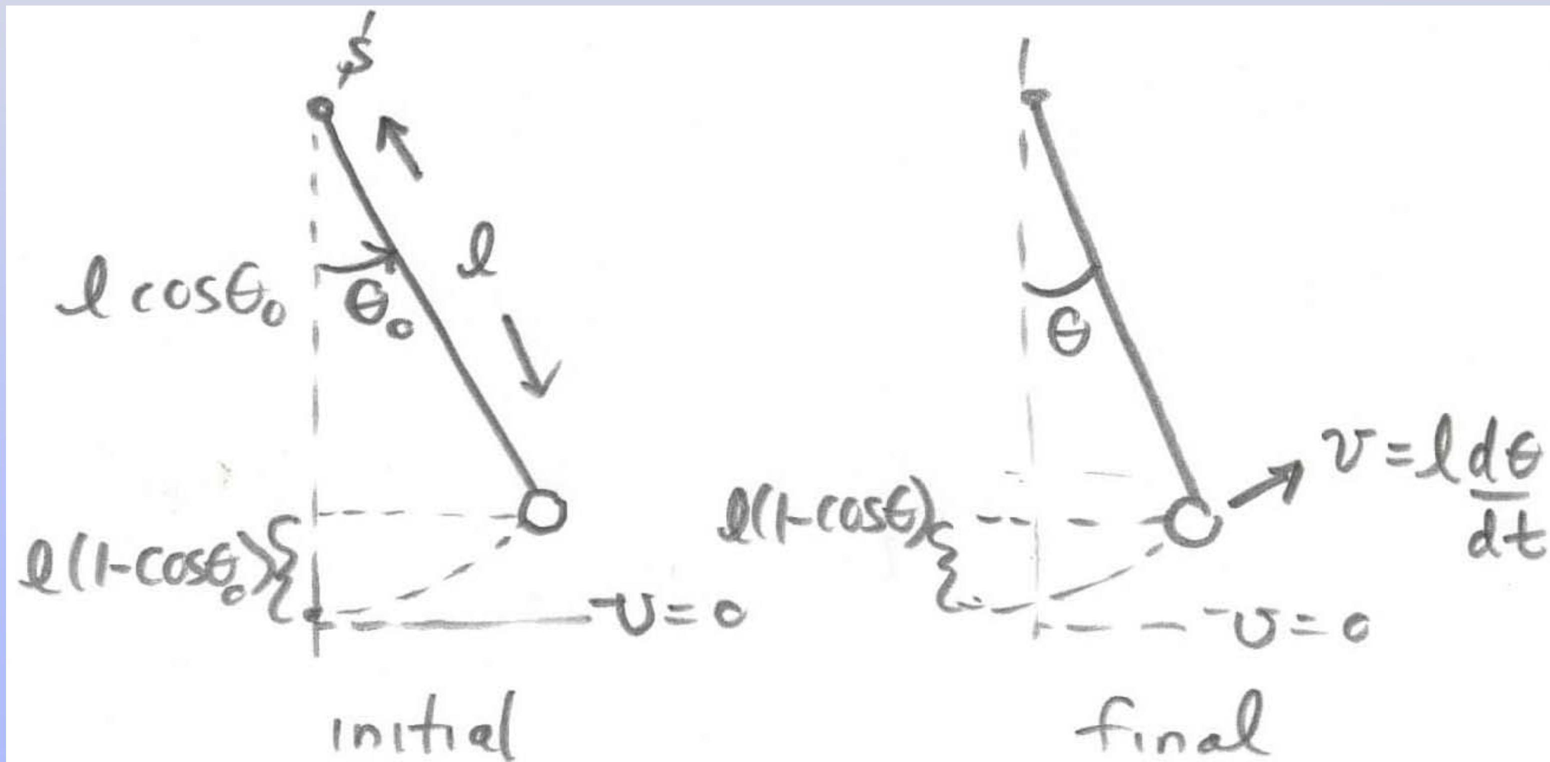
$$\left.\frac{d\theta}{dt}\right|_0 \equiv \frac{d\theta}{dt}(t=0) = \frac{2\pi}{T}B$$

