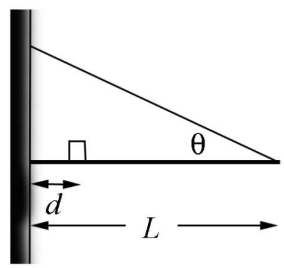
Static Equilibrium
Challenge Problem Solutions

Problem 1: Static Equilibrium: ***Steel Beam and Cable***

A uniform steel beam of mass $m_1 = 2.0\times10^2$ kg is held up by a steel cable that is connected to the beam a distance $L = 5.0$ m from the wall, at an angle $\theta = 30^\circ$ as shown in the sketch. The beam is bolted to the wall with an unknown force $\vec{\mathbf{F}}$ exerted by the wall on the beam. An object of mass $m_2 = 6.0\times10^1$ kg, resting on top of the beam, is placed a distance $d = 1.0$ m from the wall. Use $g = 9.8\ \text{m}\cdot\text{s}^{-2}$ for the gravitational acceleration.



a) Draw a free-body diagram for the beam.

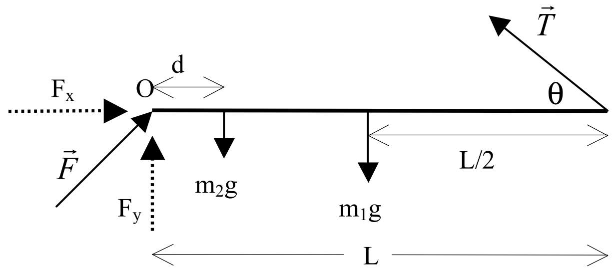
b) Find equations for static equilibrium for the beam (this will involve force equations and torque relations).

c) Find the tension in the cable.

d) Find the horizontal and vertical components of the force that the wall exerts on the beam.

Problem 1 Solution:

(a) Free-body diagram for the beam:



Here we have taken the unknown force to be $\vec{F} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ represent the unit vectors along the x- and y-coordinate axes respectively, the x-direction being to the right in Figure 1 and the y-direction being vertically upward.

(b) Equations for Static Equilibrium for the beam:

The equations for force equilibrium are:

$$\begin{aligned} \hat{\mathbf{i}}: &\quad F_x - T\cos\theta = 0 \\ \hat{\mathbf{j}}: &\quad F_y - m_2 g - m_1 g + T\sin\theta = 0. \end{aligned} \tag{1.1}$$

The equations for torque equilibrium are (taking torques point O in Figure 1):

$$L\,T\sin\theta - (L/2)\times m_1\,g - d\times m_2\,g = 0 \tag{1.2}$$

The equations in (1.1) and (1.2) are three equations in the three unknowns F_x, F_y and T (the distances L and d, and the masses m_1 and m_2 are given). By choosing the point O as the point about which to find torques, the force components F_x and F_y do not appear in the torque equation (1.2), allowing immediate calculation of the tension T.

(c) From Equation (1.2) we have

$$T = \left(\frac{m_1 g}{2} + \frac{m_2 g\,d}{L}\right)/\sin\theta = g\frac{(m_1/2 + m_2\,d/L)}{\sin\theta}. \tag{1.3}$$

Insertion of the given numerical values gives

$$T = \left(9.8\ \mathrm{m \cdot s^{-2}}\right)\frac{\left(2.0\times10^{2}\ \mathrm{kg}/2 + 6.0\times10^{1}\ \mathrm{kg}\left(1.0\ \mathrm{m}/\ 5.0\ \mathrm{m}\right)\right)}{\sin 30^{\circ}} = 2195.2\,\mathrm{N}, \qquad (1.4)$$

or $2.2\times10^{3}\mathrm{N}$ to the two significant figures given in the problem.

(d) Finding components of the force between the beam and the wall:

Using the first expression in (1.1),

$$F_x = T\cos\theta = 2195.2\ \mathrm{N}\times\cos(30°) = 1901\,\mathrm{N}, \qquad (1.5)$$

or $F_x = 1.9\times10^{3}\ \mathrm{N}$ to the two significant figures given in the problem.

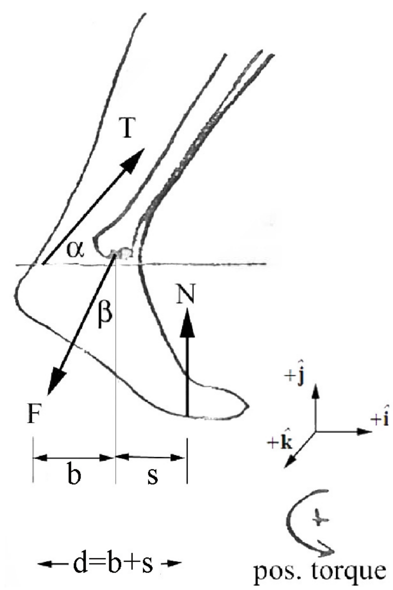
Using the second expression in (1.1),

$$\begin{aligned} F_y &= m_2 g + m_1 g - T\sin\theta \\ &= \left(200\,\mathrm{kg} + 60\,\mathrm{kg}\right)\times 9.8\ \mathrm{m\cdot s^{-2}} - 2195\,\mathrm{N}\times\sin 30° = 1450.4\,\mathrm{N}, \end{aligned} \qquad (1.6)$$

or $1.5\times10^{3}\ \mathrm{N}$ to the two significant figures given in the problem.

Problem 2: Static Equilibrium *The Ankle*

A person of mass $m = 75\ \text{kg}$ is crouching with his/her weight evenly distributed on both tiptoes. The forces on the skeletal part of the foot are shown in the diagram. In this position, the tibia acts on the foot with a force $\vec{\mathbf{F}}$ of magnitude $F = |\vec{\mathbf{F}}|$ and which makes an unknown angle β with the vertical. This force acts on the ankle a horizontal distance $s = 4.8\ \text{cm}$ from the point where the foot contacts the floor. The Achilles tendon is under considerable tension $\vec{\mathbf{T}}$ and makes a given angle $\alpha = 37^0$ with the horizontal. The tendon acts on the ankle a horizontal distance $b = 6.0\ \text{cm}$ from the point where the tibia acts on the foot. You may ignore the weight of the foot. Let $g = 9.8\ \text{m}\cdot\text{s}^{-2}$ be the gravitational constant. In this problem you will express your answers symbolically. You may want to substitute in numbers if you have the time.



a) Find the magnitude of the tension in the Achilles tendon, $T \equiv |\vec{\mathbf{T}}|$.

b) Find the magnitude, $F \equiv |\vec{\mathbf{F}}|$, and the angle, β, of the tibia force on the ankle.

Problem 2 Solution:

Understand.

