Two Dimensional Kinematics
Challenge Problem Solutions

Problem 1:

Suppose a MIT student wants to row across the Charles River. Suppose the water is moving downstream at a constant rate of 1.0 m/s. A second boat is floating downstream with the current. From the second boat's viewpoint, the student is rowing perpendicular to the current at 0.5 m/s. Suppose the river is 800 m wide.

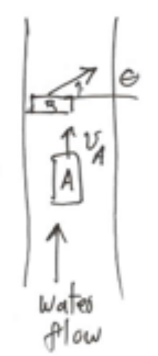
a) What is the direction and magnitude of the velocity of the student as seen from an observer at rest along the bank of the river?

b) How far down river does the student land on the opposite bank?

c) How long does the student take to reach the other side?

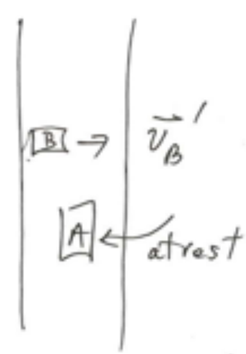
Problem 1 Solutions:

From observer on land



Boat B moves at an angle with respect to the land.

From observer on Boat A



Boat B moves horizontally to the current

$$\vec{v}_B = \vec{v}'_B + \vec{v}$$

$$\vec{v} = v_A \hat{j} \quad , \quad \vec{v}'_B = v'_B \hat{\imath}$$

$$\vec{v}_B = v'_B \hat{\imath} + v_A \hat{j}$$

$$\left|\vec{v}_B\right| = \left(\left(v'_B\right) + \left(v_A^2\right)\right)^{\frac{1}{2}}$$

$$v_A = 1 m \cdot s^{-1} \quad , \quad v'_B = 0.5 m \cdot s^{-1}$$

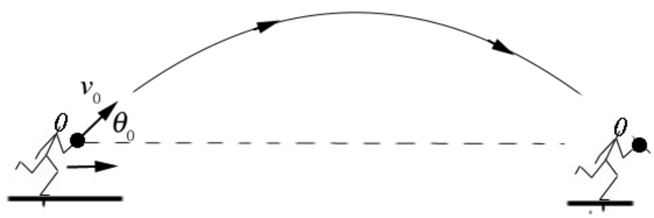
$$\left|\vec{v}_B\right| = \left(\left(1 m \cdot s^{-1}\right)^2 + \left(0.5 m \cdot s^{-1}\right)^2\right)^{\frac{1}{2}} = 1.12 m \cdot s^{-1}$$

$$\tan\theta - v_A / v'_B = 1.0 m \cdot s^{-1} / 0.5 m \cdot s^{-1} = 2$$

$$\theta \tan^{-1}\left(2\right) = 63.4^{\circ}$$

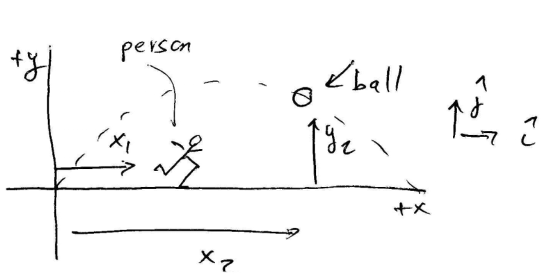
Problem 2:

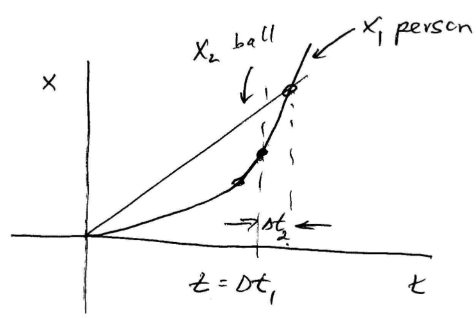
A person initially at rest throws a ball upward at an angle θ_0 with an initial speed v_0. He tries to catch up to the ball by accelerating with a constant acceleration a for a time interval Δt_1 and then continues to run at a constant speed for a time interval Δt_2. He catches the ball at exactly the same height he threw the ball. Let g be the gravitational constant. What was the person's acceleration a?



Problem 2 Solutions:

In this problem there are two objects moving. The person and the ball. The ball undergoes projectile motion so we have the kinematic equations for the ball. The person undergoes two stages of motion. The first stage is constant acceleration and the second stage is constant velocity so we can write separate equations describing the position for each stage noting that the final position and velocity at the end of the first stage are the initial conditions at the beginning of the second stage. The constraint is that the ball and the person intersect at the end of the second stage. We first draw a coordinate system and a graph of the motion of the two objects. Let's choose the origin at the point the ball was released and assume that the person catches the ball at the same height above the ground as it was released.





The ball is in flight for a total time $t_f = \Delta t_1 + \Delta t_2$. The equations for the x and y positions of the ball are then

$$x_2(t) = v_0 \cos\theta_0 t \,, \tag{2.1}$$

$$y_2(t) = v_0 \sin\theta_0 t - \frac{1}{2}gt^2 . \tag{2.2}$$

At the final instant $t_f = \Delta t_1 + \Delta t_2$, the ball is located at

$$x_2(t_f) = v_0 \cos\theta_0(\Delta t_1 + \Delta t_2) \tag{2.3}$$

Note that at $t_f = \Delta t_1 + \Delta t_2$,

$$0 = y_2(t_f) = v_0 \sin\theta_0 t_f - \frac{1}{2}g t_f^{\,2} \tag{2.4}$$

which we can solve for the t_f

$$t_f = \frac{2v_0 \sin\theta_0}{g} = \Delta t_1 + \Delta t_2 \tag{2.5}$$

This means that the initial speed and angle are related according to

$$\theta_0 = \sin^{-1}\left(\frac{g(\Delta t_1 + \Delta t_2)}{2v_0}\right) \tag{2.6}$$

$$v_0 = \frac{g(\Delta t_1 + \Delta t_2)}{2\sin\theta_0} \tag{2.7}$$

We will leave the rest of our results in terms of the angle θ_0 and v_0.

