

## Module 23: Static Equilibrium

### 23.1 Introduction Static Equilibrium

We have already seen in section 4.4 that if the vector sum of the forces acting on a point-like object is zero then the object will continue in its state of rest, or of uniform motion in a straight line. If the object is in uniform motion we can always change reference frames so that the object will be at rest. In section 9.2, we showed that for a collection of point-like objects the sum of the external forces may be regarded as acting at the center of mass (Equation 9.2.20). So if that sum is zero the center of mass will continue in its state of rest, or of uniform motion in a straight line. In section 13.1 we introduced the idea of a rigid body, and again showed that in addition to the fact that the sum of the external forces may be regarded as acting at the center of mass, forces like the gravitational force that acts at every point in the body may be treated as acting at the center of mass. However for an extended rigid body it matters where the force is applied because even though the sum of the forces on the body may be zero, a non-zero sum of torques on the body may still produce angular acceleration. In particular for fixed axis rotation, the torque on the object is proportional to the angular acceleration (Equation 13.3.15). It is possible for a body that is not constrained to rotate about a fixed axis that the sum of the torques may be zero and the body still undergoes rotation. So we would like restrict ourselves to the special case in which in an inertial reference frame both the center of mass of the body is at rest and the body does not undergo any rotation, a condition that is called **static equilibrium of an extended object**.

The two sufficient and necessary conditions for a rigid body to be in static equilibrium are:

- (1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}}. \quad (23.1.1)$$

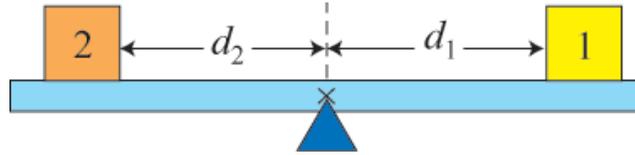
- (2) The vector sum of the torques about any point  $S$  in a rigid body is zero,

$$\vec{\tau}_{S,\text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \cdots = \vec{\mathbf{0}}. \quad (23.1.2)$$

When a body is in static equilibrium, the torques about any two points are equal (see [Appendix 14.A](#)). As a result, when solving static equilibrium problems, wisely choosing the point about which to compute torque can greatly simplify a given problem.

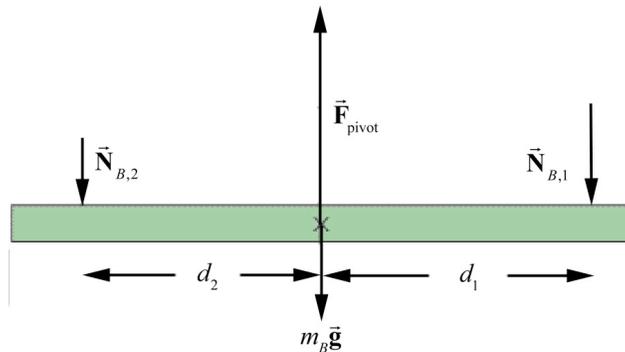
### 23.2 Lever Law

Let's consider a uniform rigid beam of mass  $m_B$  balanced on a pivot near the center of mass of the beam. We place two point-like bodies 1 and 2 of masses  $m_1$  and  $m_2$  on the beam, at distances  $d_1$  and  $d_2$  respectively from the pivot, so that the beam is static (that is, the beam is not rotating). See Figure 23.1. The finite extent of the bodies, as represented in the figure, is not part of this derivation).



**Figure 23.1** Pivoted Lever.

Let's consider the forces acting on the beam. The earth attracts the beam downward. This gravitational force acts on every atom in the beam, but we can summarize its action by stating that the gravitational force near the surface of the earth  $\vec{F}_{\text{gravity}} = m_B \vec{g}$  is concentrated at a point in the beam called the *center of gravity* of the beam, which is identical to the center of mass of the uniform beam. There is also a contact force  $\vec{F}_{\text{pivot}}$  between the pivot and the beam, acting upwards on the beam at the pivot point. The bodies 1 and 2 exert normal forces downwards on the beam,  $\vec{N}_{B,1} \equiv \vec{N}_1$ , and  $\vec{N}_{B,2} \equiv \vec{N}_2$ , with magnitudes  $N_1$ , and  $N_2$ , respectively. Note that the normal forces are not the gravitational forces acting on the bodies, but contact forces between the beam and the body. (In this case, they are mathematically the same, due to the horizontal configuration of the beam and the fact that all objects are in static equilibrium.) The distances  $d_1$  and  $d_2$  are called the *moment arms* with respect to the pivot point for the forces  $\vec{N}_1$  and  $\vec{N}_2$ , respectively. The force diagram on the beam is shown in Figure 23.2. Note that the pivot force  $\vec{F}_{\text{pivot}}$  and the force of gravity  $m_B \vec{g}$  each has a zero moment arm about the pivot point.

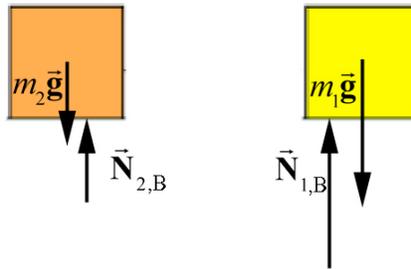


**Figure 23.2** The Lever Law in action.

Because we assume the beam is not moving, the sum of the forces in the vertical direction acting on the beam is therefore zero:

$$F_{\text{pivot}} - m_B g - N_1 - N_2 = 0. \quad (23.2.1)$$

The force diagrams on the bodies are shown in Figure 23.3. Note the magnitude of the normal forces on the bodies are also  $N_1$  and  $N_2$  since these are each part of an action-reaction pair,  $\vec{N}_{B,1} = -\vec{N}_{1,B}$ , and  $\vec{N}_{B,2} = -\vec{N}_{2,B}$ .