From the description of the problem, you are trying to determine a symbolic expression for the forces that are distributed over a foot that is in static equilibrium, and hence the two laws of static equilibrium apply:

(1) *The sum of the forces acting on the rigid body is zero,*

$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}} \tag{2.1}$$

(2) *The vector sum of the torques about any point S in a rigid body is zero,*

$$\vec{\tau}_S^{\text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \cdots = \vec{\mathbf{0}} \tag{2.2}$$

In order to apply these laws you must do the following:

i) Determine precisely which part of the body you will choose to consider as the system on which the forces are acting,

ii) Because the foot is not an isolated body, you must consider exactly which forces are acting on the foot, and at which points they act.

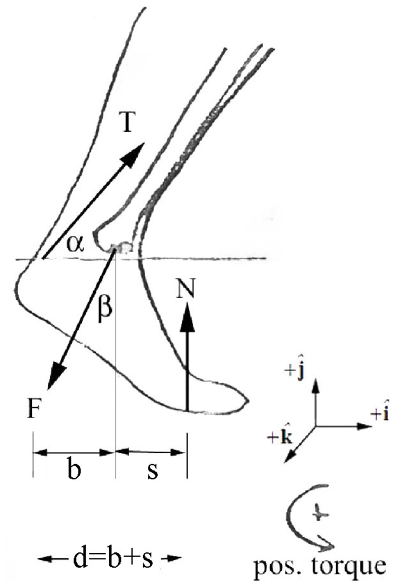
iii) Determine which point about which to calculate torques, and a choice for positive direction for the torque (counterclockwise in the above diagram). Remember that if you choose a point where a force acts then that force has zero torque about that point. You should see if you can find some special point which would simplify your torque calculation.

iv) When calculating torque about a chosen point, you also need to decide for each force whether the given information of the problem makes it easier to compute the moment arm of the force about your chosen point or the perpendicular component of the force with respect to a line drawn from your chosen point to the point where the force acts.

Design a Strategy:

System: Choose the foot as the system noting that the tibia bone and the Achilles tendon are not part of the system.

Forces: There are three forces acting on the foot. The normal force of the floor acts on the foot. You can actually determine the normal force by considering the entire body. Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or $N = (1/2)mg$. The Achilles tendon exerts a force of unknown magnitude $T \equiv |\vec{\mathbf{T}}|$ but at a known angle $\alpha = 37^0$ with respect to the horizontal. Finally the tibia exerts an unknown force $\vec{\mathbf{F}}$ with unknown magnitude $F = |\vec{\mathbf{F}}|$ and an unknown angle β with respect to the vertical. The problem states that you can ignore the weight of the foot.



The condition Equation (2.1) that the sum of the forces is zero becomes

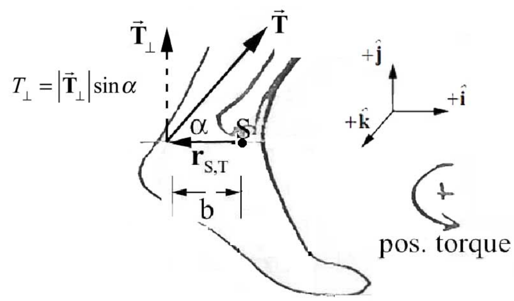
$$\hat{\mathbf{i}}: -F\sin\beta + T\cos\alpha = 0 \tag{2.3}$$

$$\hat{\mathbf{j}}: N - F\cos\beta + T\sin\alpha = 0 \tag{2.4}$$

As explained above, the normal force on one foot is equal to half the weight, or $N = (1/2)mg$. Then Equation (2.4) in the vertical direction becomes

$$\hat{\mathbf{j}}: (1/2)mg - F\cos\beta + T\sin\alpha = 0\,. \tag{2.5}$$

Torque: Choose the point of action of the tibia on the ankle as the point S to compute the torque about. Note that the force $\vec{\mathbf{F}}$ that the tibia exerts on the ankle will make no contribution to the torque about this point S. Choose counterclockwise as the positive direction for the torque, as in the above diagrams; this is the positive $\hat{\mathbf{k}}$-direction. The torque diagram on the ankle is shown below.



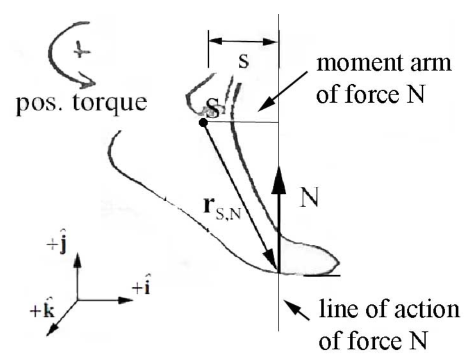
We shall first calculate the torque due to the Achilles tendon. The first thing to notice is that distance b along the horizontal line between the point of action of the force $\vec{\mathbf{T}}$ and the point S is a known quantity, and the angle α with respect to this horizontal line is also a known quantity. Therefore the magnitude of the torque is just the magnitude of the perpendicular component of $T_\perp = \left|\vec{\mathbf{T}}_\perp\right| \sin\alpha$ and the distance b. The direction of the torque is into the plane of the page in the figure above which is the $-\hat{\mathbf{k}}$-direction. The torque due to the force of the tendon $\vec{\mathbf{T}}$ on the ankle about the point where the tibia exerts a force on the foot is then

$$\vec{\tau}_{S,T} = -bT\sin\alpha\,\hat{\mathbf{k}} \tag{2.6}$$

(Note: Using the definition of torque as a cross product, you can calculate the same result;

$$\vec{\tau}_{S,T} = \vec{\mathbf{r}}_{S,T} \times \vec{\mathbf{T}} = -b\hat{\mathbf{i}} \times (T\cos\alpha\hat{\mathbf{i}} + T\sin\alpha\hat{\mathbf{j}}) = -bT\sin\alpha\,\hat{\mathbf{k}}\ .) \tag{2.7}$$

The torque diagram for the normal force is shown in the figure below;



The moment arm of the normal force is a known distance s. The magnitude of the torque of the normal force about the point S is the product of the moment arm with the magnitude of the normal force, $s\,N = (1/2)s\,mg$. The direction of the torque is out of the plane of the figure above, the $+\hat{\mathbf{k}}$-direction, so the torque of the normal force about the point S is

$$\vec{\tau}_{S,N} = s\,N\,\hat{\mathbf{k}} = (1/2)s\,mg\,\hat{\mathbf{k}} \tag{2.8}$$