Stage 1: The equations for position and velocity of the person are:

$$x_1(t) = \frac{1}{2}at^2 , \tag{2.8}$$

$$v_{x1}(t) = at \tag{2.9}$$

At the end of stage at $t = \Delta t_1$, the position and velocity of the person is given by

$$x_1(\Delta t_1) = \frac{1}{2}a(\Delta t_1)^2 , \tag{2.10}$$

$$v_{x1}(\Delta t_1) = a\Delta t_1 \tag{2.11}$$

Stage 2: Let's reset our clock $t = 0$ when the first stage is done. Then initial position at this instant is $x_{10} = \frac{1}{2}a(\Delta t_1)^2$ and the x-component of the velocity at this instant is $v_{x10} = a\Delta t_1$. The the equations for the position of the person for the second stage are then

$$x_1(t) = x_{10} + v_{x01}t = \frac{1}{2}a(\Delta t_1)^2 + a\Delta t_1 t \tag{2.12}$$

In particular after an interval $t = \Delta t_2$ has elapsed the person is at the position

$$x_1(\Delta t_2) = \frac{1}{2}a(\Delta t_1)^2 + a\Delta t_1 \Delta t_2 \tag{2.13}$$

.

Since the person and the ball are at the same position, we can equate the right hand sides of Eq. (2.3) and Eq. (2.13) and find that

$$v_0 \cos\theta_0 (\Delta t_1 + \Delta t_2) = \frac{1}{2}a(\Delta t_1)^2 + a\Delta t_1 \Delta t_2 \tag{2.14}$$

We can now solve Eq. (2.14) for the acceleration of the person

$$a = \frac{v_0 \cos\theta_0 (\Delta t_1 + \Delta t_2)}{\frac{1}{2}(\Delta t_1)^2 + \Delta t_1 \Delta t_2} \tag{2.15}$$

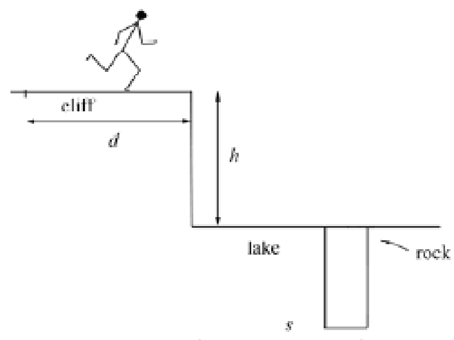
Note that we could substitute Eq. (2.6) for the initial angle or Eq. (2.7) for the initial speed into Eq. (2.15).

Problem 3:

A person, standing on a vertical cliff a height h above a lake, wants to jump into the lake but notices a rock just at the surface level with its furthest edge a distance s from the shore. The person realizes that with a running start it will be possible to just clear the rock, so the person steps back from the edge a distance d and starting from rest, runs at an acceleration that varies in time according to

$$a_x = b_1 t$$

and then leaves the cliff horizontally. The person just clears the rock. Find s in terms of the given quantities d, b_1, h, and the gravitational constant g. You may neglect all air resistance.



Problem 3 Solution:

While the person is running, since the acceleration varies in time, in order to find the x-component of the velocity, we integrate the x-component of the acceleration with respect to time

$$\begin{aligned} v_x(t) - v_{x0} &= \int_0^t a_x(t)dt = \int_0^t (b_1 t)dt \\ &= \tfrac{1}{2} b_1 t^2 \end{aligned} . \quad (3.1)$$

Since the runner started from rest $v_{x0} = 0$ and therefore

$$v_x(t) = \tfrac{1}{2} b_1 t^2 . \quad (3.2)$$

We now integrate the x-component of the velocity with respect to time to find the displacement

$$x(t) - x_0 = \int_0^t v_x(t)dt = \int_0^t (\tfrac{1}{2} b_1 t^2)dt \\ = \tfrac{1}{6} b_1 t^3 \quad (3.3)$$

Choose an origin from the point the runner began, so $x_0 = 0$, and the position function is

$$x(t) = \tfrac{1}{6} b_1 t^3 . \quad (3.4)$$

Let $t = t_1$ denote the instant the runner reaches the cliff, then

$$x(t_1) = d = \tfrac{1}{6} b_1 t_1^{\ 3} . \quad (3.5)$$

Hence

$$t_1 = \left(\frac{6d}{b_1} \right)^{1/3} . \quad (3.6)$$

The x-component of the velocity of the runner at $t = t_1$ is tehn

$$v_x(t_1) = \tfrac{1}{2} b_1 t_1^{\ 2} = \tfrac{1}{2} b_1 \left(\frac{6d}{b_1} \right)^{2/3} . \quad (3.7)$$

Let's restart our clock at $t = 0$ when the runner leaves the cliff. Let's pick our origin at the point of departure. Since the person is in free fall when in the air and we can neglect air resistance, the runner has acceleration $a_y = -g$. So the y-component of the position of the runner is given by

$$y(t) = y_0 - \frac{1}{2} g\, t^2 . \quad (3.8)$$

At $t = 0$, $y_0 = 0$, so

$$y(t) = h - \frac{1}{2} g\, t^2 . \quad (3.9)$$

Let $t = t_2$ denote the instant the runner hits the water, then

$$y(t_2) = -h = -\frac{1}{2} g\, t_2^{\ 2} . \quad (3.10)$$

So we can solve for the time of flight,

$$t_2 = \left(\frac{2h}{g} \right)^{1/2} . \qquad (3.11)$$

The horizontal position of the runner from the edge of the cliff is given by

$$x(t) = v_{x0} t . \qquad (3.12)$$

Since we know the runner leaves the cliff with an x-component of the velocity equal to the x-component of the velocity of the runner at $t = t_1$, we have that

$$x(t) = \tfrac{1}{2} b_1 \left(\frac{6d}{b_1} \right)^{2/3} t . \qquad (3.13)$$

At $t = t_2$, the horizontal distance to the rock is then

$$x(t_2) = s = \tfrac{1}{2} b_1 \left(\frac{6d}{b_1} \right)^{2/3} t_2 = \tfrac{1}{2} b_1 \left(\frac{6d}{b_1} \right)^{2/3} \left(\frac{2h}{g} \right)^{1/2} . \qquad (3.14)$$

or

$$s = \tfrac{1}{2} b_1 \left(\frac{6d}{b_1} \right)^{2/3} \left(\frac{2h}{g} \right)^{1/2} . \qquad (3.15)$$

Problem 4:

A bicyclist has an acceleration in the x-direction given by

$$a_x = b_1 t^2, \quad 0 \le t \le t_1$$

where $b_1 > 0$. At $t = 0$ the bicyclist is located at $x_0 = 0$ and starts from rest. At that same instant ($t = 0$) a ball, located along the path of motion of the bicycle at $x_{ball}(t = 0) = d$, is thrown vertically upwards with a non-zero y-component of velocity given by v_{y0}. At some later time $t = t_1$ the bicyclist catches the ball on the way down from the same height that is was thrown. You may ignore any effects of air resistance. Find an expression for b_1 in terms of d, v_{y0}, and g but do not use t_1 in your answer. What are the dimensions of b_1?