**Figures 14.3** Force diagrams for each body.

The condition that the forces sum to zero is not sufficient to completely predict the motion of the beam. All we can deduce is that the center of mass of the system is at rest (or moving with a uniform velocity). In order for the beam not to rotate the sum of the torques about any point must be zero. In particular the sum of the torques about the pivot point must be zero. Since the moment arm of the gravitational force and the pivot force is zero, only the two normal forces produce a torque on the beam. If we choose out of the page as positive direction for the torque (or equivalently counterclockwise rotations are positive) then the condition that the sum of the torques about the pivot point is zero becomes

$$d_2 N_2 - d_1 N_1 = 0. \quad (23.2.2)$$

So the magnitudes of the two torques about the pivot point is zero, a condition known as the lever law.

### Lever Law:

*A beam of length  $l$  is balanced on a pivot point that is placed directly beneath the center of mass of the beam. The beam will not undergo rotation if the product of the normal force with the moment arm to the pivot is the same for each body,*

$$d_1 N_1 = d_2 N_2. \quad (23.2.3)$$

### 23.2.1 Example Lever Law

Suppose a uniform beam of length  $l = 1.0$  m and mass  $m_b = 2.0$  kg is balanced on a pivot point, placed directly beneath the center of the beam. We place body 1 with mass  $m_1 = 0.3$  kg a distance  $d_1 = 0.4$  m to the right of the pivot point, and a second body 2 with  $m_2 = 0.6$  kg a distance  $d_2$  to the left of the pivot point, such that the beam neither translates nor rotates.

- What is the force  $\vec{F}_{\text{pivot}}$  that the pivot exerts on the beam?
- What is the distance  $d_2$  that maintains static equilibrium?

**Solution:**

a) By Newton's Third Law, the beam exerts equal and opposite normal forces of magnitude  $N_1$  on body 1, and  $N_2$  on body 2. The condition for force equilibrium applied separately to the two bodies yields

$$N_1 - m_1 g = 0, \tag{23.2.4}$$

$$N_2 - m_2 g = 0. \tag{23.2.5}$$

Thus the total force acting on the beam is zero,

$$F_{\text{pivot}} - (m_b + m_1 + m_2)g = 0, \tag{23.2.6}$$

and the pivot force is

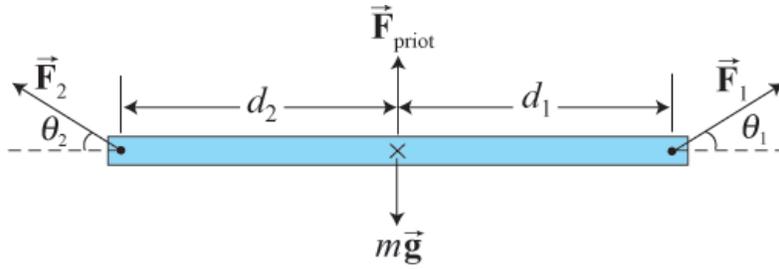
$$\begin{aligned} F_{\text{pivot}} &= (m_b + m_1 + m_2)g \\ &= (2.0 \text{ kg} + 0.3 \text{ kg} + 0.6 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) = 2.8 \times 10^1 \text{ N}. \end{aligned} \tag{23.2.7}$$

b) We can compute the distance  $d_2$  from the Lever Law,

$$d_2 = \frac{d_1 N_1}{N_2} = \frac{d_1 m_1 g}{m_2 g} = \frac{d_1 m_1}{m_2} = \frac{(0.4 \text{ m})(0.3 \text{ kg})}{0.6 \text{ kg}} = 0.2 \text{ m}. \tag{23.2.8}$$

### 23.3 Generalized Lever Law

We can extend the Lever Law to the case in which two external forces  $\vec{F}_1$  and  $\vec{F}_2$  are acting on the beam at angles  $\theta_1$  and  $\theta_2$  with respect to the horizontal as shown in Figure 23.4. Throughout this discussion the angles will be limited to the range  $[0 \leq \theta_1, \theta_2 \leq \pi]$ .

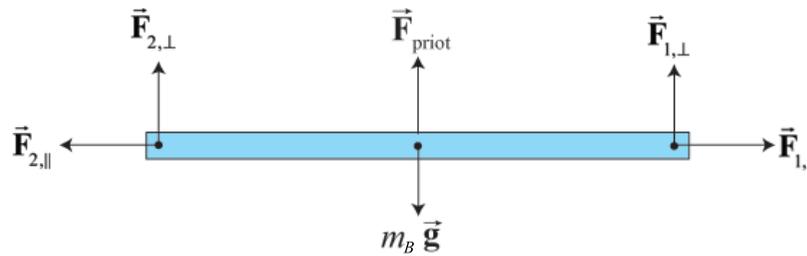


**Figure 23.4** Forces acting at angles to a pivoted beam.

Since force is a vector quantity, the forces  $\vec{F}_1$  and  $\vec{F}_2$  can be decomposed into separate vectors components respectively  $(\vec{F}_{1,\parallel}, \vec{F}_{1,\perp})$  and  $(\vec{F}_{2,\parallel}, \vec{F}_{2,\perp})$ , where  $F_{1,\parallel}$  and  $F_{2,\parallel}$  are the horizontal vector projections of the two forces with respect to the direction formed by the length of the beam, and  $\vec{F}_{1,\perp}$  and  $\vec{F}_{2,\perp}$  are the perpendicular vector projections respectively to the beam (Figure 23.5), with

$$\mathbf{F}_1 = \mathbf{F}_{1,\parallel} + \mathbf{F}_{1,\perp}, \quad (23.3.1)$$

$$\mathbf{F}_2 = \mathbf{F}_{2,\parallel} + \mathbf{F}_{2,\perp}. \quad (23.3.2)$$



**Figure 23.5** Vector decomposition of forces.

The horizontal components of the forces are

$$F_{1,\parallel} = F_1 \cos\theta_1, \quad (23.3.3)$$

$$F_{2,\parallel} = -F_2 \cos\theta_2, \quad (23.3.4)$$

where our choice of positive horizontal direction is to the right. Neither horizontal force component contributes to possible rotational motion of the beam. The sum of these horizontal forces must be zero,

$$F_1 \cos\theta_1 - F_2 \cos\theta_2 = 0. \quad (23.3.5)$$

The perpendicular component forces are

$$F_{1,\perp} = F_1 \sin \theta_1, \quad (23.3.6)$$

$$F_{2,\perp} = F_2 \sin \theta_2, \quad (23.3.7)$$

where the positive vertical direction is upwards. The perpendicular components of the forces must also sum to zero,

$$F_{\text{pivot}} - m_B g + F_1 \sin \theta_1 + F_2 \sin \theta_2 = 0. \quad (23.3.8)$$

Only the vertical components  $F_{1,\perp}$  and  $F_{2,\perp}$  of the external forces are involved in the lever law (but the horizontal components must balance, as in Equation (23.3.5), for equilibrium). Thus the Lever Law can be extended as follows.