(you may want to show that the cross product definition of torque gives

$$\vec{\tau}_{S,N} = \vec{\mathbf{r}}_{S,N} \times N\,\hat{\mathbf{j}} = (s\hat{\mathbf{i}} - h\hat{\mathbf{j}}) \times N\,\hat{\mathbf{j}} = s\,N\,\hat{\mathbf{k}} = (1/2)\,s\,mg\,\hat{\mathbf{k}}\,). \tag{2.9}$$

Substitute Equations (2.6) and (2.8) into the condition that the total torque about the point S vanishes (Equation (2.2));

$$\vec{\tau}_S^{\,\text{total}} = \vec{\tau}_{S,T} + \vec{\tau}_{S,N} = \vec{\mathbf{0}} \tag{2.10}$$

becomes

$$-b\,T\sin\alpha\hat{\mathbf{k}} + (1/2)\,s\,mg\,\hat{\mathbf{k}} = \vec{\mathbf{0}}\,. \tag{2.11}$$

We can use this torque condition (Equation (2.11)) to find the tension in the tendon,

$$T = \frac{(1/2)s\,mg}{b\sin\alpha} = \frac{(1/2)(4.8\times 10^{-2}\,\text{m})(75\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})}{(6.0\times 10^{-2}\,\text{m})\sin(37^\circ)} = 4.9\times 10^{2}\,\text{N}\,. \tag{2.12}$$

We can now solve for the angle β of the force of the tibia $\vec{\mathbf{F}}$ on the ankle as follows. Rewrite the two force equations Equations (2.3) and (2.5) as

$$F\sin\beta = T\cos\alpha \tag{2.13}$$

$$F\cos\beta = (1/2)mg + T\sin\alpha \tag{2.14}$$

Dividing these equations yields

$$\frac{F\cos\beta}{F\sin\beta} = \text{cotan}\,\beta = \frac{(1/2)mg + T\sin\alpha}{T\cos\alpha} \tag{2.15}$$

Now take the inverse cotangent to solve for the angle β;

$$\beta = \text{cotan}^{-1}\left(\frac{(1/2)mg + T\sin\alpha}{T\cos\alpha}\right). \tag{2.16}$$

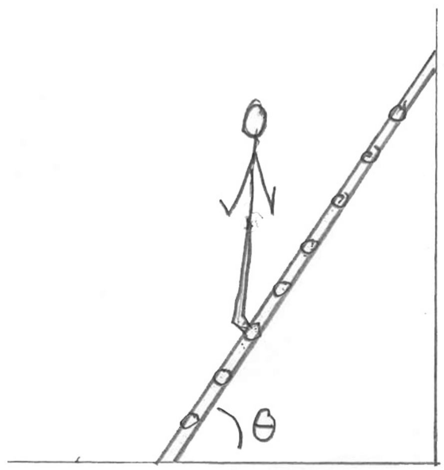
$$\beta = \text{cotan}^{-1}\left(\frac{(1/2)(75\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2}) + (4.9\times 10^{2}\,\text{N})\sin(37^\circ)}{(4.9\times 10^{2}\,\text{N})\cos(37^\circ)}\right) = 31^\circ\,. \tag{2.17}$$

We can now use the horizontal force Equation (2.13) to calculate the magnitude $\left|\vec{\mathbf{F}}\right|$ of the force of the tibia $\vec{\mathbf{F}}$ on the ankle,

$$F = \frac{T\cos\alpha}{\sin\beta} = \frac{(4.9\times10^{2}\ \mathrm{N})\cos(37^{\circ})}{\sin(31^{\circ})} = 7.7\times10^{2}\ \mathrm{N}\,. \qquad (2.18)$$

Problem 3:

A person of mass m_p is standing on a rung, one third of the way up a ladder of length d. The mass of the ladder is m_l, uniformly distributed. The ladder is initially inclined at an angle θ with respect to the horizontal. Assume that there is no friction between the ladder and the wall but that there is friction between the base of the ladder and the floor with a coefficient of static friction μ_s. In this problem you will try to find the minimum coefficient of friction between the ladder and the floor so that the person and ladder do not slip.



Problem 3 Solution:

We shall apply the two conditions for static equilibrium on the ladder,

(1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}}. \quad \textbf{(3.1)}$$

(2) The vector sum of the torques about any point S on a rigid body is zero,

$$\vec{\tau}_S^{\text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \cdots = \vec{\mathbf{0}}. \quad \textbf{(3.2)}$$

Strategy

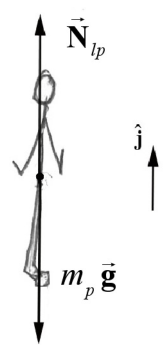
1. Determine the forces and where they act on the ladder, remembering that the gravitational force acts at the center of mass of the ladder. You should draw a free body force diagram, clearly indicating the forces, and include unit vectors on the diagram.

2. Identify any possible third law interaction pairs.

3. Determine which point to calculate torque about. Remember that if you choose a point where a force acts then that force has zero torque about that point. When calculating torque about a chosen point, you can always formally calculate the cross product $\vec{\tau}_s = \vec{r}_{sF} \times \vec{F}$. You may also argue geometrically if the given information of the problem makes it easier to compute the moment arm of the force about your chosen point or the perpendicular component of the force with respect to a line drawn from your chosen point to the point where the force acts. You still need to determine the direction of the torque.

Forces:

Consider the forces acting on the person. The gravitational force acts at the center of mass of the person and the force on the person due to the contact between the person and the ladder is an upward normal force, denoted $\vec{\mathbf{N}}_{lp}$ in the diagrams in these solutions. The force diagram on the person is shown below.



The equation for static equilibrium of forces on the person is

$$\hat{\mathbf{j}}:\ N_{lp} - m_p g = 0 \tag{3.3}$$

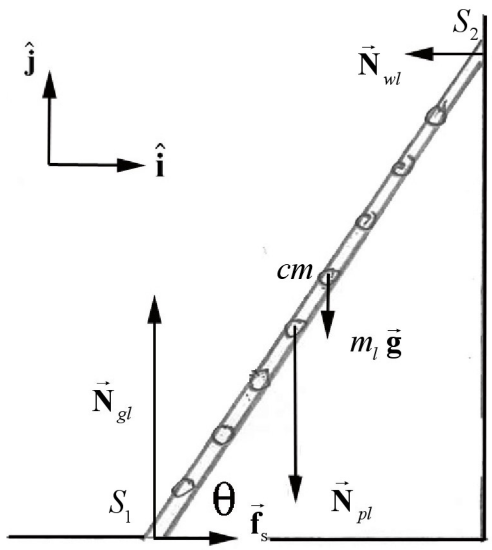
The normal force $\vec{\mathbf{N}}_{pl}$ that the person exerts on the ladder is part of a third law interaction pair. From Newton's Third Law,