Problem 4 Solution:

We integrate the x-component of the acceleration with respect to time to find change in the x-component of the velocity

$$\begin{aligned} v_x(t) - v_{x0} &= \int_0^t a_x(t)dt = \int_0^t \left(b_1 t^2\right) dt \\ &= \tfrac{1}{3} b_1 t^3 \end{aligned} \quad (4.1)$$

Since the bicyclist started from rest $v_{x0} = 0$ and therefore

$$v_x(t) = \tfrac{1}{3} b_1 t^3. \quad (4.2)$$

We now integrate the x-component of the velocity with respect to time to find the displacement

$$\begin{aligned} x(t) - x_0 &= \int_0^t v_x(t)dt = \int_0^t (\tfrac{1}{3} b_1 t^3) dt \\ &= \tfrac{1}{12} b_1 t^4 \end{aligned} \quad (4.3)$$

Since the bicyclist started from the origin $x_0 = 0$, the position function is

$$x(t) = \tfrac{1}{12} b_1 t^4. \quad (4.4)$$

Thus at $t = t_1$, the bicyclist is located at

$$x(t_1) = d = \tfrac{1}{12} b_1 t_1^{\,4} \,. \tag{4.5}$$

The ball has acceleration $a_y = -g$ so the y-component of the velocity of the ball is given by

$$v_y(t) = v_{y0} - g\,t \,. \tag{4.6}$$

At the top of its flight, the y-component of the velocity is zero so

$$0 = v_y(t_{top}) = v_{y0} - g t_{top} \tag{4.7}$$

which we can solve for t_{top}:

$$t_{top} = v_{y0} / g \,. \tag{4.8}$$

Therefore the ball takes a time

$$t_1 = 2\, t_{top} = 2 v_{y0} / g \,. \tag{4.9}$$

to return to its initial height. Thus the bicyclist has traveled a distance

$$d = \tfrac{1}{12} b_1 \left(\frac{2 v_{y0}}{g} \right)^4 . \tag{4.10}$$

We can solve this equation for b_1:

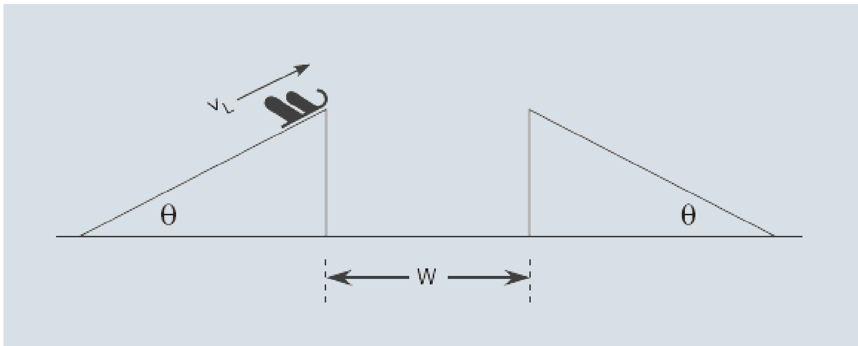
$$b_1 = \frac{3 d g^4}{4 v_{y0}^{\,4}} . \tag{4.11}$$

The dimensions of b_1 are the dimensions of acceleration divided by the dimensions of time to the squared power:

$$\dim(b_1) = (M \cdot T^{-2}) / T^2 = (M \cdot T^{-4}) \,. \tag{4.12}$$

Problem 5:

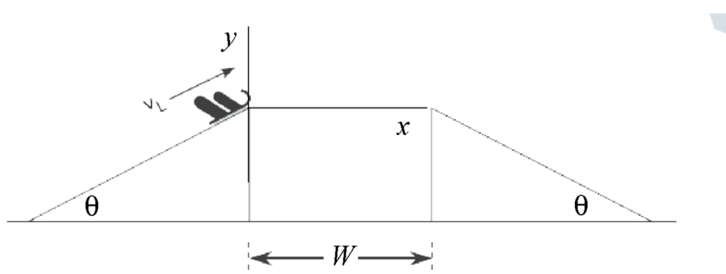
Calvin and Hobbes are riding on a sled. They are trying to jump the gap between two symmetrical ramps of snow separated by a distance W as shown above. Each ramp makes an angle θ with the horizontal. They launch off the first ramp with a speed v_L. Calvin, Hobbs and the sled have a total mass m.



a) What value of the initial launch speed v_L will result in the sled landing exactly at the peak of the second ramp? Express your answer in terms of some (or all) of the parameters m, θ, W, and the acceleration of gravity g. Include in your answer a brief description of the strategy that you used and any diagrams or graphs that you have chosen for solving this problem. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities.

Problem 5 Solution:

The sled is undergoing projectile motion with an initial launch speed v_L in the direction given by a positive θ with respect to the horizontal. We choose a coordinate system as shown in the figure below:



Then the components of the acceleration are given by $a_x = 0$ and $a_y = -g$. We are given the range W, so we can use the equations for the y-components of position to find the time of flight and then use the x-component of position to find the initially launch speed v_L.