### Generalized Lever Law

*A beam of length  $l$  is balanced on a pivot point that is placed directly beneath the center of mass of the beam. Suppose a force  $\vec{F}_1$  acts on the beam a distance  $d_1$  to the right of the pivot point. A second force  $\vec{F}_2$  acts on the beam a distance  $d_2$  to the left of the pivot point. The beam will remain in static equilibrium if the following two conditions are satisfied:*

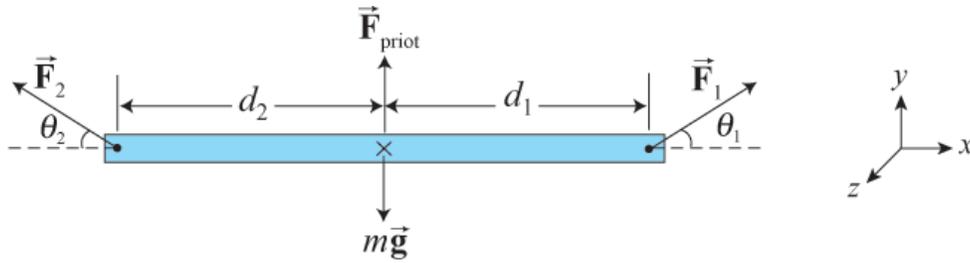
- 1) *The total force on the beam is zero,*
- 2) *The product of the magnitude of the perpendicular component of the force with the distance to the pivot is the same for each force,*

$$d_1 |F_{1,\perp}| = d_2 |F_{2,\perp}|. \quad (23.3.9)$$

*The Generalized Lever Law can be stated in an equivalent form,*

$$d_1 F_1 \sin \theta_1 = d_2 F_2 \sin \theta_2. \quad (23.3.10)$$

We shall now show that the generalized lever law can be reinterpreted as the statement that the vector sum of the torques about the pivot point  $S$  is zero when there are just two forces  $\vec{F}_1$  and  $\vec{F}_2$  acting on our beam as shown in Figure 23.6.



**Figure 23.6** Force and torque diagram.

Let's choose the  $+z$  -direction to point out of the plane of the page then torque pointing out of the page will have a positive  $z$  -component of torque (counterclockwise rotations are positive). From our definition of torque about the pivot point (Equation 13.2.2), the magnitude of torque due to force  $\vec{F}_1$  is given by

$$\tau_{s,1} = d_1 F_1 \sin \theta_1. \quad (23.3.11)$$

From the right hand rule this is out of the page (in the counterclockwise direction) so the component of the torque is positive, hence,

$$(\tau_{s,1})_z = d_1 F_1 \sin \theta_1. \quad (23.3.12)$$

The torque due to  $\vec{F}_2$  about the pivot point is into the page (the clockwise direction) and the component of the torque is negative and given by

$$(\tau_{s,2})_z = -d_2 F_2 \sin \theta_2. \quad (23.3.13)$$

The total component of the torque is the sum of the individual torques and is zero,

$$(\tau_{s,\text{total}})_z = (\tau_{s,1})_z + (\tau_{s,2})_z = d_1 F_1 \sin \theta_1 - d_2 F_2 \sin \theta_2 = 0. \quad (23.3.14)$$

which is equivalent to the Generalized Lever Law, Equation (23.3.10),

$$d_1 F_1 \sin \theta_1 = d_2 F_2 \sin \theta_2.$$

## 23.4 Worked Examples

### 23.4.1 Example Estimate Torque on a Tire iron

Your car has a flat and you try to loosen the lugs on the wheel with a tire iron shown in the figure on the left below. You can't budge the lugs but then you try the 4-way wrench shown in the figure on the right and the lugs loosen. Based on the Figure 23.9, estimate how much torque you needed to apply to loosen the lugs.



**Figure 23.9** Force and torque diagram.

**Possible Answer:** There are many considerations in this problem, and if you've had some experience with changing flat tires, you're aware of what they are. Your experience may suggest that the 4-way is overall easier to use.

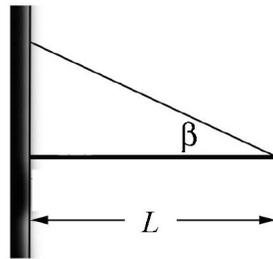
Start by keeping in mind that the car should be up on a jack, and many jacks don't supply good overall stability. Applying too much net force, up or down, on the wheel could cause the car to fall off the jack. As an example, for a 1983 VW Rabbit (since departed), the car could be lifted off the jack. Typically, a person can exert an upward force of 2-3 times the person's weight. So, using the simple wrench on the left and pulling up with both hands warrants caution. Pushing down, the maximum downward force exerted can't exceed the person's weight. As an estimate of the net torque, use a force with magnitude equal to your weight and a moment arm of about half a meter. For me, that's about  $500 \text{ N}\cdot\text{m}$ . Of course, the angle between the applied force and the wrench should be  $90^\circ$ ; the wrench arm should be horizontal.

For the 4-way wrench, assume the same magnitude of force applied on each of the two arms used to turn the wrench, one force directed up and the other down. Ideally, this would mean that you apply no net force the car, and falling off the jack is less likely. Also, you would be able to position yourself more or less symmetric about the wheel, making the process a bit more comfortable. (But the forces on your feet wouldn't be the same – see Problem 3.) The forces you exert then form a “couple,” and the torque would be the product of the applied force and the moment arm, times 2 for the two arms of the wrench. From the figure, it looks like the moment arms are about the same as for the simple wrench, so the net torque is roughly your weight times the moment arm of about half a meter times 2, or  $1000 \text{ N}\cdot\text{m}$ ; much easier to undo the lug nuts.