$$\vec{\mathbf{N}}_{lp} = -\vec{\mathbf{N}}_{pl} . \tag{3.4}$$

Denote the magnitude of $\vec{\mathbf{N}}_{pl}$ by N_{pl}. Equation (3.3) then becomes

$$N_{pl} = m_p g \, . \tag{3.5}$$

There are four forces acting on the ladder. The person exerts a downward contact force on the ladder at the point of contact on the rung of the ladder, $\vec{\mathbf{N}}_{pl}$, a distance $d/3$ from the contact point with the floor. The gravitational force between the earth and the ladder , $m_l \vec{\mathbf{g}}$, acts at the center of mass of the ladder, which is a distance $d/2$ from the contact point with the floor since the mass of the ladder is assumed to be uniformly distributed. At the point where the ladder is in contact with the wall, the contact force of the wall with ladder $\vec{\mathbf{N}}_{wl}$ is only perpendicular because we have assumed that the contact surface is frictionless. At the point where the ladder is contact with the floor, the contact force has both a vertical component, the normal force $\vec{\mathbf{N}}_{gl}$ and a horizontal component pointing toward the wall, the static friction, $\vec{\mathbf{f}}_s$. The force diagram is shown in the figure below



Key point: The magnitude of the static friction force depends on the other forces and where they act. As the person walks up the ladder, the normal force of the person $\vec{\mathbf{N}}_{pl}$ changes position and hence the friction force will change in magnitude possibly causing the ladder to slip.

The equations for static equilibrium of forces on the ladder (using Equation (3.5) for the magnitude of the normal force of the person on the ladder) becomes

$$\hat{\mathbf{j}}: \;\; N_{gl} - m_p g - m_l g = 0 \tag{3.6}$$

$$\hat{\mathbf{i}}: \;\; f_s - N_{wl} = 0 \, . \tag{3.7}$$

Solve Equation. (3.6) for the upward normal force of the ground on the ladder, which has magnitude

$$N_{gl} = m_p g + m_l g \,. \tag{3.8}$$

We can use Equation (3.7) to find a relationship between the friction force of the ground on the ladder and the normal force of the wall on the ladder,

$$f_s = N_{wl} \,. \tag{3.9}$$

Torques:

Because the sum of the forces are zero, the torque about any point will be zero so we shall calculate the torque about several points: the point of contact between the ladder and the ground, the point of contact between the wall and the ladder, and the center of mass of the ladder.

a) **Torque about the contact point between the ladder and the ground**

The torque equation about the contact point S_1 between the ladder and the floor is given by

$$\vec{\tau}_{s_1} = \vec{r}_{s_1 p} \times \vec{N}_{pl} + \vec{r}_{s_1 cm} \times m_l \vec{g} + \vec{r}_{s_1 w} \times \vec{N}_{wl} = \vec{0} \,. \tag{3.10}$$

We now explicitly write out the vectors from our choice of point to where the forces are acting:

$$\vec{r}_{s_1 p} = \frac{d}{3}\cos\theta\,\hat{\mathbf{i}} + \frac{d}{3}\sin\theta\,\hat{\mathbf{j}} \,, \tag{3.11}$$

$$\vec{r}_{s_1 cm} = \frac{d}{2}\cos\theta\,\hat{\mathbf{i}} + \frac{d}{2}\sin\theta\,\hat{\mathbf{j}} \,, \tag{3.12}$$

$$\vec{r}_{s_1 w} = d\cos\theta\,\hat{\mathbf{i}} + d\sin\theta\,\hat{\mathbf{j}} \,. \tag{3.13}$$

The forces are

$$\vec{N}_{pl} = -m_p g\,\hat{\mathbf{j}} \,, \tag{3.14}$$

$$\vec{N}_{wl} = -N_{wl}\,\hat{\mathbf{i}} \,, \tag{3.15}$$

$$m_l\vec{g} = -m_l g\hat{\mathbf{j}}\,. \qquad \textbf{(3.16)}$$

Then the torque about S_1 is

$$\begin{aligned} 0 \stackrel{\rightharpoonup}{=} \vec{\tau}_{s_1} = &\left(\frac{d}{3}\cos\theta\hat{\mathbf{i}} + \frac{d}{3}\sin\theta\hat{\mathbf{j}}\right)\times -m_p g\hat{\mathbf{j}} + \left(\frac{d}{2}\cos\theta\hat{\mathbf{i}} + \frac{d}{2}\sin\theta\hat{\mathbf{j}}\right)\times -m_l g\hat{\mathbf{j}} \\ &+ \left(d\cos\theta\hat{\mathbf{i}} + d\sin\theta\hat{\mathbf{j}}\right)\times -N_{wl}\hat{\mathbf{i}} \end{aligned}\,. \qquad \textbf{(3.17)}$$

Calculating the cross products we get

$$\vec{\tau}_{s_1} = -\frac{d}{3}\cos\theta m_p g\,\hat{\mathbf{k}} - \frac{d}{2}\cos\theta m_l g\,\hat{\mathbf{k}} + d\sin\theta N_{wl}\hat{\mathbf{k}} = 0\,. \qquad \textbf{(3.18)}$$

We can solve this last equation for the magnitude of the normal force of the wall on the ladder

$$N_{wl} = g\text{cotan}\theta\left(\frac{m_p}{3} + \frac{m_l}{2}\right). \qquad \textbf{(3.19)}$$

We can now substitute Equation (3.19) into Equation (3.9) and solve for the magnitude of the friction force

$$f_{\text{s}} = \frac{(1/3)m_2 g\cos\theta + (1/2)m_1 g\cos\theta}{\sin\theta} = g\cot\theta\left(m_2/3 + m_1/2\right). \qquad \textbf{(3.20)}$$

b) The minimum coefficient of friction between the ladder and the floor so that the person and ladder do not slip is given by the condition that

$$f_{\text{s}} = \mu_{\text{s}} N_{wl}\,. \qquad \textbf{(3.21)}$$

Substituting Equation (3.8) into Equation (3.21) gives

$$f_s = \mu_s(m_p g + m_l g)\,. \qquad \textbf{(3.22)}$$

We can now equate the expressions in Equations (3.22) and (3.20) and solve for the minimum coefficient of static friction such that the ladder just starts to slip,

$$\mu_{\text{s}} = \frac{m_p/3 + m_l/2}{m_p + m_l}\text{cotan}\theta\,. \qquad \textbf{(3.23)}$$

b) Torque about the center of mass

Let's compute the torque about the center of mass cm. Then

$$\vec{\tau}_{cm} = \vec{r}_{cmg} \times \left(\vec{N}_{gl} + \vec{f}_{gl}\right) + \vec{r}_{cm,p} \times \vec{N}_{pl} + \vec{r}_{cmw} \times \vec{N}_{wl} = \vec{0}, \qquad \textbf{(3.24)}$$

where

$$\vec{r}_{cmg} = -\frac{d}{2}\cos\theta\hat{\mathbf{i}} - \frac{d}{2}\sin\theta\hat{\mathbf{j}}, \qquad \textbf{(3.25)}$$

$$\vec{r}_{cmp} = -\frac{d}{6}\cos\theta\hat{\mathbf{i}} - \frac{d}{6}\sin\theta\hat{\mathbf{j}}, \qquad \textbf{(3.26)}$$

$$\vec{r}_{cmw} = \frac{d}{2}\cos\theta\hat{\mathbf{i}} + \frac{d}{2}\sin\theta\hat{\mathbf{j}}. \qquad \textbf{(3.27)}$$