The y-component of the position is given by

$$y(t) = v_L \sin\theta t - \frac{1}{2} g t^2 \tag{5.1}$$

Since the sled returns to it's initial starting point at the instant $t = t_f$, Eq. (5.1) becomes

$$y(t = t_f) = 0 = v_L \sin\theta t_f - \frac{1}{2} g t_f^{\,2} \tag{5.2}$$

We can solve Eq. (5.2) for the time of flight

$$t_f = \frac{2 v_L \sin\theta}{g} \tag{5.3}$$

The x-component of the position is given by

$$x(t) = v_L \cos\theta t \tag{5.4}$$

At the end of its flight. Eq. (5.4) becomes

$$x(t_f) = W = v_L \cos\theta t_f \tag{5.5}$$

Substitute Eq. (5.3) into Eq. (5.5) to find that

$$W = \frac{2 v_L^{\,2} \cos\theta \sin\theta}{g} = \frac{v_L^{\,2} \sin 2\theta}{g} \tag{5.6}$$

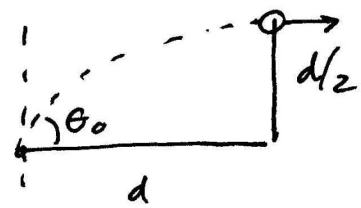
We can now solve Eq. (5.6) for the launch speed

$$v_L = \sqrt{\frac{gW}{\sin 2\theta}} . \tag{5.7}$$

Hobbes was clutching a bag of tiger food when they left the first ramp. He got so excited while in the air that he let go of the bag at the top of the flight, lightening the total mass attached to the sled by 10%. Explain qualitatively how this will affect the results you found in a).

Problem 6:

A person is a playing a game that requires throwing an object onto a ledge. The ledge is a distance d and a height $d/2$ above the release point. You may neglect air resistance. You may use g for the magnitude of the gravitational acceleration.



(a) At what angle must the person throw the object and with what magnitude of the velocity if the object is to be exactly at the top of its flight when it reaches the ledge? Briefly describe how you will model this problem and your strategy for finding the answer. Express your answer in terms of the given quantities d and g, as needed.

(b) What is the horizontal component of the acceleration? Again briefly describe your model and strategy for solving the problem. Express your answer in terms of the given quantities s, d, and g.

Problem 6 Solutions:

a) The object is undergoing projectile motion and there are three conditions that are fixed when the ball reaches the top of the ledge at the instant $t = t_{top}$: $v_y(t = t_{top}) = 0$, $y(t = t_{top}) = \frac{d}{2}$, and $x(t = t_{top}) = d$. We can use these conditions to solve the projectile motion equations for the initial x- and y-component of the velocity, v_{x0} and v_{y0}, in terms of the given quantities d and g. But we will first solve for the time it takes to reach the top of the ledge. We can then find the initial angle by solving $\theta_0 = \tan^{-1}\frac{v_{y0}}{v_{x0}}$.

Let's choose our origin at the release point (set the clock to zero at this instant) in the figure above. Since the object is undergoing parabolic motion the equations for position components and the y-component of the velocity are given by

$$x(t) = v_{x0}t \tag{6.1}$$

$$y(t) = v_{y0}t - \frac{1}{2}gt^2. \tag{6.2}$$

$$v_y(t) = v_{y0} - gt \tag{6.3}$$

When the object reaches the top of the ledge at the instant $t = t_{top}$, the y-component of the velocity is zero

$$0 = v_y(t = t_{top}) = v_{y0} - g t_{top} \tag{6.4}$$

We can solve this equation for the $t = t_{top}$,

$$t_{top} = \frac{v_{y0}}{g}. \tag{6.5}$$

When the object reaches the top of the ledge at the instant $t = t_{top}$, the y-component of the position is

$$y(t = t_{top}) = \frac{d}{2} = v_{y0} t_{top} - \frac{1}{2} g t_{top}{}^2 \tag{6.6}$$

We now substitute Eq. (6.5) into Eq. (6.6) and solve for

$$\frac{d}{2} = v_{y0} \frac{v_{y0}}{g} - \frac{1}{2} g \left(\frac{v_{y0}}{g} \right)^2 = \frac{1}{2g} (v_{y0})^2 \tag{6.7}$$

We can then solve this equation for the y-component of the initial velocity

$$v_{y0} = \sqrt{gd}\,. \tag{6.8}$$

We now substitute Eq. (6.8) into Eq. (6.5) and find that

$$t_{top} = \frac{v_{y0}}{g} = \frac{\sqrt{gd}}{g} = \sqrt{\frac{d}{g}} \tag{6.9}$$

We also know that when the object reaches the top of the ledge at the instant $t = t_{top}$, the x-component of the position is $x(t = t_{top}) = d$. Therefore Eq. (6.1) becomes at

$$x(t = t_{top}) = d = v_{x0} t_{top} \tag{6.10}$$

We can solve Eq. (6.10) for the x-component of the velocity which yields

$$v_{x0} = \frac{d}{t_{top}} \tag{6.11}$$

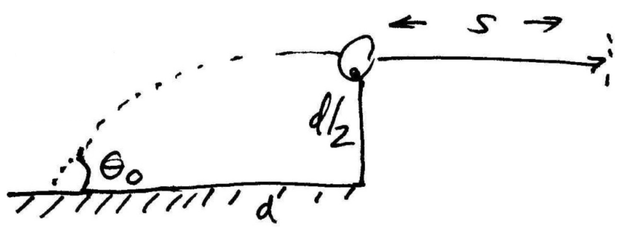
We now substitute in Eq. (6.9) and find that

$$v_{x0} = \frac{d}{t_{top}} = d\sqrt{g/d} = \sqrt{gd} \qquad (6.12)$$

Comparing Eq (6.12) and Eq. (6.8), we see that $v_{x0} = v_{y0} = \sqrt{gd}$ and hence $\theta_0 = \tan^{-1}(v_{y0}/v_{x0}) = \tan^{-1}(1) = 45^\circ$. Then the initial speed can be found by noting that

$$v_0 = \frac{v_{x0}}{\cos\theta_0} = v_0\cos\theta_0 = \frac{\sqrt{gd}}{\sqrt{2}/2} = \sqrt{2gd} \qquad (6.13)$$

Once the object reaches the ledge it slows down with a constant deceleration and comes to a stop after sliding a distance s.



b) Solution: When the object slows down, we are assuming the acceleration is constant (this time negative). So once again we can use the equations for the position and velocity for an object moving with constant (and negative) x-component of the acceleration. Let's chjoose a new origin at the start of the ledge and reset our clocks to $t = t_0$ when the object just reaches the top of the ledge. We are given that it comes to a rest after moving a distance s during a time interval $t = t_f$. So the final x-component of the velocity is zero. Thus $x_0 = 0$, $x(t = t_f) = s$ and $v_x(t = t_f) = 0$. The x-component of the velocity when the object jyst reaches the ledge is given by Eq. (6.12), $v_{x0} = \sqrt{gd}$. So we should be able to solve the equations for the position and the x-component of the velocity of the object to find the horizontal component of the acceleration.