### 23.4.2 Example Suspended Rod

A uniform rod of length  $l = 2.0 \text{ m}$  and mass  $m = 4.0 \text{ kg}$  is hinged to a wall at one end and suspended from the wall by a cable that is attached to the other end of the rod at an angle of  $\beta = 30^\circ$  to the rod (see figure below). Assume the cable has zero mass. There is a contact force at the pivot on the rod. The magnitude and direction of this force is unknown. One of the most difficult parts of these types of problems is to introduce an angle for the pivot force and then solve for that angle if possible. In this problem you will

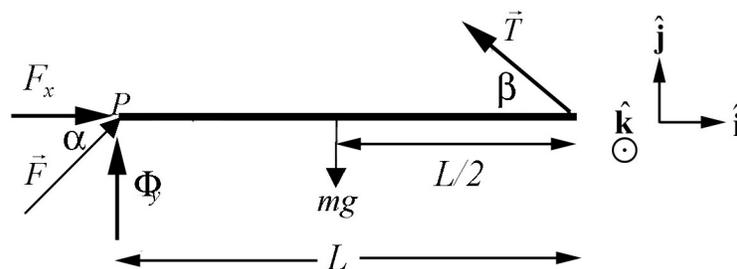
solve for the magnitude of the tension in the cable and the direction and magnitude of the pivot force.



- Draw a free body force diagram for the beam. Clearly indicate your choice of angle for the pivot force.
- What are the equations for static equilibrium of the forces?
- About which point will you choose to analyze the torques? What are the equations for static equilibrium of the torques?
- What is the tension in the cable?
- What angle does the pivot force make with the beam?
- What is the magnitude of the pivot force?

**Solution:**

- The force diagram is shown in the figure below.



Take the positive  $\hat{i}$ -direction to be to the right in the figure above, and take the positive  $\hat{j}$ -direction to be vertically upward. The forces on the rod are: the gravitational force  $m\vec{g} = -m g \hat{j}$ , acting at the center of the rod; the force that the cable exerts on the rod,  $\vec{T} = T(-\cos\beta\hat{i} + \sin\beta\hat{j})$ , acting at the right end of the rod; and the pivot force  $\vec{F}_{\text{pivot}} = F(\cos\alpha\hat{i} + \sin\alpha\hat{j})$ , acting at the left end of the rod. If  $0 < \alpha < \pi/2$ , the pivot force is directed up and to the right in the figure. If  $0 > \alpha > -\pi/2$ , the pivot force is

directed down and to the right. We have no reason, at this point, to expect that  $\alpha$  will be in either of the quadrants, but it must be in one or their other.

b) For static equilibrium, the sum of the forces must be zero, and hence the sums of the components of the forces must be zero;

$$\begin{aligned}0 &= -T \cos \beta + F \cos \alpha \\0 &= -m g + T \sin \beta + F \sin \alpha.\end{aligned}$$

c), d) While any point may be chosen as the origin for the purposes of finding torques, consideration of the problem may give some idea of what the most advantageous point would be. Since part d) asks for the tension  $T$ , choosing a point that does not involve the other unknown  $\vec{F}_{\text{pivot}}$  should allow determination of  $T$ . Accordingly, find the torques with respect to the pivot point.

With respect to the pivot point, and taking positive torques to be counterclockwise, the gravitational force exerts a negative torque of magnitude  $m g(l/2)$  and the cable exerts a positive torque of magnitude  $T l \sin \beta$ . The pivot force exerts no torque about the pivot (which is sort of the idea). Setting the net torque equal to zero then gives

$$\begin{aligned}0 &= T l \sin \beta - m g(l/2) \\T &= \frac{m g}{2 \sin \beta}.\end{aligned}$$

This result has many features we would expect; proportional to the weight of the rod and inversely proportional to the sine of the angle made by the cable with respect to the horizontal. Inserting numerical values gives

$$T = \frac{m g}{2 \sin \beta} = \frac{(4.0 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2})}{2 \sin 30^\circ} = 39.2 \text{ N}.$$

e) There are many ways to find the angle  $\alpha$ . The method presented here uses the results of the previous parts. Specifically, substituting the above expression for the tension into the force equations yields

$$\begin{aligned}F \cos \alpha &= T \cos \beta = \frac{m g}{2} \cot \beta \\F \sin \alpha &= m g - T \sin \beta = \frac{m g}{2}.\end{aligned}$$

From the above, dividing one equation by the other, we see that  $\tan \alpha = \tan \beta$ ,  $\alpha = \beta$ .

As an alternative, if we had not done the previous parts, we could find torques about the point where the cable is attached to the wall. The cable exerts no torque about this point, and the  $y$ -component of the pivot force exerts no torque as well. The moment arm of the  $x$ -component of the pivot force is  $l \tan \beta$  and the moment arm of the weight is  $l/2$ . Equating the magnitudes of these two torques gives

$$F \cos \alpha l \tan \beta = mg \frac{l}{2},$$

equivalent to one of the results found above. Similarly, evaluating torques about the right end of the rod, the cable exerts no torques and the  $x$ -component of the pivot force exerts no torque. The moment arm of the  $y$ -component of the pivot force is  $l$  and the moment arm of the weight is  $l/2$ . Equating the magnitudes of these two torques gives

$$F \sin \alpha l = mg \frac{l}{2},$$

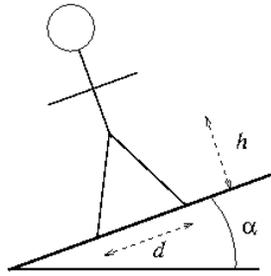
again reproducing a previous result.

The point of this alternative solution is to show that if the questions had been asked in a different order, choosing a different origin (or even more than one origin) in order to remove an unknown force from the torques equations might give a desired result more directly.

f) The horizontal forces on the rod must cancel. The tension force and the pivot force act with the same angle (but in opposite horizontal directions) and hence must have the same magnitude;  $F = T = 39.2 \text{ N}$ .