Noting that

$$\vec{N}_{gl} = N_{gl}\,\hat{\mathbf{j}}, \qquad \textbf{(3.28)}$$

$$\vec{f}_{gl} = f_s\,\hat{\mathbf{i}}. \qquad \textbf{(3.29)}$$

The torque about the center of mass is then

$$\begin{aligned}\vec{0} = \vec{\tau}_{cm} &= \left(-\frac{d}{2}\cos\theta\hat{\mathbf{i}} - \frac{d}{2}\sin\theta\hat{\mathbf{j}}\right)\times\left(N_{gl}\,\hat{\mathbf{j}} + f_s\,\hat{\mathbf{i}}\right) + \left(-\frac{d}{6}\cos\theta\hat{\mathbf{i}} - \frac{d}{6}\sin\theta\hat{\mathbf{j}}\right)\times -m_p g\hat{\mathbf{j}} \\ &\quad + \left(\frac{d}{2}\cos\theta\hat{\mathbf{i}} + \frac{d}{2}\sin\theta\hat{\mathbf{j}}\right)\times -N_{wl}\hat{\mathbf{i}}\end{aligned} \qquad \textbf{.(3.30)}$$

We now calculate the cross products and get

$$\vec{\tau}_{cm} = \left(-\frac{d}{2}\cos\theta N_{gl} + \frac{d}{2}\sin\theta f_s\right)\hat{\mathbf{k}} + \left(\frac{d}{6}\cos\theta m_p g\right)\hat{\mathbf{k}} + \left(\frac{d}{2}\sin\theta N_{wl}\right)\hat{\mathbf{k}} = \vec{0}\,\textbf{.(3.31)}$$

Recall from Eqs. (3.8) and (3.9)that $N_{gl} = m_p g + m_l g$ and $f_s = N_{wl}$, Eq. becomes

$$\vec{\tau}_{cm} = \left(-\frac{d}{2}\cos\theta\left(m_p g + m_l g\right) + \frac{d}{2}\sin\theta N_{wl}\right)\hat{\mathbf{k}} + \left(\frac{d}{6}\cos\theta m_p g\right)\hat{\mathbf{k}} + \left(\frac{d}{2}\sin\theta N_{wl}\right)\hat{\mathbf{k}} = \vec{0}\,\textbf{(3.32)}$$

We can solve this equation for N_{wl}:

$$N_{wl} = \text{cotan}\theta g\left(\frac{m_p}{3} + \frac{m_l}{2}\right) \quad \textbf{(3.33)}$$

identical to our previous result (Eq. (3.19)).

c) Torque about the contact point S_2 between the wall and the ladder

The torque equation about S_2 is given by

$$\vec{\tau}_{s_2} = \vec{r}_{s_2 p} \times \vec{N}_{pl} + \vec{r}_{s_2 cm} \times m_l \vec{g} + \vec{r}_{s_2 g} \times \left(\vec{N}_{gl} + \vec{f}_{gl}\right) = \vec{0}\,. \quad \textbf{(3.34)}$$

We now explicitly write out the vectors from our choice of point to where the forces are acting:

$$\vec{r}_{s_2 p} = -\frac{2d}{3}\cos\theta\hat{\mathbf{i}} - \frac{2d}{3}\sin\theta\hat{\mathbf{j}}\,, \quad \textbf{(3.35)}$$

$$\vec{r}_{s_2 cm} = -\frac{d}{2}\cos\theta\hat{\mathbf{i}} + -\frac{d}{2}\sin\theta\hat{\mathbf{j}}\,, \quad \textbf{(3.36)}$$

$$\vec{r}_{s_2 g} = -d\cos\theta\hat{\mathbf{i}} - d\sin\theta\hat{\mathbf{j}}\,. \quad \textbf{(3.37)}$$

Then the torque about S_2 is

$$\begin{aligned}\vec{0} = \vec{\tau}_{s_1} &= \left(-\frac{2d}{3}\cos\theta\hat{\mathbf{i}} - \frac{2d}{3}\sin\theta\hat{\mathbf{j}}\right)\times -m_p g\hat{\mathbf{j}} + \left(-\frac{d}{2}\cos\theta\hat{\mathbf{i}} - \frac{d}{2}\sin\theta\hat{\mathbf{j}}\right)\times -m_l g\hat{\mathbf{j}} \\ &+ \left(-d\cos\theta\hat{\mathbf{i}} - d\sin\theta\hat{\mathbf{j}}\right)\times\left(N_{gl}\,\hat{\mathbf{j}} + f_s\,\hat{\mathbf{i}}\right)\end{aligned}\,. \quad \textbf{(3.38)}$$

Calculating the cross products we get

$$\vec{0} = \frac{2d}{3}\cos\theta m_p g\hat{\mathbf{k}} + \frac{d}{2}\cos\theta m_l g\hat{\mathbf{k}} - d\cos\theta N_{gl}\hat{\mathbf{k}} + d\sin\theta f_s\hat{\mathbf{k}}\,. \quad \textbf{(3.39)}$$

We need to use Eqs. (3.8) and (3.9): $N_{gl} = m_p g + m_l g$ and $f_s = N_{wl}$, so that Eq. (3.39) becomes

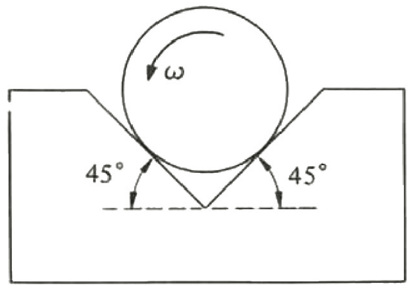
$$\vec{0} = \frac{2d}{3}\cos\theta m_p g\hat{\mathbf{k}} + \frac{d}{2}\cos\theta m_l g\hat{\mathbf{k}} - d\cos\theta\left(m_p g + m_l g\right)\hat{\mathbf{k}} + d\sin\theta N_{wl}\hat{\mathbf{k}}\,. \quad \textbf{(3.40)}$$

We can solve this last equation for the magnitude of the normal force of the wall on the ladder, finding our same result that

$$N_{wl} = g\text{cotan}\theta\left(\frac{m_p}{3} + \frac{m_l}{2}\right). \quad \textbf{(3.41)}$$

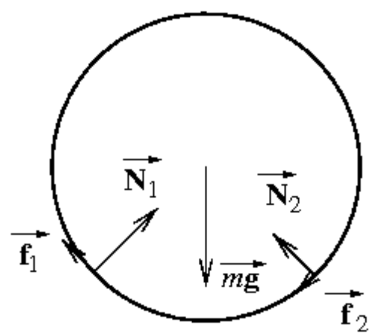
Problem 4: V-Groove Frictional Torque and Fixed Axis Rotation

A cylinder of mass m and radius R is rotated in a V-groove with constant angular velocity ω_0. The coefficient of friction between the cylinder and the surface is μ. What external torque must be applied to the cylinder to keep it rolling?