The equations are:

$$x(t) = v_{x0}t + \frac{1}{2}a_x t^2 = \sqrt{gd}t + \frac{1}{2}a_x t^2, \qquad (6.14)$$

$$v_x(t) = v_{x0} + a_x t. \qquad (6.15)$$

At $t = t_f$, these equations become

$$s = x(t = t_f) = \sqrt{gd}\ t_f + \frac{1}{2} a_x t_f^{\ 2}, \tag{6.16}$$

$$0 = v_x(t = t_f) = \sqrt{gd} + a_x t_f\,. \tag{6.17}$$

We can solve Eq. (6.17) for the time interval it takes to come to rest,

$$t_f = -\frac{\sqrt{gd}}{a_x}\,. \tag{6.18}$$

We substitute Eq. (6.18) into Eq. (6.16) to find that

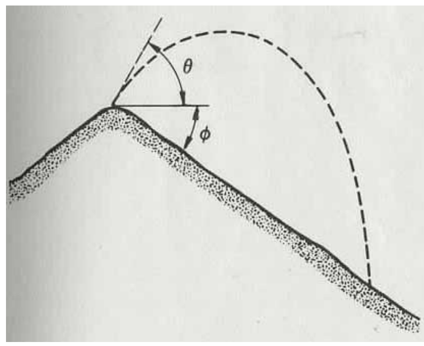
$$s = \sqrt{gd}\left(-\frac{\sqrt{gd}}{a_x}\right) + \frac{1}{2} a_x \left(-\frac{\sqrt{gd}}{a_x}\right)^2 = -\frac{gd}{2a_x}\,. \tag{6.19}$$

Finally we can solve Eq. (6.19) for the x-component of the acceleration

$$a_x = -\frac{gd}{2s}\,. \tag{6.20}$$

Problem 7: Two-Dimensional Kinematics: ***Projectile motion***

A person is standing on top of a hill that slopes downwards uniformly at an angle ϕ with respect to the horizontal. The person throws a stone at an initial angle θ_0 from the horizontal with an initial speed of v_0. You may neglect air resistance. What is the horizontal range of the stone when the stone strikes the ground?



Problem 7 Solution:

In the absence of air friction or other forces other than gravity, the stone's horizontal component of velocity will be constant, $v_x = v_0 \cos\theta$. Taking the top of the hill as the origin, the stone's x-coordinate as a function of time is $x = \left(v_0 \cos\theta\right)t$. The stone's vertical component of velocity is

$$v_y = v_{0y} - g\,t = v_0 \sin\theta - g\,t \tag{7.1}$$

and its vertical position is

$$\begin{aligned} y &= \left(v_0 \sin\theta\right)t - \frac{1}{2}g\,t^2 \\ &= \left(v_0 \sin\theta\right)\frac{x}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta}\right)^2 \\ &= x\tan\theta - x^2 \frac{g}{2\,v_0^{\;2}\cos^2\theta}, \end{aligned} \tag{7.2}$$

where $t = x / \left(v_0 \cos\theta\right)$ has been used to express y in terms of x.

The position of any point on the downward slope is

$$y = -x\;\tan\phi \tag{7.3}$$

and the stone will land at the point where the expressions in Equations (7.2) and (7.3) are equal,

$$x\tan\theta - x^2 \frac{g}{2{v_0}^2\cos^2\theta} = -x\tan\phi\,. \tag{7.4}$$

The $x = 0$ root of Equation (7.4) represents the stone being at the top of the hill, and may be neglected. Solving Equation (7.4) for x gives

$$\begin{aligned} x &= \left(\tan\theta + \tan\phi\right)\cos^2\theta \frac{2{v_0}^2}{g} \\ &= \left(\sin\theta\cos\theta + \cos^2\theta\tan\phi\right)\frac{2{v_0}^2}{g}. \end{aligned} \tag{7.5}$$

Problem 8: Instructions to Shoot Down a Rising Weather Balloon

You must give the crew of a howitzer instructions for shooting down an enemy weather balloon (diameter $d = 5\text{ m}$) launched a known distance ($D = 5\text{ km to }10\text{ km}$) from their gun. They can control the velocity and angle of their gun within the limit that it has a 25 km range (but can be fired at a controllably lower velocity) and maximum elevation of 45 degrees, and can fire the gun any time. It takes 30 seconds to change the speed of the projectile (aiming takes zero time; it can track a target's motion), and 50 seconds for loading or reloading. After the balloon has risen 5 meters, it rises at a constant speed v_B, which is typically in the range 100 to 300 meters per minute. Try to come up with as simple instructions as possible. You can assume the gun crew has optical surveying equipment. The weather team does not launch balloons if the wind is over 0.3 meters per second.

Write as simple instructions as possible, and show calculations that indicate the projectile will hit the target and that you have considered and solved the likely complications.

Problem 8 Solutions:

The 25 km range implies $v_0 \simeq 500\text{ m s}^{-1}$. So it can take about 20 sec for the projectile to reach the balloon. During 20 s, the balloon may rise 33 m to 100 m. The uncertainty in the height is much more than the diameter of the balloon. So the instructions must include a way for the crew to estimate the vertical speed of the balloon.

1. Load the gun for maximum velocity and set the elevation angle for the gun an the angle that will hit the balloon at an elevation of 350 meters that is a horizontal distance D from the gun. Calculate the time interval Δt_{proj} that it will the projectile to reach the range of the target balloon.

2. To get good velocity measurement of the balloon, the crew must measure the time interval Δt_1 between when they see it at 50 m and 200 m altitude (precompute the angles for the crew), which should be larger than 30 s.