### 23.4.3 Example Person standing on a Hill

A person is standing on a hill that is sloped at an angle of  $\alpha$  with respect to the horizontal (Figure 23.11). The person's legs are separated by a distance  $d$ , with one foot uphill and one downhill. The center of mass of the person is at a distance  $h$  above the ground, perpendicular to the hillside, midway between the person's feet. Assume that the coefficient of static friction between the person's feet and the hill is sufficiently large that the person will not slip.

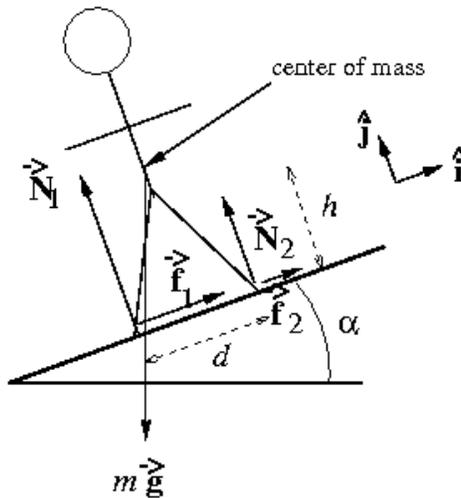


**Figure 23.11** Person standing on hill

- What is the magnitude of the normal force on each foot?
- How far must the feet be apart so that the normal force on the upper foot is just zero? This is the moment when the person starts to rotate and fall over.

**Solution:**

The force diagram on the person is shown in Figure 23.12. Note that the contact forces have been decomposed into components perpendicular and parallel to the hillside. A choice of unit vectors and positive direction for torque is also shown.



**Figure 23.12** Free body diagram for person standing on hill

Applying Newton's Second Law to the two components of the net force,

$$\hat{\mathbf{j}}: N_1 + N_2 - mg \cos \alpha = 0 \quad (23.4.1)$$

$$\hat{\mathbf{i}}: f_1 + f_2 - mg \sin \alpha = 0. \quad (23.4.2)$$

These two equations imply that

$$N_1 + N_2 = mg \cos \alpha \quad (23.4.3)$$

$$f_1 + f_2 = mg \sin \alpha . \quad (23.4.4)$$

Evaluating torques about the center of mass,

$$h(f_1 + f_2) + (N_2 - N_1)\frac{d}{2} = 0 . \quad (23.4.5)$$

Equation (23.4.5) can be rewritten as

$$N_1 - N_2 = \frac{2h(f_1 + f_2)}{d} . \quad (23.4.6)$$

Substitution of Equation (23.4.4) into Equation (23.4.6) yields

$$N_1 - N_2 = \frac{2h(mg \sin \alpha)}{d} . \quad (23.4.7)$$

We can solve for  $N_1$  by adding Equations (23.4.3) and (23.4.7) and dividing by 2, giving

$$N_1 = \frac{1}{2}mg \cos \alpha + \frac{h(mg \sin \alpha)}{d} = mg \left( \frac{1}{2} \cos \alpha + \frac{h}{d} \sin \alpha \right) . \quad (23.4.8)$$

Similarly, we can solve for  $N_2$  by subtracting Equation (23.4.7) from Equation (23.4.3) and dividing by 2, giving

$$N_2 = mg \left( \frac{1}{2} \cos \alpha - \frac{h}{d} \sin \alpha \right) . \quad (23.4.9)$$

The normal force  $N_2$  as given in Equation (23.4.9) vanishes when

$$\frac{1}{2} \cos \alpha = \frac{h}{d} \sin \alpha , \quad (23.4.10)$$

which can be solved for the minimum distance between the legs,

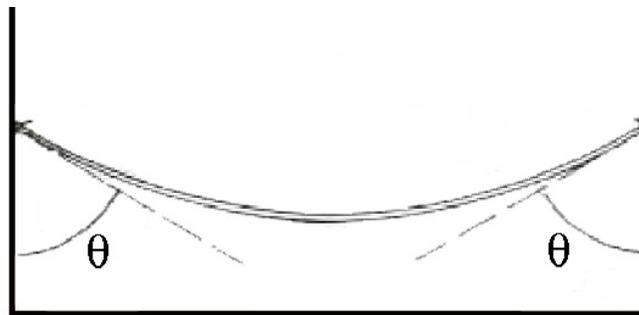
$$d = 2h(\tan \alpha) . \quad (23.4.11)$$

In the above figures,  $\alpha = 20^\circ$ ,  $2 \tan \alpha = 0.73$  and the stick-figure person is very close to tipping over.

It should be noted that no specific model for the friction force was used, that is, no coefficient of static friction entered the problem. The two friction forces  $f_1$  and  $f_2$  were not determined separately; only their sum entered the above calculations.

### 23.4.4 Example Static Equilibrium: Rope Between Trees

Suppose a rope of mass  $m = 0.1\text{kg}$  is connected at the same height to two walls and is allowed to hang under its own weight. At both contact points between the rope and the wall, the rope makes an angle  $\theta = 60^\circ$  with respect to the vertical (Figure 23.13). In order to find the tension in the rope at the ends and at the middle of the rope, you will need to think cleverly about what to include as the system in your free body diagram.



**Figure 23.13** Rope suspended between two trees

- What is the tension at the ends of the rope where they are connected to the wall? Include in your answer your free body force diagram. Show all the forces acting on the rope and your choice of unit vectors.
- What is the tension in the rope at the point midway between the walls? Include in your answer your free body force diagram. Show all the forces acting on the rope and your choice of unit vectors.