Problem 4 Solution:

A choice of coordinate system is essential. There are many ways to do this problem, and several ways to select a coordinate system. The solution presented here refers to vertical and horizontal forces. Any way the problem is done, it must be recognized that there are two contact forces, $\vec{\mathbf{C}}_1$ at the left and $\vec{\mathbf{C}}_2$ at the right. Each contact force has two constituents, a normal force ($\vec{\mathbf{N}}_1$ or $\vec{\mathbf{N}}_2$) with upward vertical components and a frictional force ($\vec{\mathbf{f}}_1$ or $\vec{\mathbf{f}}_2$), each with a horizontal component directed to the left in the figure. (In an attempt to keep the figure to scale, the vector representing $\vec{\mathbf{f}}_2$ in the figure below is more or less just an arrowhead.)



Given the coefficient of friction, we have $f_1 = \mu N_1$, $f_2 = \mu N_2$.

The horizontal force components must add to zero;

$$N_1\frac{1}{\sqrt{2}}-f_1\frac{1}{\sqrt{2}}-N_2\frac{1}{\sqrt{2}}-f_2\frac{1}{\sqrt{2}}=0$$
$$N_1\frac{1}{\sqrt{2}}(1-\mu)-N_2(1+\mu)\frac{1}{\sqrt{2}}=0. \tag{4.1}$$

The vertical components of the normal forces must sum to the weight of the cylinder;

$$N_1\frac{1}{\sqrt{2}}+f_1\frac{1}{\sqrt{2}}+N_2\frac{1}{\sqrt{2}}-f_2\frac{1}{\sqrt{2}}=mg$$
$$N_1(1+\mu)\frac{1}{\sqrt{2}}+N_2(1-\mu)\frac{1}{\sqrt{2}}=mg. \tag{4.2}$$

The second expressions Equations (4.1) and (4.2) are two linear equations in the unknowns N_1 and N_2. There are many ways to solve such pairs of equations; the first method presented here is perhaps pedestrian, but it works. Multiply (4.1) by $1-\mu$ and (4.2) by $1+\mu$ and add to eliminate the terms containing N_2; the result is

$$\frac{N_1}{\sqrt{2}}\left[(1-\mu)^2+(1+\mu)^2\right]=mg(1+\mu)$$
$$N_1=\frac{mg}{\sqrt{2}}\frac{1+\mu}{1+\mu^2}. \tag{4.3}$$

Similarly, multiplying (4.1) by $1+\mu$ and (4.2) by $1-\mu$ and subtracting to eliminate the terms containing N_1 gives

$$N_2=\frac{mg}{\sqrt{2}}\frac{1-\mu}{1+\mu^2}. \tag{4.4}$$

The magnitude of the frictional torque is then seen to be

$$\begin{aligned}\tau_{\text{fric}}&=(f_1+f_2)R\\&=\mu(N_1+N_2)R\\&=\mu\left(\frac{mg}{\sqrt{2}}\frac{2}{1+\mu^2}\right)R\\&=mgR\sqrt{2}\frac{\mu}{1+\mu^2}\end{aligned} \tag{4.5}$$

and this must also be the magnitude of the applied torque to maintain constant angular velocity.

Note that the moment of inertia of the cylinder, that is, whether the cylinder is hollow, solid, or something in between, does not enter the solution of this problem.

Alternate Solution:

Not really an alternate solution, but different algebra.
Re-express the second expressions in each of (4.1) and (4.2) as

$$\begin{aligned}(N_1 - N_2) - \mu(N_1 + N_2) &= 0\\ (N_1 + N_2) + \mu(N_1 - N_2) &= \sqrt{2}\,mg.\end{aligned} \tag{4.6}$$

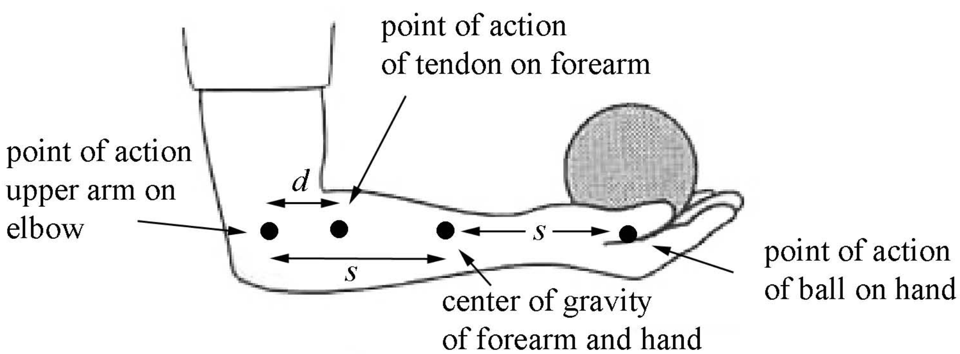
Multiply the first equation in (4.6) by $-\mu$ and add to the second to obtain

$$(N_1 + N_2)(1+\mu^2) = \sqrt{2}\,mg\,, \tag{4.7}$$

which leads to the results in (4.5).
Similarly, in the first expressions in Equations (4.1) and (4.2) we could have substituted $N_1 = f_1/\mu$, $N_2 = f_2/\mu$ and solved for the two friction forces directly.