General Solution:

$$\theta(t) = \theta_0\cos\left(\frac{2\pi}{T}t\right) + \frac{T}{2\pi}\left.\frac{d\theta}{dt}\right|_0 \sin\left(\frac{2\pi}{T}t\right)$$

Simple Pendulum: Mechanical Energy

- released from rest at an angle θ_0



Simple Pendulum: Mechanical Energy

- Velocity

$$v_{\tan} = l \frac{d\theta}{dt}$$

- Kinetic energy

$$K_f = \frac{1}{2} m v_{\tan}{}^2 = \frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2$$

- Initial energy

$$E_0 = K_0 + U_0 = mgl(1 - \cos\theta_0)$$

- Final energy

$$E_f = K_f + U_f = \frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2 + mgl(1 - \cos\theta\ )$$

- Conservation of energy

$$\frac{1}{2} m \left(l \frac{d\theta}{dt} \right)^2 + mgl(1 - \cos\theta) = mgl(1 - \cos\theta_0)$$

Simple Pendulum: Angular Velocity Equation of Motion

- Angular velocity

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}}\sqrt{(\cos\theta - \cos\theta_0)}$$

- Integral form

$$\int \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_0)}} = \int \sqrt{\frac{2g}{l}}\,dt$$

Simple Pendulum: First Order Correction

- period $$T = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\left(\theta_0/2\right)+\cdots\right) \qquad T_0 = 2\pi\sqrt{\frac{l}{g}}$$
- initial angle is small $$\sin^2\left(\theta_0/2\right) \cong \theta_0^{\ 2}/4$$
- Approximation $$T \cong 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_0^{\ 2}\right) = T_0\left(1+\frac{1}{16}\theta_0^{\ 2}\right)$$
- First order correction $$\Delta T_1 \cong \frac{1}{16}\theta_0^{\ 2}T_0$$

Mini-Experiment: Simple Pendulums

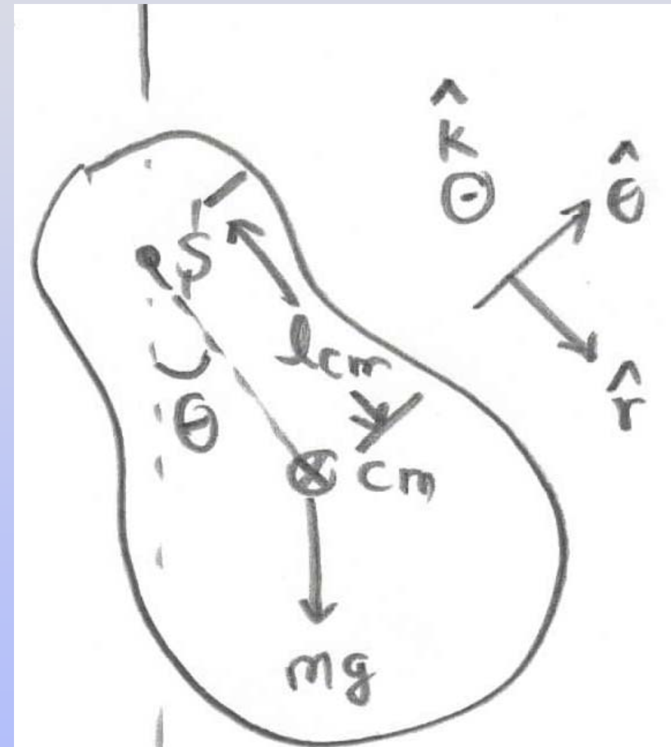
Time 10 oscillations of a simple pendulum (bob on a string) for two different initial angles: i) 5^0 and ii) 20^0. Calculate periods and compare results. Can you explain any differences? Are they what you might expect?