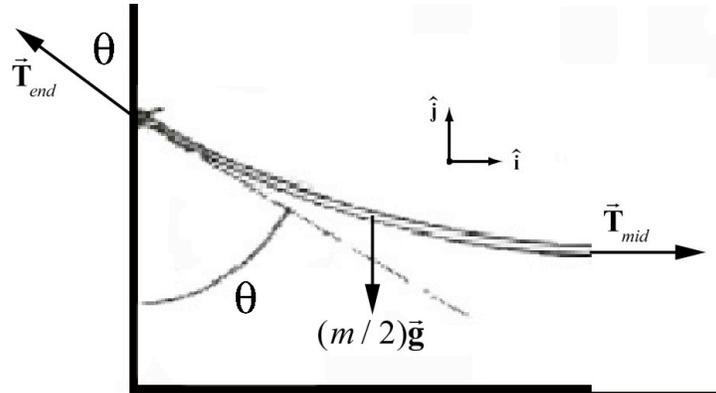
#### **Solution:**

The key to this problem is in understanding how to choose a force diagram for an extended body. Note that we are trying to find the tension at the ends of the rope and at the midpoint. We defined tension at a point in a rope to be:

The *tension*  $T(x)$  in a rope at a distance  $x$  from one end of the rope is the magnitude of the action-reaction pair of forces acting at the point  $x$ ,

$$T(x) = \left| \vec{F}_{\text{left, right}}(x) \right| = \left| \vec{F}_{\text{right, left}}(x) \right|. \quad (23.4.12)$$

This definition suggests that we need to slice the rope at the midpoint to calculate the tension at the midpoint. Let's consider then only half the rope. The forces on half the rope are the tension  $\vec{T}_{\text{end}}$  at the end, the tension  $\vec{T}_{\text{mid}}$  at the midpoint and the gravitational force between the half of the rope and the earth,  $(m/2)\vec{g}$ . The free body diagram for the left half of the rope is shown in Figure 23.14, with a suitable choice of unit vectors.



**Figure 23.14** Free body diagram for left half of rope

Since the rope is in static equilibrium, the sum of the forces is zero. The sum of the components of the forces in the  $\hat{j}$ -direction is zero,

$$\hat{j}: \left| \vec{T}_{\text{end}} \right| \cos \theta - (m/2)g = 0. \quad (23.4.13)$$

Therefore the tension at the end is

$$\left| \vec{T}_{\text{end}} \right| = \frac{(m/2)g}{\cos \theta} = \frac{((0.1\text{kg})/2)(9.8 \text{ m} \cdot \text{s}^{-2})}{\cos 60^\circ} = 0.98 \text{ N}. \quad (23.4.14)$$

The sum of the components of the forces in the  $\hat{i}$ -direction is zero.

$$\hat{i}: -\left| \vec{T}_{\text{end}} \right| \sin \theta + \left| \vec{T}_{\text{mid}} \right| = 0. \quad (23.4.15)$$

Substitute Equation (23.4.14) into Equation (23.4.15), yielding

$$\frac{-(m/2)g \sin \theta}{\cos \theta} + \left| \vec{T}_{\text{mid}} \right| = 0. \quad (23.4.16)$$

Then solve for tension in the middle of the rope,

$$\begin{aligned} |\vec{T}_{\text{mid}}| &= (m/2)g \tan \theta = ((0.1 \text{ kg})/2)(9.8 \text{ m} \cdot \text{s}^{-2}) \tan 60^\circ = |\vec{T}_{\text{mid}}| \sin 60^\circ \\ &= 0.85 \text{ N.} \end{aligned} \quad (23.4.17)$$

### 23.4.5 Example The Knee

A man of mass  $m = 70 \text{ kg}$  is about to start a race. Assume the runner's weight is equally distributed on both legs. The patellar ligament in the knee is attached to the upper tibia and runs over the kneecap. When the knee is bent, a tensile force,  $\vec{T}$ , that the ligament exerts on the upper tibia, is directed at an angle of  $\theta = 40^\circ$  with respect to the horizontal. The femur exerts a force  $\vec{F}$  on the upper tibia. The angle,  $\alpha$ , that this force makes with the vertical will vary and is one of the unknowns to solve for. Assume that the ligament is connected a distance,  $d = 3.8 \text{ cm}$ , directly below the contact point of the femur on the tibia. The contact point between the foot and the ground is a distance  $s = 3.6 \times 10^1 \text{ cm}$  from the vertical line passing through contact point of the femur on the tibia. The center of mass of the lower leg lies a distance  $x = 1.8 \times 10^1 \text{ cm}$  from this same vertical line. Suppose the mass  $m_L$  of the lower leg is a 1/10 of the mass of the body (Figure 23.15).

- Find the magnitude  $T$  of the force  $\vec{T}$  of the patellar ligament on the tibia.
- Find the direction (the angle  $\alpha$ ) of the force  $\vec{F}$  of the femur on the tibia.
- Find the magnitude  $F$  of the force  $\vec{F}$  of the femur on the tibia.

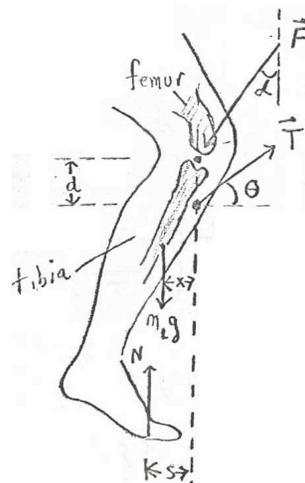


Figure 23.15 Knee

For a more detailed picture of this “articulation,” see [Illustrations. Fig. 345. Gray, Henry. 1918. Anatomy of the Human Body.](#)

**Solutions:**

a) Choose the unit vector  $\hat{\mathbf{i}}$  to be directed horizontally to the right and  $\hat{\mathbf{j}}$  directed vertically upwards. The two conditions for static equilibrium are

(1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = \vec{\mathbf{0}}. \quad (23.4.18)$$

(2) The vector sum of the torques about any point S in a rigid body is zero,

$$\vec{\tau}_{S,\text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \vec{\tau}_{S,3} = \vec{\mathbf{0}}. \quad (23.4.19)$$

The first condition that the sum of the forces is zero becomes

$$\begin{aligned} \hat{\mathbf{i}}: & \quad -F \sin \alpha + T \cos \theta = 0 \\ \hat{\mathbf{j}}: & \quad N - F \cos \alpha + T \sin \theta - (1/10)mg = 0. \end{aligned} \quad (23.4.20)$$