4. After the balloon reaches 200 m it will travel another 150 m to reach 350 m so it will take an additional time interval Δt_1 to reach 350 meters elevation.

5. The time interval Δt_1 it take the balloon to travel upwards 150 m is greater than the time interval Δt_{proj} in takes the projectile to reach the balloon, therefore the crew must wait a time interval $\Delta t_1 - \Delta t_{proj}$ after the balloon reaches 200m and then fire the gun.

Wind: The balloon may move ~6 m due to wind. For motion left and right, the gunner can guess where to shoot from the lateral displacement measured in 2 and 3. Movement toward/away from the gun won't matter as the projectile is moving close to horizontal.

The wind can move the balloon up to 200 feet, so this makes an error that needs correction for also, but only approximately.

Problem 9: Hitting the Bucket

A person is standing on a ladder holding a pail. The person releases the pail from rest at a height above the ground. A second person standing a horizontal distance s_2 from the pail aims and throws a ball the instant the pail is released in order to hit the pail. The person throws the ball at a height h_2 above the ground, with an initial speed v_0, and at an angle θ_0 with respect to the horizontal. You may ignore air resistance.

Questions:

a) Find an expression for the angle θ_0 that the person aims the ball in order to hit the pail as a function of the other variables given in the problem

b) If the person aims correctly, find an expression for the range of speeds that the ball must be thrown at in order to hit the pail?

c) Find an expression for the angle θ_0 that the person throws the ball as a function of h_1, h_2, and s_2.

d) Find an expression for the time of collision as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and s_2.

e) Find an expression for the height above the ground where the collision occurred as a function of the initial speed of the ball v_0, and the quantities h_1, h_2, and.

f) Find an expression for the range of speeds (as a function of h_1, h_2, and s_2) that the ball can be thrown in order that the ball will collide with the pail?

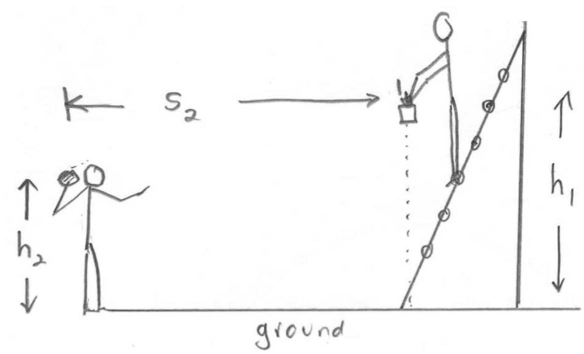
Problem 9 Solutions:

In case you need some help, try answering these questions.

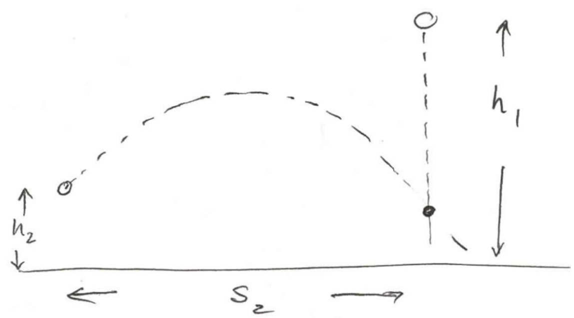
I. Understand – get a conceptual grasp of the problem

Sketch the motion of all the bodies in this problem. Introduce a coordinate system.

Sketch and Coordinate system:



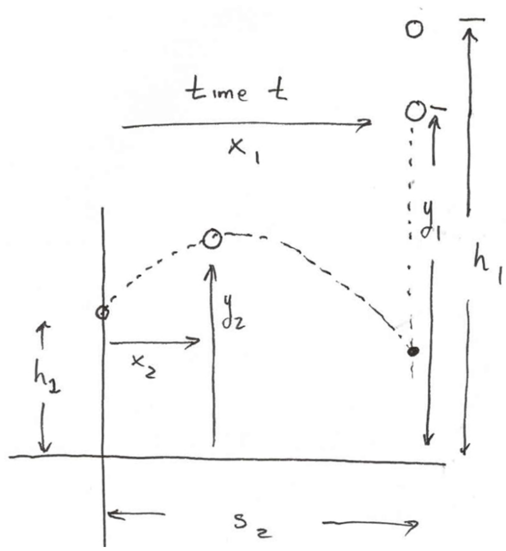
Sketch of motion:



There are two objects involved in this problem. Each object is undergoing free fall, so there is only one stage each. The pail is undergoing one dimensional motion. The ball is undergoing two dimensional motion. The parameters h_1, h_2, and s_2 are unspecified, so our answers will be functions of those symbolic expressions for the quantities.

Since the acceleration is unidirectional and constant, we will choose Cartesian coordinates, with one axis along the direction of acceleration. Choose the origin on the ground directly underneath the point where the ball is released. We choose upwards for the positive y-direction and towards the pail for the positive x-direction.

We choose position coordinates for the pail as follows. The horizontal coordinate is constant and given by $x_1 = s_2$. The vertical coordinate represents the height above the ground and is denoted by $y_1(t)$. The ball has coordinates $(x_2(t), y_2(t))$. We show these coordinates in the figure below.



II. Devise a Plan - set up a procedure to obtain the desired solution

Question: What equations of motion follow from your model for the position and velocity functions of each body?

Model: The pail undergoes constant acceleration $(a_y)_1 = -g$ in the vertical direction downwards and the ball undergoes uniform motion in the horizontal direction and constant acceleration downwards in the vertical direction, with $(a_x)_2 = 0$ and $(a_y)_2 = -g$.

Equations of Motion for Pail:

The initial conditions for the pail are $(v_{y,0})_1 = 0$, $x_1 = s_2$, $(y_0)_1 = h_1$. Since the pail moves vertically, the pail always satisfies the constraint condition $x_1 = s_2$ and $v_{x,1} = 0$. The equations for position and velocity of the pail simplify to

$$y_1(t) = h_1 - \frac{1}{2}gt^2$$

$$v_{y,1}(t) = -gt$$

Equations of Motion for Ball:

The initial position is given by $(x_0)_2 = 0$, $(y_0)_2 = h_2$. The components of the initial velocity are given by $(v_{y,0})_2 = v_0 \sin(\theta_0)$ and $(v_{x,0})_2 = v_0 \cos(\theta_0)$, where v_0 is the magnitude of the initial velocity and θ_0 is the initial angle with respect to the horizontal. So the equations for position and velocity of the ball simplify to

$$x_2(t) = v_0 \cos(\theta_0)t$$

$$v_{x,2}(t) = v_0 \cos(\theta_0)$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2$$

$$v_{y,2}(t) = v_0 \sin(\theta_0) - gt$$

Question: How many independent equations and unknowns do you have? Should the quantities h_1, h_2, and s_2 be treated as knowns or unknowns.