Physical Pendulum

Pendulum pivoted about point S

Gravitational force acts center of mass

Center of mass distance from the pivot point ℓ_{cm}



Physical Pendulum

torque about pivot point

$$\vec{\boldsymbol{\tau}}_S = \vec{\mathbf{r}}_{s,cm} \times m\vec{\mathbf{g}} = l_{cm}\hat{\mathbf{r}} \times mg(-\sin\theta\,\hat{\boldsymbol{\theta}} + \cos\hat{\mathbf{r}}) = -l_{cm}mg\sin\theta\,\hat{\mathbf{k}}$$

moment of inertial about pivot point I_S

Example: uniform rod of mass m and length l .

$$I_S = \frac{1}{3}ml^2$$

Physical Pendulum

Rotational dynamical equation $\vec{\boldsymbol{\tau}}_S = I_S \vec{\boldsymbol{\alpha}}$

Small angle approximation $\sin\theta \cong \theta$

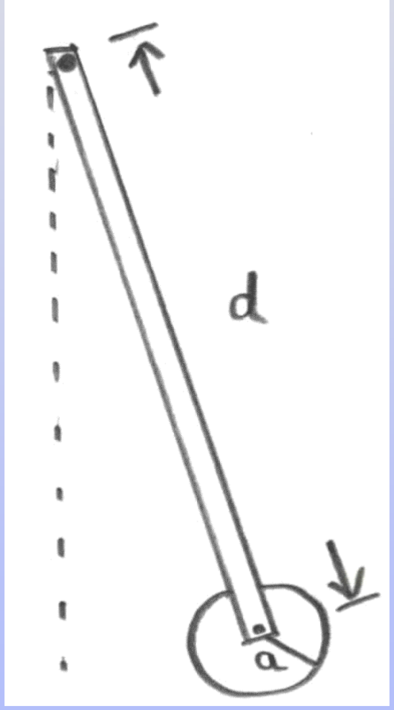
Equation of motion $\dfrac{d^2\theta}{dt^2} \cong -\dfrac{l_{cm} m g}{I_S}\theta$

Angular frequency $\omega_0 \cong \sqrt{\dfrac{l_{cm} m g}{I_S}}$

Period $T = \dfrac{2\pi}{\omega_0} \cong 2\pi\sqrt{\dfrac{I_S}{l_{cm} m g}}$

Concept Question: Physical Pendulum

A physical pendulum consists of a uniform rod of length l and mass m pivoted at one end. A disk of mass m1 and radius a is fixed to the other end. Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin. Does the period of the pendulum



1. increase?
2. stay the same?
3. decrease?

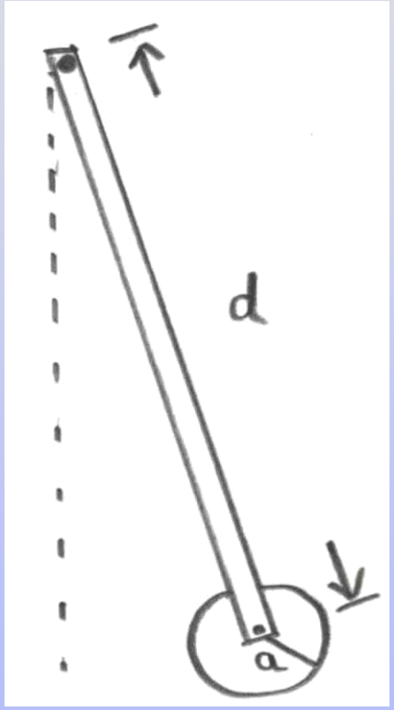
Checkpoint Problem: Physical Pendulum

A physical pendulum consists of a uniform rod of length l and mass m pivoted at one end. A disk of mass m_1 and radius a is fixed to the other end.

a) Find the period of the pendulum.

Suppose the disk is now mounted to the rod by a frictionless bearing so that is perfectly free to spin.

b) Find the new period of the pendulum.



Checkpoint Problem: Torsional Oscillator

A disk with moment of inertia I_0 rotates in a horizontal plane. It is suspended by a thin, massless rod. If the disk is rotated away from its equilibrium position by an angle θ, the rod exerts a restoring torque given by $\tau = -\gamma\theta$. At $t = 0$ the disk is released from rest at an angular displacement of θ_0. Find the subsequent time dependence of the angular displacement $\theta(t)$.