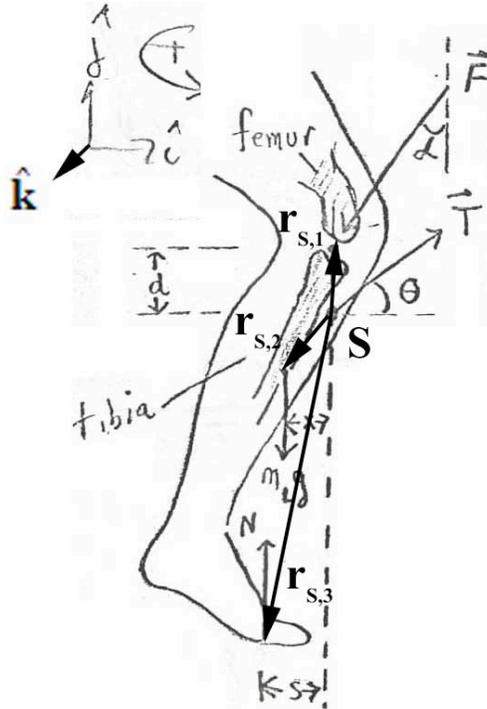
Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or

$$N = (1/2)mg; \quad (23.4.21)$$

the second equation in (23.4.20) becomes

$$\begin{aligned} \hat{\mathbf{j}}: & \quad (1/2)mg - F \cos \alpha + T \sin \theta - (1/10)mg = 0 \\ & \quad (2/5)mg - F \cos \alpha + T \sin \theta = 0. \end{aligned} \quad (23.4.22)$$

The torque-force diagram on the knee is shown in Figure 23.16.



**Figure 23.16** Torque-force diagram for knee

Choose the point of action of the ligament on the tibia as the point  $S$  about which to compute torques. Note that the tensile force,  $\vec{T}$ , that the ligament exerts on the upper tibia will make no contribution to the torque about this point  $S$ . This may help slightly in doing the calculations. Choose counterclockwise as the positive direction for the torque; this is the positive  $\hat{\mathbf{k}}$  - direction.

Then the torque due to the force  $\vec{F}$  of the femur on the tibia is

$$\vec{\tau}_{S,1} = \vec{r}_{S,1} \times \vec{F} = d \hat{\mathbf{j}} \times (-F \sin \alpha \hat{\mathbf{i}} - F \cos \alpha \hat{\mathbf{j}}) = d F \sin \alpha \hat{\mathbf{k}}. \quad (23.4.23)$$

The torque due to the mass of the leg is

$$\vec{\tau}_{S,2} = \vec{r}_{S,2} \times (-mg/10) \hat{\mathbf{j}} = (-x \hat{\mathbf{i}} - y_L \hat{\mathbf{j}}) \times (-mg/10) \hat{\mathbf{j}} = (1/10)x mg \hat{\mathbf{k}}. \quad (23.4.24)$$

The torque due to the normal force of the ground is

$$\vec{\tau}_{S,3} = \vec{r}_{S,3} \times N \hat{\mathbf{j}} = (-s \hat{\mathbf{i}} - y_N \hat{\mathbf{j}}) \times N \hat{\mathbf{j}} = -s N \hat{\mathbf{k}} = -(1/2)s mg \hat{\mathbf{k}}. \quad (23.4.25)$$

(In Equations (23.4.24) and (23.4.25),  $y_L$  and  $y_N$  are the vertical displacements of the point where the weight of the leg and the normal force with respect to the point  $S$ ; as can be seen, these quantities do not enter directly into the calculations.)

The condition that the total torque about the point  $S$  vanishes,

$$\vec{\tau}_{S, \text{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \vec{\tau}_{S,3} = \vec{0}, \quad (23.4.26)$$

then becomes

$$d F \sin \alpha \hat{\mathbf{k}} + (1/10)x mg \hat{\mathbf{k}} - (1/2)s mg \hat{\mathbf{k}} = \vec{0}. \quad (23.4.27)$$

The torque equation to be used is then

$$d F \sin \alpha + (1/10)x mg - (1/2)s mg = 0 \quad (23.4.28)$$

The three equations in the three unknowns are summarized below:

$$\begin{aligned} -F \sin \alpha + T \cos \theta &= 0 \\ (2/5)mg - F \cos \alpha + T \sin \theta &= 0 \\ d F \sin \alpha + (1/10)x mg - (1/2)s mg &= 0. \end{aligned} \quad (23.4.29)$$

The horizontal force equation, the first in (23.4.29), implies that

$$F \sin \alpha = T \cos \theta. \quad (23.4.30)$$

Substituting this into the torque equation, the third of (23.4.29), yields

$$dT \cos \theta + (1/10)x mg - s(1/2)mg = 0. \quad (23.4.31)$$

It is essential that you understand that Equation (23.4.31) is the equation that would have been obtained if we had chosen the contact point between the tibia and the femur as the point about which to determine torques. Had we chosen this point, we would have saved one minor algebraic step.

We can solve this Equation (23.4.31) for the magnitude  $T$  of the force  $\vec{T}$  of the patellar ligament on the tibia,

$$T = \frac{s(1/2)mg - (1/10)x mg}{d \cos \theta}. \quad (23.4.32)$$

Inserting numerical values into Equation (23.4.32),

$$\begin{aligned} T &= (70 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) \frac{(3.6 \times 10^{-1} \text{ m})(1/2) - (1/10)(1.8 \times 10^{-1} \text{ m})}{(3.8 \times 10^{-2} \text{ m}) \cos(40^\circ)} \\ &= 3.8 \times 10^3 \text{ N}. \end{aligned} \quad (23.4.33)$$

b) We can now solve for the direction  $\alpha$  of the force  $\vec{F}$  of the femur on the tibia as follows. Rewrite the two force equations in (23.4.29) as

$$\begin{aligned} F \cos \alpha &= (2/5)mg + T \sin \theta \\ F \sin \alpha &= T \cos \theta. \end{aligned} \quad (23.4.34)$$