Answer: Note that the quantities h_1, h_2, and s_2 should be treated as known quantities although no numerical values were given, only symbolic expressions. There are six independent equations with 9 as yet unspecified quantities $y_1(t)$, t, $y_2(t)$, $x_2(t)$, $v_{y,1}(t)$, $v_{y,2}(t)$, , v_0, θ_0.

$$y_1(t) = h_1 - \frac{1}{2}gt^2$$

$$v_{y,1}(t) = -gt$$

$$x_2(t) = v_0 \cos(\theta_0)t$$

$$v_{x,2}(t) = v_0 \cos(\theta_0)$$

$$y_2(t) = h_2 + v_0 \sin(\theta_0)t - \frac{1}{2}gt^2$$

$$v_{y,2}(t) = v_0 \sin(\theta_0) - gt$$

So we need two more conditions, in order to find expressions for the initial angle, θ_0, the time of collision, t_a, and the spatial location of the collision point specified by $y_1\left(t_a\right)$ or $y_2\left(t_a\right)$ in terms of the one unspecified parameter v_0.

Question: What mathematical formulae follow from the phase “hits the pail”?

Answer: At the collision time $t = t_a$, the collision occurs when the two balls are located at the same position. Therefore

$$y_1(t_a) = y_2(t_a),\ \text{and}\ x_2(t_a) = x_1 = s_2\,.$$

Question: Clean up your equations. What strategy can you design for finding the angle the second person needs to aim the ball?

Answer:

We shall apply the conditions we found for the ball hitting the pail.

$$h_1 - \frac{1}{2} g t_a^{\ 2} = h_2 + v_0 \sin(\theta_0) t_a - \frac{1}{2} g t_a^{\ 2}$$

$$s_2 = v_0 \cos(\theta_0) t_a$$

From the first equation, the term $\frac{1}{2} g t_a^{\ 2}$ cancels from both sides. Therefore we have that

$$h_1 = h_2 + v_0 \sin(\theta_0) t_a$$

$$s_2 = v_0 \cos(\theta_0) t_a\,.$$

We will now solve these equations for $\tan(\theta_0) = \sin(\theta_0) / \cos(\theta_0)$, and thus the angle the person throws the ball in order to hit the pail.

III. Carry our your plan – solve the problem!

We rewrite these equations as

$$v_0 \sin(\theta_0) t_a = h_1 - h_2$$

$$v_0 \cos(\theta_0) t_a = s_2$$

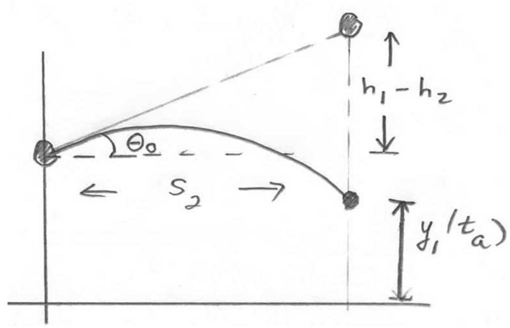
Dividing these equations yields

$$\frac{v_0 \sin(\theta_0) t_a}{v_0 \cos(\theta_0) t_a} = \tan(\theta_0) = \frac{h_1 - h_2}{s_2}\,.$$

So the initial angle is independent of v_0, and is given by

$$\theta_0 = \tan^{-1}\left(\left(h_1 - h_2\right)/s_2\right)$$

From the figure below we can see that $\tan(\theta_0) = (h_1 - h_2)/s_2$, implies that the second person aims the ball at the initial position of the pail.



Question: When does the ball collide with the pail?

Answer: In order to find the time of collision as a function of the initial speed, we begin with our results that

$$v_0 \sin(\theta_0) t_a = h_1 - h_2$$

$$v_0 \cos(\theta_0) t_a = s_2$$

We square both of the equations above and utilize the trigonometric identity

$$\sin^2(\theta_0) + \cos^2(\theta_0) = 1 .$$

So our squared equations become

$$v_0^{\ 2} \sin^2(\theta_0) t_a^{\ 2} = \left(h_1 - h_2\right)^2$$

$$v_0^{\ 2} \cos^2(\theta_0) t_a^{\ 2} = s_2^{\ 2}$$

Adding these equations together yields

$$v_0^{\ 2} \sin^2(\theta_0) t_a^{\ 2} + v_0^{\ 2} \cos^2(\theta_0) t_a^{\ 2} = v_0^{\ 2} t_a^{\ 2} = s_2^{\ 2} + \left(h_1 - h_2\right)^2 .$$

We can solve this for the time of collision

$$t_a = \left(\frac{s_2^{\ 2} + (h_1 - h_2)^2}{v_0^{\ 2}} \right)^{1/2}.$$

Question: At what height does the ball collide with the pail?

Answer: We can use the y-coordinate function of either the ball or the pail at $t = t_a$. Since it had no initial y velocity, it's easier to use the pail,

$$y_1(t_b) = h_1 - \frac{g\left(s_2^{\ 2} + (h_1 - h_2)^2\right)}{2v_0^{\ 2}}$$

Question: What is the minimum speed the person must throw the ball in order to ensure that there will be a collision?

Answer: Suppose the ball and the pail collide exactly at the ground at the time $t = t_b$. The condition is that the speed v_0 must be great enough such that the ball reaches $x_2(t_b) = s_2$ before the ball hits the ground, $y_2(t_b) = y_1(t_b) = 0$. .