Problem 5: Static Equilibrium *Arm*

You are holding a ball of mass m_2 in your hand. In this problem you will solve for the upward force $\vec{\mathbf{T}}$ that the tendon of your biceps muscle exerts to keep the forearm horizontal and the downward force $\vec{\mathbf{F}}$ that the upper arm exerts on the forearm at the elbow joint. Assume the outstretched arm has a mass of m_1, the center of mass of the outstretched arm is a distance s from the elbow, the tendon attaches to the bone a distance d from the elbow, and the ball is a distance $2s$ from the elbow. (Taking $\vec{\mathbf{T}}$ to be upward and $\vec{\mathbf{F}}$ to be downward, with no horizontal components, indicates that this is a simplified model.) A schematic representation, but not showing the forces, of this situation is shown below:



a) What is the magnitude of the tension $T \equiv \left|\vec{\mathbf{T}}\right|$ in the tendon?

b) What is the magnitude of the force $\vec{\mathbf{F}}$ that the upper arm exerts on the forearm at the elbow joint?

Problem 5 Solutions:

Understand the problem: When the outstretched hand supports a ball, the arm, hand , and ball are all in static equilibrium. So we can apply the equations of static equilibrium,

(1) *The sum of the forces acting on the rigid body is zero,*

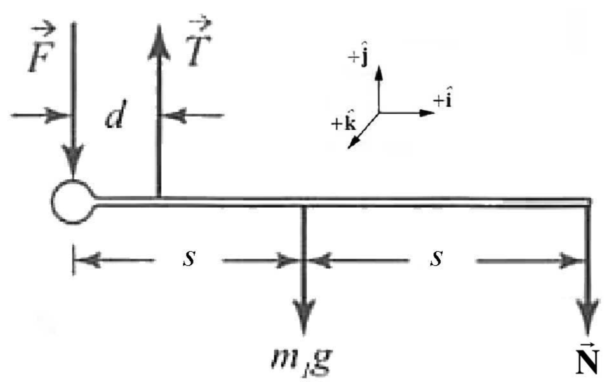
$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}}. \qquad (5.1)$$

(2) *The vector sum of the torques about any point S in a rigid body is zero,*

$$\vec{\tau}_S^{\,\text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \cdots = \vec{\mathbf{0}}. \qquad (5.2)$$

Design a strategy: In this problem you are trying to find the magnitude of the tension $T \equiv |\vec{\mathbf{T}}|$ in the tendon, and the magnitude of the force $\vec{\mathbf{F}}$ that the upper arm exerts on the forearm at the elbow joint. The key to the problem is to identify a suitable system and then identify all the forces acting on that system, and finally draw a free-body force diagram. Once you have identified a suitable system, you need to choose a point about which to compute the torques due to all the forces acting on that system. Then apply the two conditions of static equilibrium to determine a set of equations involving the forces. You then need to determine whether you have enough information to solve the system of equations for the desired quantities.

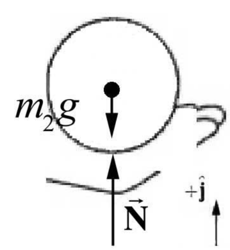
Implement the strategy: Choose the forearm and hand from the elbow to the hand as the system. This means that the upper arm and the ball are not part of the system. The forces acting on the system are the downward force $\vec{\mathbf{F}}$ that the upper arm exerts on the elbow, the downward normal force $\vec{\mathbf{N}}$ that the ball exerts on the hand, the gravitational interaction between the system and the earth, and the upward force $\vec{\mathbf{T}}$ that the tendon exerts to keep the forearm horizontal. The free body force diagram on the system is shown below with a choice of unit vectors.



Force: The forces are all in the vertical direction, so the condition that the sum of the forces is zero is then

$$\hat{\mathbf{j}}: -N - F + T - m_1 g = 0. \tag{5.3}$$

From the figure below,



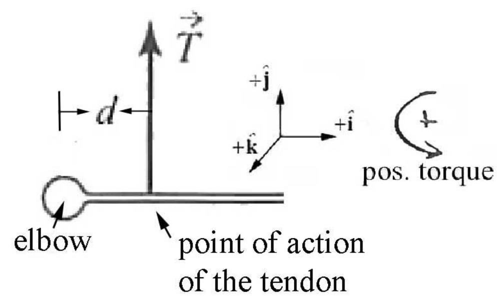
the forces acting on the ball in the vertical direction are

$$\hat{\mathbf{j}}: N - m_2 g = 0. \tag{5.4}$$

Thus the normal force is $N = m_2 g$. So the equation for force equilibrium becomes

$$-m_2 g - F + T - m_1 g = 0. \tag{5.5}$$

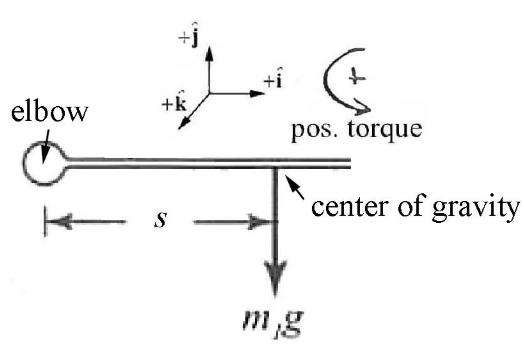
Torque: From the free body diagram, choose the point S where the upper arm exerts a force on the elbow to calculate torques. Choose counterclockwise as the positive direction for the torque; this is the positive $\hat{\mathbf{k}}$-direction. The torque diagram for the action of the tendon is shown below.



The magnitude of the torque about the elbow is the product of the distance d with the magnitude of the tendon force $T \equiv |\vec{\mathbf{T}}|$. The direction of the torque is out of the plane of the figure above, in the positive $\hat{\mathbf{k}}$-direction. So the torque of the tendon about the elbow is given by

$$\vec{\tau}_{S,T} = d\,T\,\hat{\mathbf{k}}. \tag{5.6}$$

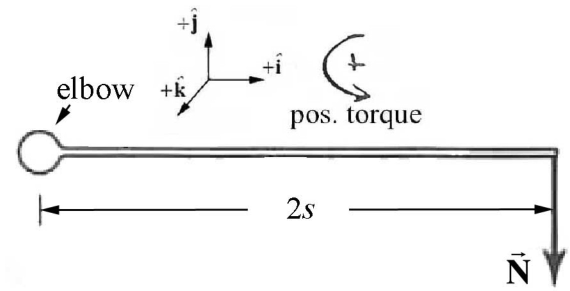
The torque diagram due to the gravitational interaction between the system and the earth is shown below