Dividing these equations yields

$$\frac{F \cos \alpha}{F \sin \alpha} = \cotan \alpha = \frac{(2/5)mg + T \sin \theta}{T \cos \theta}, \quad (23.4.35)$$

And so

$$\begin{aligned} \alpha &= \cotan^{-1} \left( \frac{(2/5)mg + T \sin \theta}{T \cos \theta} \right) \\ \alpha &= \cotan^{-1} \left( \frac{(2/5)(70 \text{ kg})(9.8 \text{ m} \cdot \text{s}^{-2}) + (3.4 \times 10^3 \text{ N}) \sin(40^\circ)}{(3.4 \times 10^3 \text{ N}) \cos(40^\circ)} \right) = 47^\circ. \end{aligned} \quad (23.4.36)$$

c) We can now use the horizontal force equation to calculate the magnitude  $F$  of the force of the femur  $\vec{F}$  on the tibia from Equation (23.4.30),

$$F = \frac{(3.8 \times 10^3 \text{ N}) \cos(40^\circ)}{\sin(47^\circ)} = 4.0 \times 10^3 \text{ N}. \quad (23.4.37)$$

Note you can find a symbolic expression for  $\alpha$  that did not involve the intermediate numerical calculation of the tension. This is rather complicated algebraically; basically, the last two equations in (23.4.29) are solved for  $F$  and  $T$  in terms of  $\alpha$ ,  $\theta$  and the other variables (Cramer's Rule is suggested) and the results substituted into the first of (23.4.29). The resulting expression is

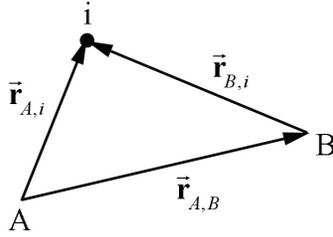
$$\begin{aligned} \cot \alpha &= \frac{(s/2 - x/10) \sin(40^\circ) + ((2d/5) \cos(40^\circ))}{(s/2 - x/10) \cos(40^\circ)} \\ &= \tan(40^\circ) + \frac{2d/5}{s/2 - x/10} \end{aligned} \quad (23.4.38)$$

which leads to the same numerical result,  $\alpha = 47^\circ$ .

## Appendix 23.A The Torques About any Two Points are Equal for a Body in Static Equilibrium

When the net force on a body is zero, the torques about any two points are equal. To show this, consider any two points  $A$  and  $B$ . Choose a coordinate system with origin  $O$  and denote the constant vector from  $A$  to  $B$  by  $\vec{r}_{A,B}$ . Suppose a set of forces  $\vec{F}_i$  are acting at the points  $\vec{r}_{O,i}$ ,  $i = 1, 2, \dots$ . The vectors from the point  $A$  to the points where the respective forces act are denoted by  $\vec{r}_{A,1}, \vec{r}_{A,2}, \dots$ , and the vectors from the point  $B$  to the points where the forces act are denoted by  $\vec{r}_{B,1}, \vec{r}_{B,2}, \dots$ . Note that (Figure 23.A.1) for each point  $i = 1, 2, \dots$ ,

$$\vec{r}_{A,i} = \vec{r}_{A,B} + \vec{r}_{B,i} . \quad (23.A.1)$$



**Figure 23.A.1** Location of a body  $i$  with respect to the points  $A$  and  $B$ .

Then the total torque about the point  $A$  is given by

$$\vec{\tau}_{A,\text{total}} = \vec{r}_{A,1} \times \vec{F}_1 + \vec{r}_{A,2} \times \vec{F}_2 + \dots = \sum_{i=1}^{i=N} \vec{r}_{A,i} \times \vec{F}_i . \quad (23.A.2)$$

The total torque about the point  $B$  is given by

$$\vec{\tau}_{B,\text{total}} = \vec{r}_{B,1} \times \vec{F}_1 + \vec{r}_{B,2} \times \vec{F}_2 + \dots = \sum_{i=1}^{i=N} \vec{r}_{B,i} \times \vec{F}_i . \quad (23.A.3)$$

We can now substitute Equation (23.A.1) into Equation (23.A.2) and find that

$$\vec{\tau}_{A,\text{total}} = \sum_{i=1}^{i=N} \vec{r}_{A,i} \times \vec{F}_i = \sum_{i=1}^{i=N} (\vec{r}_{A,B} + \vec{r}_{B,i}) \times \vec{F}_i = \sum_{i=1}^{i=N} \vec{r}_{A,B} \times \vec{F}_i + \sum_{i=1}^{i=N} \vec{r}_{B,i} \times \vec{F}_i . \quad (23.A.4)$$

In the next-to-last term in Equation (23.A.4), the vector  $\vec{r}_{A,B}$  is constant and so may be taken outside the summation,

$$\sum_{i=1}^{i=N} \vec{r}_{A,B} \times \vec{F}_i = \vec{r}_{A,B} \times \sum_{i=1}^{i=N} \vec{F}_i. \quad (23.A.5)$$

We are assuming that there is no net force on the body, and so the sum of the forces on the body is zero,

$$\vec{r}_{A,B} \times \sum_{i=1}^{i=N} \vec{F}_i = \vec{0}. \quad (23.A.6)$$

Therefore the torque about point  $A$ , Equation (23.A.2), becomes

$$\vec{\tau}_{A,\text{total}} = \sum_{i=1}^{i=N} \vec{r}_{B,i} \times \vec{F}_i = \vec{\tau}_{B,\text{total}}. \quad (23.A.7)$$

For static equilibrium problems, the result of Equation (23.A.7) tells us that it does not matter which point we use to determine torques. In fact, note that the position of the chosen origin did not affect the result at all. Choosing the point about to calculate torques (variously called “ $A$ ”, “ $B$ ”, “ $S$ ” or sometimes “ $O$ ” in these notes) so that unknown forces do not exert torques about that point may often greatly simplify calculations.

Later in these notes, we will want to consider rigid bodies that may be accelerating, so that in Equation (23.A.6) we would need to replace the sum of the forces by the product of the total mass and the vector acceleration, with interesting results.

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