So we require that

$$x_2(t_b) = v_0 \cos(\theta_0) t_b \geq s_2$$

This condition is easiest to apply when solving for the time, $t = t_b$, that the pail hits the ground,

$$y_1(t_b) = h_1 - \frac{1}{2} g t_b^{\ 2} = 0.$$

Thus

$$t_b = \left(\frac{2h_1}{g} \right)^{1/2}.$$

Therefore the condition becomes

$$x_2(t_b) = v_0 \cos(\theta_0) \left(\frac{2h_1}{g} \right)^{1/2} \geq s_2$$

or

$$v_0 \geq \left(\frac{g}{2h_1}\right)^{1/2} \frac{s_2}{\cos(\theta_0)} = \left(\frac{g}{2h_1}\right)^{1/2} \frac{s_2}{\cos\left(\tan^{-1}\left((h_1 - h_2)/s_2\right)\right)}.$$

IV. Look Back – check your solution and method of solution

Question: check your algebra and your units. Any obvious errors?

Answer: No obvious errors.

Question: Can you see that the answer is correct now that you have it – often simply by retrospective inspection?

Answer: The person aims at the pail at the point where the pail was released. Both undergo free fall so the key result was that the vertical position obeys

$$h_1 - \frac{1}{2}g t_a^{\,2} = h_2 + v_0 \sin(\theta_0) t_a - \frac{1}{2}g t_a^{\,2}.$$

The distance traveled due to gravitational acceleration are the same for both so all that matters is the contribution form the initial positions and the vertical component of velocity

$$h_1 = h_2 + v_0 \sin(\theta_0) t_a.$$

Since the time is related to the horizontal distance by

$$s_2 = v_0 \cos(\theta_0) t_a$$

This is now as if both objects were moving at constant velocity.

Question: Substitute some values for the initial conditions that test the limits of your answer. Make the pail very far away or very close and what your answer predicts about the time of flight or the collision height.

Answer: Let $s_2 = 10\text{ m}$, $v_0 = 20\text{ m}\cdot\text{s}^{-1}$, let $h_2 = 2\text{ m}$, and $h_1 = 4\text{ m}$. Then the condition for the initial speed is satisfied since

$$v_0 \geq \left(\frac{g}{2h_1}\right)^{1/2} \frac{s_2}{\cos\left(\tan^{-1}\left(\left(h_1 - h_2\right)/s_2\right)\right)}$$

$$= \left(\frac{10\ \text{m}\cdot\text{s}^{-2}}{2(4\ \text{m})}\right)^{1/2} \frac{10\ \text{m}}{\cos\left(\tan^{-1}\left(\left(4\ \text{m} - 2\ \text{m}\right)/10\ \text{m}\right)\right)} = 11.4\ \text{m}\cdot\text{s}^{-1}$$

Question: Can you solve it a different way? Is the problem equivalent to one you've solved before if the variables have some specific values?

Answer: This is an unusual application of moving to a reference frame accelerating downwards with $A_y = -g$. Then the problem is simply

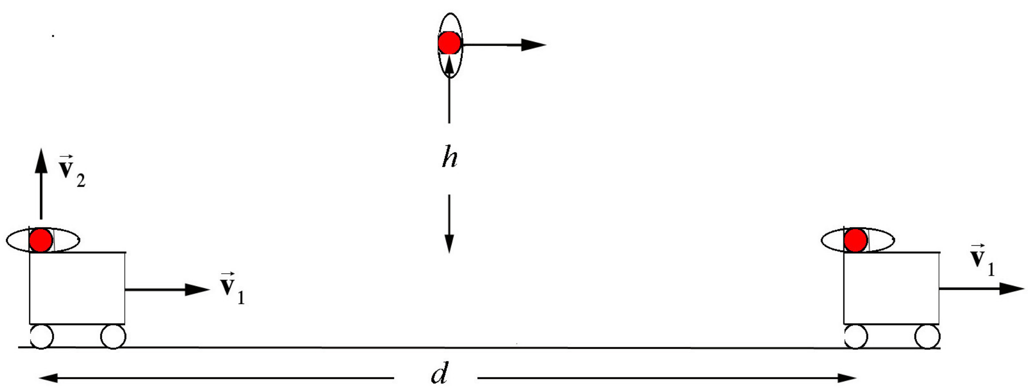
$$y_1'(t) = h_1$$

$$x_2'(t) = v_0 \cos(\theta_0) t$$

$$y_2'(t) = h_2 + v_0 \sin(\theta_0) t$$

Problem 10: Moving cart and hoop Kinematics

A cart moving with horizontal velocity $\vec{\mathbf{v}}_1$ with magnitude v_1 ejects a ball with a velocity $\vec{\mathbf{v}}_2$ (magnitude v_2) in the vertical direction <u>relative</u> to the moving cart. The ball must pass <u>horizontally</u> through a small stationary hoop appropriately oriented and located at a height h above the launch point. Later the ball passes through a second hoop that is fixed horizontally on the cart and centered at the launch point. Let g be the gravitational acceleration. Express your answers in terms of the given quantities v_1, v_2, and g. You may ignore air resistance, and any rolling friction on the cart.



a) Why will the ball pass through the hoop mounted on the cart as it comes back down? Briefly explain your answer.

b) Find an expression for the time interval it takes the ball to move from its launch point to the center of the top hoop. You may neglect the thickness of the hoop. Show all your work.

c) Find an expression for the vertical distance h between the center of top hoop and the launch point. Show all your work.

d) Find an expression for the horizontal distance d the ball has traveled relative to the ground when it returns to its launch point on the cart. Show all your work.

Problem 10 Solutions:

a) The key is that the ball passes through the upper hoop *horizontally*. For purposes of this problem, this would mean that the ball has been moving upwards for the same amount of time it has been moving down, and the cart has therefore traveled the same horizontal distance between the time of the launch and the time the ball returns to the cart.

b) Given that the ball moves through the upper hoop *horizontally*, the vertical component of the ball's velocity has decreased from v_2 to zero between the time of launch and the time when the ball passed through the upper hoop. The time interval for this to occur is then $t_1 = v_2 / g$.

c) The height h is then $(1/2)g t_1^2 = \dfrac{v_2^2}{2g}$.

d) The result of part a) above allows us to say that the cart has traveled a distance $d = 2 v_1 t_1 = 2 v_1 v_2 / g$.