The magnitude of the torque due to the gravitational force on the system about the elbow is the product of the distance s with the magnitude of the gravitational force $m_1 g$. The direction of the torque is into the plane of the figure above, in the negative $\hat{\mathbf{k}}$-direction. The torque of the gravitational force about the elbow is then given by

$$\vec{\tau}_{S,\text{grav}} = -s\, m_1 g\, \hat{\mathbf{k}}. \tag{5.7}$$

Finally, the torque diagram associated with the normal force is



The magnitude of the torque due to the normal force of the ball about the elbow is the product of the distance $2s$ with the magnitude of the normal force $N = m_2 g$. The direction of the torque is into the plane of the figure above, in the negative $\hat{\mathbf{k}}$-direction. So the torque of the gravitational force about the elbow is given by

$$\vec{\tau}_{S,N} = -2\, s\, N = -2\, s\, m_2 g\, \hat{\mathbf{k}}. \tag{5.8}$$

The condition that the total torque about the point S vanishes,

$$\vec{\tau}_S^{\,\text{total}} = \vec{\tau}_{S,T} + \vec{\tau}_{S,\text{grav}} + \vec{\tau}_{S,N} = \vec{\mathbf{0}} \tag{5.9}$$

becomes

$$\vec{\tau}_{S,N} = d\, T\, \hat{\mathbf{k}} - s\, m_1 g\, \hat{\mathbf{k}} - 2\, s\, m_2 g\, \hat{\mathbf{k}} = \vec{\mathbf{0}}. \tag{5.10}$$

You can solve this equation for the magnitude of the tendon force

$$T = sg(m_1 + 2m_2)/d. \tag{5.11}$$

Note that Equations (5.5) and (5.11) are two equations in the two unknowns F and T, so we have enough information to solve the problem. You can now substitute Equation (5.11), the expression for the tendon force, into the equation for force equilibrium, Equation (5.5), yielding

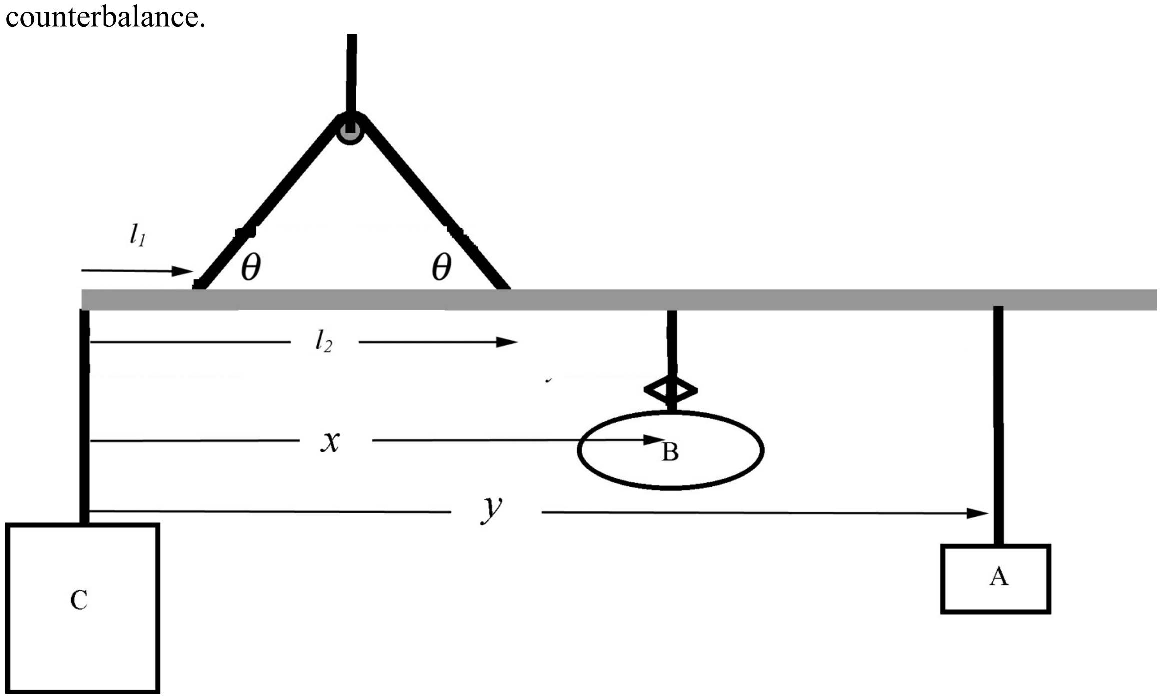
$$-m_2 g - F + sg(m_1 + 2m_2)/d - m_1 g = 0. \tag{5.12}$$

You can then solve Equation (5.12) for the magnitude of the force of the upper arm on the elbow,

$$F = -(m_1 + m_2)g + sg(m_1 + 2m_2)/d. \tag{5.13}$$

Problem 6: Crane

A crane is configured as below, with the beam (which we can define as massless, for simplicity) suspended at two points l_1 and l_2 by each end of a cable passing over a frictionless pulley. The two ends of the cable each make an angle θ with the beam. A counterbalance object C with mass m_C is fixed at one end of the beam. A balance object B of mass m_B is attached to the beam and can move horizontally in order to maintain static equilibrium. The crane lifts an object A with mass m_A at a distance y from the counterbalance.



a) Draw a free body force diagram for the beam.
b) What is the tension in the cable that runs over the pulley, as a function of the masses of the hanging objects and the angle θ between the cable and the beam? Show all your work. Answers without work will not receive any credit.

c) At what horizontal position should one put the balance object B such that the crane doesn't tilt? Show all your work. Answers without work will not receive any credit.

Problem 6 Solutions:

a) (Figure needed) There are five forces, three gravitational forces and two tension forces. The pulley being given as frictionless (probably bad planning for a crane), the magnitudes of the tensions could be taken to be the same at this point. In fact, this is assumed in the phrasing of part b).

b) The net upward force supplied by the cable is $2T\sin\theta$. For equilibrium, this must be equal to the total weight, $(m_A + m_B + m_C)g$. Solving for the tension yields

$$T = \frac{(m_A + m_B + m_C)g}{2\sin\theta}. \tag{6.1}$$

(Note what trouble could occur if the angle θ is too small.)

c) Although any point could be used as the origin for calculating torques, the distances are all measured from the left end of the beam in the above diagram, so using this point as the origin will simplify calculations. Also, the counterbalance (with mass m_C) exerts no torque about this point. There are then four forces exerting torques. Taking the counterclockwise direction to be the positive direction for torques, the two torques due to the cable exert positive torques and the two other weights exert negative torques.
The component of the force perpendicular to the beam from the cable is $T\sin\theta$ for each. The net torque is then

$$\begin{aligned}\tau_{net} &= 0 = l_1 T\sin\theta + l_2 T\sin\theta - x(m_B g) - y(m_A g) \\ &= (l_1 + l_2)(m_A + m_B + m_C)g/2 - x(m_B g) - y(m_A g)\end{aligned} \tag{6.2}$$

where the result from part b) for the tension has been used. Solving the last equation in (6.2) for x, and noting that the factors of g cancel,

$$x = \left((l_1 + l_2)(m_A + m_B + m_C)/2 - y\,m_A\right)/m_B. \tag{6.3}$$